Processing-induced anisotropy

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**ABSTRACT**
Processing of seismic data is often performed under the assumption that the velocity distribution in the subsurface can be approximated by a macro-model composed of isotropic homogeneous layers or blocks. Despite being physically unrealistic, such models are believed to be sufficient for describing the kinematics of reflection arrivals.

Here, we examine the distortions in normal-moveout (NMO) velocities caused by the vertical heterogeneity unaccounted for in velocity analysis. To match P-wave moveout measurements from a horizontal or a dipping reflector overlaid by a vertically heterogeneous isotropic medium, the effective homogeneous model has to be anisotropic. This apparent anisotropy is caused not only by velocity monotonically increasing with depth, but also by random velocity variations similar to those routinely observed in well logs.

Assuming that the effective medium is transversely isotropic with a vertical symmetry axis (VTI), we express the VTI parameters through the depth-dependent isotropic velocity function. If the reflector is horizontal, combining the NMO and vertical velocities always results in non-negative values of the Thomsen's coefficient δ. For a dipping reflector, the inversion of the P-wave NMO ellipse yields a non-negative Alkhalifah-Tsvankin coefficient η that increases with dip. The values of η obtained by two other methods – 2-D dip-moveout inversion and nonhyperbolic moveout analysis – are also non-negative but generally differ from that needed to fit the NMO ellipse. These conclusions indicate a potential bias toward positive δ and η estimates in the velocity analysis of reflection data for VTI media.

**Key words:** P-waves, vertically heterogeneous media, effective anisotropy

**Introduction**
Rapid progress in the development of anisotropic velocity-analysis methods has made transverse isotropy with a vertical symmetry axis (VTI media) a common model in time and depth processing of P-wave data (e.g., Alkhalifah et al., 1996; Williamson et al., 1997; Toldi et al., 1999). P-wave depth imaging in VTI media requires estimates of the P-wave vertical velocity V₀ and Thomsen's (1986) anisotropic coefficients ε and δ, while time imaging for models with a laterally homogeneous overburden is controlled by the normal-moveout (NMO) velocity (i.e., stacking velocity in the limit as offset approaches zero) for horizontal reflectors V_nmo and the Alkhalifah-Tsvankin (1995) coefficient η, defined as

\[ η = \frac{ε - δ}{2} \]  (1)

Both time-imaging parameters can be obtained from P-wave reflection traveltimes using either dip-dependent P-wave NMO velocity or nonhyperbolic moveout from horizontal interfaces (e.g., Alkhalifah and Tsvankin, 1995; Grechka and Tsvankin, 1998a; Toldi et al., 1999; for a detailed overview, see Tsvankin, 2001).

In contrast, estimation of the parameters V₀, ε and δ from surface P-wave data is possible for only a certain class of models with dipping intermediate interfaces or some other types of lateral heterogeneity in the overburden (Grechka et al., 2000a, b). Therefore, values of δ are often determined at borehole locations by combining the NMO (stacking) velocity V_nmo measured from surface seismic data and the vertical velocity V₀ derived from check shots or well logs. For a single homogeneous VTI layer, the two velocities are related by...
If the medium is horizontally layered, the interval NMO velocity can be found by the conventional Dix differentiation and used to obtain the interval parameter $\delta$ from equation (2). After estimating both $\delta$ and $\eta$, it is possible to find the remaining parameter $\epsilon$ using equation (1).

In addition to their key role in anisotropic processing, the parameters $\epsilon$, $\delta$, and $\eta$ contain useful information about the petrophysical and lithologic properties of the subsurface formations. Hence, it becomes increasingly important to understand the physical origin of the anisotropic coefficients and their relationship to lithology. Transverse isotropy in sedimentary basins is believed to be caused by two main reasons. One of them is the intrinsic anisotropy of shales resulting from preferential orientation of clay particles aligned by gravity (Sayers, 1994). The second reason is interbedding of thin (compared to the predominant seismic wavelength) isotropic layers that creates an effective TI medium in the long-wavelength limit. The anisotropic parameters of this effective model can be found from either Backus (1962) averaging or the more general procedure of Schoenberg and Muir (1989). In both cases, the properties of seismic waves propagating through the medium are indistinguishable from those observed in a homogeneous VTI model.

In this paper we discuss a different reason for effective transverse isotropy—the assumption of homogeneity commonly used in seismic processing. High-quality reflection events are recorded from only a limited number of interfaces, and it is often assumed that the medium between the interpreted reflectors is homogeneous. We show that if vertical heterogeneity in isotropic media is not properly accounted for, description of P-wave reflection traveltimes requires making the velocity field anisotropic. Estimation of the parameters of this “apparent” (or effective) VTI model using the anisotropic velocity-analysis methods discussed above leads to always non-negative values of $\delta$ (for a horizontal reflector) and $\eta$ (for a dipping reflector).

**Effective NMO velocity from a horizontal reflector**

Figure 1 shows the vertical (zero-offset) and nonzero-offset rays for the simple model of a horizontal reflector beneath a stack of homogeneous isotropic layers. If the reflector depth $D$ is known, the zero-offset (vertical) traveltime $T(0)$ can be used to compute the vertical velocity $V_0$:

$$V_0 = \frac{2D}{T(0)}.$$  \hspace{1cm} (3)

Since the model is vertically heterogeneous, $V_0$ becomes an effective quantity that averages the interval (or local) velocities $v(z)$. If the interfaces in the overburden are not strong enough to generate detectable reflection events, it is natural to treat the whole section above the reflector as homogeneous. Such an assumption, which clearly contradicts the actual model in Figure 1, is the consequence of our inability to reconstruct the vertically varying velocity $v(z)$ using the reflection traveltimes from the bottom of the layer. Note that any permutation of layers in Figure 1 produces exactly the same reflection traveltime $T(x)$ at any offset $x$; this phenomenon is called the “O-equivalence” of velocity functions (Goldin, 1986). Thus, for processing purposes it is necessary to assume a certain velocity distribution $v(z)$.

If the composite layer in Figure 1 is treated as homogeneous and isotropic in accordance with the usual practice of velocity analysis, the reflection traveltime is supposed to be a hyperbola parameterized by the vertical velocity $V_0$:

$$\tilde{T}^2(x) = T^2(0) + \frac{x^2}{V_0^2}.$$  \hspace{1cm} (4)

The traveltime $\tilde{T}$ from equation (4), however, corresponds to ray $ARB$ (Figure 1), which does not satisfy Fermat’s principle. The reflection time $T(x)$ along the actual ray $AL_ARLB_B$ is smaller (according to Fermat’s principle) than that predicted by equation (4). As a result, the NMO velocity $V_{nmo}$ for the model in Figure 1 is always greater than the vertical velocity $V_0$, and the data cannot be explained in terms of a homogeneous isotropic model.

As noticed by Fomel and Grechka (2001) and Grechka, Pech and Tsvankin (2001), the obtained relationship between $V_{nmo}$ and $V_0$ is typical for transversely isotropic media with a positive value of $\delta$ [see equation (2)]. It can be shown (Grechka et al., 2001) that $V_{nmo}^2/V_0^2$ is equal to the ratio of the arithmetic and har-
monic averages of the interval velocities, which cannot be smaller than unity. Therefore, the parameter \( \delta \) of the effective VTI model is given by

\[
1 + 2\delta = \frac{V_{\text{nmo}}^2}{V_0^2} = \frac{1}{\left[ \frac{1}{D} \int_0^D \frac{dz}{v(z)} \right]^2} \geq 1. \tag{5}
\]

This relationship might also explain a certain bias toward positive \( \delta \) values derived from stacking (NMO) and vertical velocities for purposes of anisotropic parameter estimation.

**Effective NMO ellipse from a dipping reflector**

Next, consider P-wave reflection moveout from a plane dipping reflector beneath a vertically heterogeneous isotropic medium (Figure 2). The azimuthally dependent P-wave NMO velocity for this model is described by the NMO ellipse with axes in the dip and strike directions of the reflector (Grechka and Tsvankin, 1998b). Since the dip plane of the reflector represents a plane of symmetry for the whole model, the Dix-type averaging of the interval NMO ellipses (Grechka et al., 1999) reduces to the conventional Dix (1955) formula for the NMO velocities in the dip \( (V_{\text{nmo, dip}}) \) and strike \( (V_{\text{nmo, str}}) \) directions:

\[
V_{\text{nmo, dip}}^2 = \frac{1}{T} \int_0^T v_{\text{nmo, dip}}^2(t) \, dt. \tag{6}
\]

\[
V_{\text{nmo, str}}^2 = \frac{1}{T} \int_0^T v_{\text{nmo, str}}^2(t) \, dt. \tag{7}
\]

Here \( T \) is the two-way zero-offset traveltime, and \( v_{\text{nmo, dip}} \) and \( v_{\text{nmo, str}} \) are the interval (local) dip-line and strike-line NMO velocities computed along the zero-offset ray (Figure 2), as described in Grechka et al. (1999). Since the medium is isotropic, the values of the interval NMO velocities can be adapted from Levin (1971):

\[
v_{\text{nmo, dip}}(t) = \frac{1}{q(t)}, \tag{8}
\]

\[
v_{\text{nmo, str}}(t) = v(t), \tag{9}
\]

where \( v(t) \) is the interval (isotropic) velocity and

\[
q(t) = \sqrt{\frac{1}{v^2(t)} - p^2} \tag{10}
\]

is the vertical component of the slowness vector; \( p \) is the horizontal slowness component (ray parameter).

Note that the ray parameter \( p \) is preserved along any ray propagating in vertically heterogeneous media in accordance with Snell’s law. Since we are not interested in treating inhomogeneous (evanescent) waves, \( q(t) \) is taken to be real and positive for any \( t \), which means that \( p \) satisfies the inequality

\[
p^2 v^2(t) < 1. \tag{11}
\]

Substituting equations (8)–(10) into equations (6) and (7) yields

\[
V_{\text{nmo, dip}}^2 = \frac{1}{T} \int_0^T v^2 \, dt \left[ 1 - p^2 v^2 \right], \tag{12}
\]

\[
V_{\text{nmo, str}}^2 = \frac{1}{T} \int_0^T v^2 \, dt. \tag{13}
\]

If the medium above the reflector were homogeneous, the dip and strike components of the NMO velocity would satisfy the well-known cosine-of-dip relationship that follows from equations (8) and (9),

\[
V_{\text{nmo, dip}}^2 \left( 1 - p^2 V_{\text{nmo, str}}^2 \right) = V_{\text{nmo, str}}^2. \tag{14}
\]

Using equations (12) and (13), we prove in Appendix A that

\[
V_{\text{nmo, dip}}^2 \left( 1 - p^2 V_{\text{nmo, str}}^2 \right) \geq V_{\text{nmo, str}}^2 \tag{15}
\]

for any interval-velocity function \( v(t) \). The equality \( V_{\text{nmo, dip}} = V_{\text{nmo, dip}} = V_{\text{nmo, str}} \) is reached only in the special cases of a horizontal reflector \( (p = 0) \) or homogeneous media \( (v(t) = \text{const}) \).

Figure 3 shows the two-way zero-offset traveltime \( T \) and the velocities \( V_{\text{nmo, dip}} \) and \( V_{\text{nmo, str}} \) computed

\[\text{Figure 2. P-wave NMO ellipse from a dipping reflector overlaid by a vertically heterogeneous isotropic medium.}\]
for a model that contains a plane dipping reflector $z = 1 + Y \tan \phi$ ($\phi$ is the dip) beneath an isotropic medium with a constant vertical-velocity gradient (Figure 3a). The common-midpoint (CMP) locations $Y$ in Figure 3 record reflections from the segment of the interface within the depth range $1 \text{ km} \leq z \leq 2 \text{ km}$. Note that the zero-offset time $T$ (Figure 3b) varies with $Y$ because of the combined influence of the reflector dip and vertical heterogeneity. The gradual decrease of the reflection slope $p(Y) = (1/2)(dT/dY)$ in Figure 3b and the corresponding increase of the velocities in Figure 3c may lead to the conclusion that the subsurface is laterally heterogeneous.

Also, since $V_{\text{NMO, dip}} \neq V_{\text{NMO, str}}$ at any CMP location, the medium above the reflector may be mistakenly identified as anisotropic. It is natural to assume that such a model is transversely isotropic with a vertical symmetry axis, because the actual heterogeneous isotropic medium does not produce any effective azimuthal anisotropy. Below we invert the dip- and strike-components of the NMO velocity for the effective anellipticity coefficient $\eta$ under the assumption that the model is homogeneous and has VTI symmetry.

**Estimation of the effective $V_{\text{NMO}}$ and $\eta$**

Clearly, the subsurface parameters obtained from the traveltimes and velocities in Figures 3b and 3c will depend on the selected model of the overburden. If the model is assumed (correctly) to be vertically heterogeneous and isotropic, the reflection data can be used to estimate the actual function $v(z)$ within the depth range $1 \text{ km} \leq z \leq 2 \text{ km}$ covered by the reflection points (Goldin, 1986). Still, the velocity $v(z)$ cannot be found uniquely for depths $z < 1 \text{ km}$.

Therefore, it may be more attractive from a practical point of view to adopt a model that is vertically homogeneous but changes laterally. Then, it is possible to compute medium parameters uniquely for the whole overburden, although the inverted model cannot be isotropic (indeed, $V_{\text{NMO, dip}} > V_{\text{NMO, str}}$). If we ignore the contribution of lateral heterogeneity to the NMO-velocity measurements on the scale of a single CMP gather, as is usually done in practice, the velocities $V_{\text{NMO, dip}}(Y)$ and $V_{\text{NMO, str}}(Y)$ can be inverted for the VTI parameters at each CMP location. Then, the dependence of the obtained parameters on $Y$ can be interpreted in terms of lateral heterogeneity.

Since the reflection traveltimes are symmetric with respect to the dip plane, it is natural to use the azimuthally isotropic VTI model for the inversion. We applied the algorithm of Grechka and Tsvankin (1998b) based on the exact NMO equations to estimate the zero-dip NMO velocity $V_{\text{NMO}}(Y)$ and the anellipticity coefficient $\eta(Y)$ of the effective VTI medium. Both $V_{\text{NMO}}$ and $\eta$ vary with the CMP coordinate $Y$ (Figure 4), which could be attributed to the influence of lateral heterogeneity.

As illustrated by the example in Figure 4, the effective parameter $\eta$ for our model is always non-negative. This is a general result that follows directly from inequality (15) in the limit of weak anisotropy (the proof is given in Appendix B); $\eta$ vanishes only for the trivial special case of a homogeneous medium. Thus, vertical
heterogeneity unaccounted for in velocity analysis yields an effective VTI medium with the same (positive) sign of the coefficients $\delta$ and $\eta$. It is interesting that the effective $\eta$ estimated from the long-period (nonhyperbolic) moveout of $P$-waves in vertically heterogeneous isotropic media is also non-negative (Fomel and Grechka, 2001).

The value of $\eta$ is controlled not only by the magnitude of vertical heterogeneity, but also by reflector dip. To study the dependence of the effective parameters $\eta$ and $V_{\text{nmo}}$ on dip, it is convenient to express the difference $V_{\text{nmo, dip}} - V_{\text{nmo, str}}$ as a function of the horizontal slowness $p$ of the zero-offset ray. Combining equations (15), (A2) and (A5), we find

\begin{equation}
V_{\text{nmo, dip}} - V_{\text{nmo, str}} = \frac{1}{T} \sum_{i=1}^{\infty} p^{2i} \left[ \int_{0}^{T} \int_{0}^{T} \int_{0}^{T} v^2 dt \right].
\end{equation}

On the other hand, in the limit of weak anisotropy of the effective medium the same difference can be represented as [using equations (15) and (B1)-(B3)]

\begin{equation}
V_{\text{nmo, dip}} - V_{\text{nmo, str}} = 8p^2 \eta V_{\text{nmo}}^4 \left(1 - p^2 V_{\text{nmo}}^2\right) = 8p^2 \eta V_{\text{nmo}}^4 - 8p^4 \eta V_{\text{nmo}}^5.
\end{equation}

Clearly, for a horizontal reflector ($p = 0$) the NMO ellipse degenerates into a circle, and the effective $\eta$ is undefined. For $p \to 0$, however, $\eta$ can be found by matching the coefficients of the $p^2$-term in equations (16) and (17). According to the Chebyshev inequality (A6), the term in the brackets in equation (16) is non-negative for any value of $i$, and goes to zero only for a homogeneous medium ($v = \text{const}$). Therefore, for subhorizontal reflectors beneath a vertically heterogeneous isotropic medium, the effective $\eta$ is always positive.

Furthermore, analysis of the expansion coefficients in equations (16) and (17) suggests that the computed $\eta$ increases with $p$ (i.e., with dip) for mild dips. In contrast, the effective NMO velocity $V_{\text{nmo}}$ should become slightly smaller with increasing dip. Figure 5 demonstrates that the variation of the effective $V_{\text{nmo}}$ and $\eta$ with dip is well-predicted by the weak-anisotropy approximation (17). Both $V_{\text{nmo}}$ and $\eta$ are plotted in Figure 5 as functions of the depth $z$ of reflection points because the effective velocities $V_{\text{nmo, dip}}$ and $V_{\text{nmo, str}}$ in equations (16) and (17) refer to a fixed value of $z$.

It might be thought that the non-negligible values of $\eta$ for the model from Figure 3 are associated with the monotonic increase in velocity with depth. However, since the apparent anisotropy is caused only by the different types of averaging applied to the vertically varying velocity to obtain the measured (effective) quantities, the phenomena discussed above can be observed in any $v(z)$ media. For example, Figure 6 shows the inverted effective $V_{\text{nmo}}$ and $\eta$ for the isotropic velocity $v(z)$ specified as a random Gaussian function (the mean is 1 km/s; the standard deviation is 0.1 km/s). Similar to the model from Figure 3, the velocities $V_{\text{nmo}}$ are smaller than the mean of $v(z)$ (Figure 6b), and the coefficients $\eta$ are positive (Figure 6c). As illustrated by Figure 7, for larger magnitude of random velocity variations the effective $\eta$ can reach 0.1-0.2 — values that can cause serious distortions in isotropic imaging.
Discussion and conclusions

Complicated, spatially varying isotropic velocity fields are sometimes kinematically equivalent to simpler effective anisotropic models, which poses a serious challenge for anisotropic velocity analysis. Here, we examined one of the consequences of approximate treatment of vertical heterogeneity in estimating the subsurface velocity field. If a heterogeneous medium between reflectors is treated as a homogeneous layer, the traveltimes cannot be fit without introducing apparent (non-existent) anisotropy.

The apparent (or effective) VTI model, equivalent to a vertically heterogeneous isotropic medium above a horizontal reflector, has a non-negative coefficient $\delta$. If the reflector is dipping, the relationship between the semi-axes of the $P$-wave NMO ellipse yields a non-negative effective anellipticity coefficient $\eta \equiv (\epsilon - \delta)/(1 + 2\delta)$. Therefore, Thomsen (1986) parameters of the effective VTI model always satisfy the inequality $\epsilon \geq \delta \geq 0$.

It is interesting to note that although the inequality $\eta \geq 0$ was obtained from 3-D (azimuthal) inversion using the dip- and strike-components of the $P$-wave NMO ellipse ($V_{\text{iso},\text{dip}}$ and $V_{\text{iso},\text{str}}$), the same result follows from the 2-D dip-moveout (DMO) method of Alkhalifi and
Tsvankin (1995), which operates with $V_{\text{nmo, dip}}$ and the NMO velocity from a horizontal reflector $V_{\text{nmo}}$. Indeed, $V_{\text{nmo, str}}$ and $V_{\text{nmo}}$ for our heterogeneous isotropic model are equal to each other because they should be found from the same Dix formula (7). Therefore, $V_{\text{nmo}}$ can be substituted instead of $V_{\text{nmo, str}}$ into inequality (15), which is equivalent to the condition of positive $\eta$ in the DMO inversion (Alkhalifah and Tsvankin, 1995). The values of $\eta$ obtained by the Alkhalifah–Tsvankin method, however, would differ from those shown in Figures 4b, 5b, 6c and 7c because $V_{\text{nmo, str}}$ is not equal to $V_{\text{nmo}}$ in VTI media [see equation (B2)]. To fit all three velocities ($V_{\text{nmo}}$, $V_{\text{dip}}$, and $V_{\text{nmo, str}}$) using a homogeneous anisotropic medium, it is necessary to assume orthorhombic (Grechka and Tsvankin, 1999) or even lower symmetry. Note that the effective $\eta$ needed to describe nonhyperbolic reflection moveout of $P$-waves over a heterogeneous isotropic medium is also non-negative (Fomel and Grechka, 2001).

Positive values of the effective anisotrophic coefficients stem from the common physical origin – Fermat’s principle, which requires that reflected rays bend in such a way that the traveltimes for all source-receiver pairs reach their minimum values. As discussed above, Fermat’s principle directly explains non-negative effective $\delta$ values for reflections from horizontal interfaces. When the reflector is dipping, we observe a similar phenomenon: because of ray bending, the reflection traveltime increases with offset slower than that in a reference homogenous medium, and the corresponding NMO velocity is higher. Since the influence of ray bending is more pronounced in the dip plane, the vertical heterogeneity increases the difference between the dip-line and strike-line NMO velocities, which translates into positive values of $\eta$ of the effective homogeneous VTI model.

The estimates of the effective anisotropic coefficients of the apparent VTI model also provides insight into potential biases in anisotropic velocity analysis. For example, ignoring the vertical velocity gradient between reflectors in VTI media should lead to overestimates of the parameters $\eta$ and $\delta$. This may partially explain the discrepancy between predominantly positive $\delta$ values estimated from reflection data (e.g., Alkhalifah et al., 1996; Williamson et al., 1997) and sometimes negative values of $\delta$ derived from core measurements and VSP surveys (e.g., Thomsen, 1986; Vernik and Liu, 1997; Jakobsen and Johannsen, 2000). Note that if vertical transverse anisotropy is caused by fine isotropic layering on a scale small compared to the predominant wavelength (see Backus, 1962), the parameters $\eta$ and $\epsilon$ are positive, while $\delta$ can be either positive or negative (Berryman, 1979; Berryman et al., 1999).

Problems of the type considered here are typical for seismic velocity analysis and inversion. Since the available data usually cannot constrain all components of the parameter vector $\mathbf{m}$, the actual velocity distribution $v(\mathbf{m}, x)$ is replaced by a certain model $\hat{v}(\mathbf{m}, x)$ with fewer unknowns, so that all model parameters $\mathbf{m}$ can be resolved in a unique fashion. For that reason, some anisotropy-related components of the vector $\mathbf{m}$ may be invoked in travelt ime inversion for complex spatially varying isotropic velocity fields. Improved understanding of various types of interplay between anisotropy and heterogeneity remains one of the key problems in anisotropic velocity analysis.

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APPENDIX A: Relationship between the semi-axes of the NMO ellipse

Here we prove inequality (15), which involves the semi-axes of the P-wave NMO ellipse \( V_{nmo,dip} \) and \( V_{nmo,str} \):

\[
\tilde{V}_{nmo,dip}^2 \geq V_{nmo,str}^2 \quad (A1)
\]

where \( p \) is the horizontal component of the slowness vector of the zero-offset ray. After substituting equations (12) and (13) and multiplying by \( T^2 \), inequality (A1) takes the form

\[
\left( \int_0^T \frac{v^2 \, dt}{1 - p^2 v^2} \right) \left( T - p^2 \int_0^T v^2 \, dt \right) \geq T \int_0^T v^2 \, dt. \quad (A2)
\]

Using inequality (11), the denominator of the first integral can be replaced by the converging series

\[
\frac{1}{1 - p^2 v^2} = \sum_{i=0}^\infty p^{2i} v^{2i}. \quad (A3)
\]

Since \( p \) is independent of \( t \), the left-hand side of inequality (A2) can be rewritten as

\[
F \equiv \left( \int_0^T \frac{v^2 \, dt}{1 - p^2 v^2} \right) \left( T - p^2 \int_0^T v^2 \, dt \right) = \left( \sum_{i=0}^\infty p^{2i} \int_0^T v^{2(i+1)} \, dt \right) \left( T - p^2 \int_0^T v^2 \, dt \right) = T \sum_{i=0}^\infty p^{2i} \int_0^T v^{2(i+1)} \, dt
\]

\[
= T \sum_{i=0}^\infty p^{2i} \int_0^T v^{2} \, dt + T \sum_{i=1}^\infty p^{2i} \int_0^T v^{2(i+1)} \, dt
\]

\[
- \sum_{i=1}^\infty p^{2i} \int_0^T v^{2} \, dt \int_0^T v^{2} \, dt. \quad (A4)
\]

Therefore,

\[
F = T \int_0^T v^2 \, dt + \sum_{i=1}^\infty p^{2i} \left[ T \int_0^T v^{2(i+1)} \, dt \right. - \left. \int_0^T v^2 \, dt \int_0^T v^{2i} \, dt \right]. \quad (A5)
\]
Note that the first integral in equation (A5) coincides with the right-hand side of inequality (A2). The terms in the brackets of equation (A5) are always non-negative, which follows from the Chebyshev inequality [Abramovitz and Stegun, 1965, equation (3.2.7)]:

\[
T \int_0^T v^{2(i+1)} \, dt \geq \int_0^T v^2 \, dt \int_0^T v^{2i} \, dt. \tag{A6}
\]

The equality is reached only if \( v(t) = \text{const} \). Also, note that for \( i = 1 \), inequality (A6) takes the well-known Cauchy-Schwartz form

\[
T \int_0^T v^4 \, dt \geq \left( \int_0^T v^2 \, dt \right)^2. \tag{A7}
\]

Thus, from equation (A5) we conclude that

\[
\begin{cases}
F = T \int_0^T v^2 \, dt & \text{if } v(t) = \text{const} \\
 & \text{or } p = 0 \text{ and} \\
F > T \int_0^T v^2 \, dt & \text{otherwise,}
\end{cases} \tag{A8}
\]

which proves inequality (A2).

**APPENDIX B: Inequality for the effective parameter \( \eta \)**

In this appendix, we show that the parameter \( \eta \) computed from the NMO ellipse for the effective VTI model is always non-negative. The \( P \)-wave NMO velocities in the dip \( (V_{nmo,dip}) \) and strike \( (V_{nmo,str}) \) directions satisfy inequality (A1) proved in Appendix A. Assuming that the anisotropy is weak and \( |\eta| \ll 1 \), we can use the following linearized approximations for \( V_{nmo,dip} \) and \( V_{nmo,str} \) (Alkhaliﬁah and Tsivkin, 1995; Grechka and Tsivkin, 1998b):

\[
V_{nmo,dip}^2(p) = \frac{V_{nmo}^2}{1 - \xi} \left[ 1 + \frac{2\eta \xi}{1 - \xi} (6 - 9\xi + 4\xi^2) \right], \tag{B1}
\]

\[
V_{nmo,str}^2(p) = V_{nmo}^2 [1 + 2\eta \xi (2 - \xi)] ; \tag{B2}
\]

\[
\xi \equiv p^2 V_{nmo}^2. \tag{B3}
\]

Equations (B1) and (B2) are derived under the assumption that

\[
\xi < 1, \tag{B4}
\]

which effectively removes from consideration large dips close to 90°.

Substituting equations (B1) and (B2) into inequality (A1) leads to

\[
\frac{1}{1 - \xi} \left[ 1 + \frac{2\eta \xi}{1 - \xi} (6 - 9\xi + 4\xi^2) \right] \left[ 1 - \xi - 2\eta \xi^2 (2 - \xi) \right] \geq 1 + 2\eta \xi (2 - \xi). \tag{B5}
\]

Further linearization in \( \eta \) yields

\[
\frac{1}{1 - \xi} \left[ 1 - 2\eta \xi^2 (2 - \xi) + 2\eta \xi (6 - 9\xi + 4\xi^2) \right] \geq 1 + 2\eta \xi (2 - \xi), \tag{B6}
\]

or

\[
8\eta \xi (1 - \xi) \geq 0. \tag{B7}
\]

Since \( 0 \leq \xi < 1 \) [see inequality (B4)],

\[
\eta \geq 0. \tag{B8}
\]

As discussed in Appendix A, except for the special cases \( p = \xi = 0 \) (horizontal reflector) and \( v(z) = \text{const} \) (homogeneous isotropic medium), inequality (A1) becomes

\[
V_{nmo,dip}^2 (1 - p^2 V_{nmo,str}^2) > V_{nmo,str}^2, \tag{B9}
\]

and

\[
8\eta \xi (1 - \xi) > 0. \tag{B10}
\]

Therefore, for dipping reflectors \( (\xi \neq 0) \) beneath a heterogeneous isotropic medium, the effective \( \eta \) computed from the \( P \)-wave NMO ellipse is strictly positive.

If the reflector is horizontal, \( \eta \) is not constrained by the \( P \)-wave NMO ellipse, which degenerates into a circle. Equations (16) and (17) of the main text show that

\[
\lim_{p \to 0} \eta > 0 \tag{B11}
\]

for any velocity function \( v(z) \neq \text{const} \). Clearly, for \( v(z) = \text{const} \) the effective \( \eta \) vanishes because the medium is isotropic and homogeneous.