point. If this $\xi$ is not in the range, then, for that choice of $x$, there is no stationary point and the contribution to the total integral is of lower order. We proceed as if there is an interior stationary point, $\xi_I = \xi_I(x)$.

Note that if $x$ is on the reflector, then the $\xi_I$ and $\tilde{\ell}(\xi_I)$ for which this point is the specular reflection point satisfies both stationary phase conditions. An easy way to see this is to note that in this case, the rays from $x$ and $x_R$ to the source and receiver are the same and are specular. The fact that they are specular makes $\Phi_1$ stationary; the fact that these two points are the same makes their derivatives with respect to $\xi_I$ the same and the difference of derivatives appearing in $\partial \Phi_2 / \partial \xi_I$ is then equal to zero. It is for this reason that the difference of second derivatives are combined into the expression $\Delta \tau_I$. This difference is equal to zero on the reflector and, therefore, near zero for $x$ near the reflector. This will be important, below.

In order to determine the second derivative at stationarity, the first derivative of $\tilde{\ell}$ with respect to $\xi_I$ is needed. This derivative is determined by first setting $\partial \Phi_1 / \partial x = 0$ in (24) and then differentiating implicitly with respect to $\xi_I$.

This leads to the solution,

\[
\frac{d\tilde{\ell}}{d\xi_I} = -\frac{\partial^2 \Phi_1(x_R(\tilde{\ell}(\xi_I)), \xi_I)}{\partial \xi_I \partial x_{R_j}} \frac{dx_{R_j}}{d\ell} \frac{d\ell}{d\xi_I} [\Phi''_1]^{-1}.
\]

(30)

With this result, (28) is replaced by

\[
\frac{\partial^2 \Phi_2}{\partial \xi_I^2} = -\left[ \frac{\partial^2 \Phi_1(x_R(\tilde{\ell}(\xi_I)), \xi_I)}{\partial \xi_I \partial x_{R_j}} \frac{dx_{R_j}}{d\ell} \frac{d\ell}{d\xi_I} \right]^2 [\Phi''_1]^{-1} + \Delta \tau_I.
\]

(31)

The first factor on the right, here, can be simplified as follows.

\[
\left| \frac{\partial^2 \tau_I(x_R(\tilde{\ell}(\xi_I)), \xi_I)}{\partial \xi_I \partial x_{R_j}} d\xi_I \frac{d\ell}{d\xi_I} \right| = \left| \frac{\partial \Phi_2}{\partial \xi_I} \cdot \frac{dx_{R_j}}{d\ell} \frac{d\ell}{d\xi_I} \right| = \left| \frac{\partial \nabla_x \tau_I}{\partial \xi_I} \times \hat{n}_R \right|
\]

\[
= \left| \frac{\partial \nabla_x \tau_I}{\partial \xi_I} \times \nabla_x \tau_I \right| = \frac{|H(x_R, \xi_I)|}{|\nabla_x \tau_I|}.
\]

(32)

In the second line, the two dimensional tangent to the reflector has been replaced by the two-dimensional normal to the reflector. In the next line, the collinearity (within a sign) of the surface normal and the travel time gradient at stationarity is used. The last line, in turn, rewrites this two dimensional cross product as a determinant, the same Beylkin determinant as appears in the inversion formula. However, it is now evaluated at the point, $x_R$ on the reflector, subject to the two stationarity conditions, above. Now (31) can be rewritten as

\[
\Phi''_2 = -\left| \frac{|H(x_R, \xi_I)|}{|\nabla_x \tau_I|} \right|^2 [\Phi''_1]^{-1} + \Delta \tau_I.
\]

(33)

As with $\Phi_1$, the notation, $\Phi''_2$, is introduced for the evaluation of the second derivative at the stationary point. We remark that for $x$ near the reflector, this second derivative is dominated by the first term and

\[
\text{sgn}(\Phi''_2) = -\text{sgn}(\Phi''_1),
\]

while this sign might change “sufficiently far” from the reflector, presumably, more than three wavelengths away, for the sake of asymptotic analysis. The discussion of this possible latter region is postponed until later, and the analysis proceeds in the restricted range where the signs of the second derivative satisfy the stated relationship, above. In this case, application of the method of stationary phase to (25) leads to the result,

\[
uO(\xi_O, \omega_O) = -2\pi \sqrt{\lambda \omega} e^{-i\text{sgn}(\omega_O)/4} \cdot \int F(\omega_I) d\omega_I d^2 x \frac{G e^{i\psi \phi}}{\sqrt{|\Phi'_I \Phi'_2|}}.
\]

(34)
Here, the amplitude and the phase are to be evaluated at the dual stationary points in $\ell$ and $\xi_I$.

The dependence on $\omega_I$ has now become particularly simple. There is the linear dependence in $\Phi$, as defined by (21), and also the amplitude factor, $F(\omega_I)$. If $F = 1$, the $\omega_I$-integration yields a delta function. We take the point of view that $F$ is a filter that leads to a bandlimited version of the delta function that we will denote by $\delta_B$:

$$\delta_B(t) = \frac{1}{2\pi} \int F(\omega) e^{-i\omega t} d\omega. \quad (35)$$

By using this identity to carry out the $\omega_I$ integration in (34), we obtain

$$u_O(\xi_O, \omega_O) = -4\pi^2 \sqrt{|\omega_O|} e^{-i\text{sgn}(\omega_O)/4} \int d^2x \mathcal{G} \frac{e^{i\Phi}}{\sqrt{|\Phi_1^\prime \Phi_2^\prime|}} \delta_B(\tau_I(x_R(\ell), \xi_I) - \tau_I(x, \xi_I)). \quad (36)$$

The last factor here is a scalar delta function. Its argument is zero when $x$ is on the reflector where the stationary phase conditions yield $x = x_R$ and the value of $\xi_I$ makes the corresponding source/receiver pair specular. Furthermore, this zero is isolated; the gradient of the argument is just the gradient of the travel time, which is normal to the reflector. Thus, the direction of maximal change of argument of the delta function is initially normal to the reflector. Within a scale factor, then, this delta function is the singular function of the surface, $S_R$. The scale factor is just the magnitude of the gradient of the travel time; that is,

$$\delta_B(\tau_I(x_R(\ell), \xi_I) - \tau_I(x, \xi_I)) = \sqrt{|\nabla_x \tau_I(x_R(\ell), \xi_I)|}. \quad (37)$$

Consequently, replacing the bandlimited delta function by the delta function, itself, (36) can be rewritten as

$$u_O(\xi_O, \omega_O) = -4\pi^2 \sqrt{|\omega_O|} e^{-i\text{sgn}(\omega_O)/4} \int d\ell \mathcal{G} \frac{e^{i\Phi}}{\sqrt{|\Phi_1^\prime \Phi_2^\prime|} \sqrt{|\nabla_x \tau_I(x_R(\ell), \xi_I)|}}. \quad (38)$$

In this equation, the stationarity conditions define $\xi_I = \xi_I(\ell)$, choosing the value of $\xi_I$ for which the input source/receiver path are specular at $x_R(\ell)$. Now, the amplitude in this equation must be evaluated at stationarity and for $x = x_R$. In this limit, from (22),

$$\mathcal{G} = \frac{R(x_R(\ell), x_s(\xi_I))}{4\pi} a_O(x_R, \xi_O) \tilde{n}_R \cdot \nabla_x \tau_I(x_R(\ell), \xi_I) \frac{\sqrt{\nabla_x \tau_O(x_R, \xi_O)}}{\sqrt{\nabla_x \tau_I(x_R, \xi_I)}}. \quad (39)$$

Furthermore, the term, $\Delta \tau_I$, defined by (29), is zero and, from (33),

$$\sqrt{|\Phi_1^\prime \Phi_2^\prime|} = \frac{|H(x_R, \xi_I)|}{\sqrt{\nabla_x \tau_I}}. \quad (40)$$

These results are used in (38) to obtain

$$u_O(\xi_O, \omega_O) = -\sqrt{|\omega_O|} e^{-i\text{sgn}(\omega_O)/4} \int d\ell R(x_R(\ell), x_s(\xi_I)) a_O(x_R, \xi_O)$$

$$\tilde{n}_R \cdot \nabla_x \tau_I(x_R(\ell), \xi_I) \frac{\sqrt{\nabla_x \tau_O(x_R, \xi_O)}}{\sqrt{\nabla_x \tau_I(x_R, \xi_I)}}$$

$$\sqrt{\frac{\sigma_{O_s}(x_R, \xi_O) \sigma_{O_{\ell}}(x_R, \xi_O)}{\sigma_{O_s}(x_R, \xi_O) + \sigma_{O_{\ell}}(x_R, \xi_O)}} e^{i\omega_O \tau_O(x_R, \xi_O)}, \quad (41)$$
Since the integrand is evaluated subject to the stationarity relation between \( \ell \) and \( \xi_I \),
\[
\hat{n}_R \cdot \nabla_x \tau_I(x_R(\ell), \xi_I) = -|\nabla_x \tau_I(x_R, \xi_I)|.
\] (42)

Just as in (4), we set
\[
|\nabla_x \tau_O(x_R, \xi_O)| = -\hat{n}_R \cdot \nabla_x \tau_O(x_R(\ell), \xi_O).
\] (43)

With these substitutions,
\[
u_O(\xi_O, \omega_O) = -\sqrt{|\omega_O|} e^{-i\pi \text{sgn}(\omega_O)/4} \int R(x_R(\ell), x_s(\xi_I)) \hat{n}_R \cdot \nabla_x \tau_O(x_R(\ell), \xi_O) u_O(x_R, \xi_O) \cdot \frac{\sqrt{\sigma_{Ox}(x_R, \xi_O) \sigma_{Oy}(x_R, \xi_O)}}{\sigma_{Ox}(x_R, \xi_O) + \sigma_{Oy}(x_R, \xi_O)} e^{i\omega_O \tau_O(x_R, \xi_O)} d\ell.
\] (44)

The Kirchhoff representation in the input source/receiver coordinates has been transformed into the Kirchhoff representation in the output source/receiver coordinates. In obtaining this result, a region of the \( x \)-domain where \( \text{sgn}(\Phi'_y) = \text{sgn}(\Phi'_y) \) has been neglected. In this region, the delta function in time is replaced by a principal-value-1/t function. This function also has its singular support centered around \( t = 0 \). However, the \( x \)-domain where this is the correct value for the signature is bounded away from \( t = 0 \), which corresponded to the neighborhood of the reflector. Thus, any contribution that might be obtained from this combined integration over \( \omega_I \) and \( \omega_R \) will be of lower order asymptotically than the result given here.

The source signature of the input data, \( F(\omega_I) \) in (19), was used to define the bandlimited delta function that confined the \( x \)-domain integration to the reflecting surface. This, again, is a leading order asymptotic result. In another context, Tygel has suggested that such operators should be viewed as providing a “sinc-like” interpolation of the data in the neighborhood of the peak of the signal. This means an interpolation of the spatial part of the operator, appearing in lines two and three of (16), in the neighborhood of the reflector.

It is well known that the Kirchhoff integral provides the leading order asymptotic expansion of the return from specular reflections. Hence, travel time and all geometrical spreading and curvature effects, including effects of “buried foci,”—caustics produced by synclines—will be properly transformed by the KDM process. Where the caustic pierces the upper surface, the arrival time is expected to be accurate, but no claims are made about the accuracy of the amplitude. The factor, \( \Phi'_y \), is zero in this case and the asymptotic analysis is invalid. However, it produces an integrable singularity in the Kirchhoff integral, with the correct travel times in the phase, hence, our claim that the arrival time is correct, but the amplitude need not be.

For edge-diffracted returns, the Kirchhoff integral produces the correct arrival time, but an inaccurate diffraction coefficient, except at the shadow boundary of the last reflected ray. Thus, the mapped data is expected to contain mapped diffraction arrival times with inaccurate amplitudes.

There is another source of “error” in the amplitude. Note that the reflection coefficient is evaluated at an incidence angle associated with the input source/receiver configuration, through its dependence on the stationary value of \( \xi_I \). For correctly mapped data, it would be preferable to have this dependence mapped to \( \xi_O \). However, this is simply not the case. The input reflection coefficient is preserved, not mapped. This is known from the TZO case and is therefore not surprising in this general result.

We can also see, now, how the additional integral operators \( \cos_I \) and \( \cos_O \) produce the desired result. Since the entire integrand is evaluated at its stationary value, this is true, in particular, for the travel time gradients appearing in this integral kernels. Those stationary values are, indeed, just \( 2 \cos \theta/c(x) \), subscripted either \( I \) or \( O \). Hence the ratio of operators is, asymptotically, as claimed.

In summary, we have shown that the leading order asymptotic input data is mapped to the leading order asymptotic output data, except for the reflection coefficient, which maintains its input value everywhere.
CONCLUSIONS

We have derived platforms for 3D and 2.5D KDM of scalar wave fields. The formalism assumes knowledge of a physical model for both the input and output data and prescribed input and output source/receiver configurations. By cascading an inversion formula with a modeling formula, we obtain the KDM platform formula. This cascade is a single reflector formalism in the absence of multiple reflections and multi-pathing. In that sense, it is still a somewhat limited result, at the level of generality of standard migration or DMO formalisms.

In the absence of a specific application, the formula includes a multifold integration over the physical model space that must be evaluated asymptotically for each example of KDM in order to derive a computationally feasible formalism for implementation. Application of this formalism in constant background 2.5D DMO produces the same formula as was derived in Bleistein, Cohen and Jaramillo [1999]. This is a straightforward exercise that is not included in this paper.

On the other hand, we show how Kirchhoff approximate model data in a given input configuration is mapped to Kirchhoff data in a different output configuration for the 2.5D case. We have done this in great generality, without specifying any particular configuration transformation. From this result, we conclude that the travel time and geometrical spreading effects of the input model are properly mapped to their counterparts in the output model, while the reflection coefficient is not.

We have further introduced two additional operators that allow us to estimate the cosine of the specular reflection angle of a reflection event as a ratio of integral operators on the data. As part of the proof, we provided asymptotic justification for this method.

In future papers, we will specialize the mapping platforms to achieve specific KDM formulas. Work is currently in progress on 3D constant background DMO and wave-equation-datuming.

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