Hagedoorn told us how to do Kirchhoff migration and inversion

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Migration and inversion as I expound on it, is often lost on my audience because of my tendency to couch my explanations in mathematical terms. Well, I really do come at these problems from the mathematical, rather than the (geo)physical point of view and it is kind of hard for us leopards to change our spots. On the other hand, I really would like to expand my audience of Kirchhoff inversion afficionadas, so I am always looking for alternative ways to explain our mathematical esoterica. Chalk this article up as an attempt to do just that. Some math is necessary, because I am trying to explain why a particular *mathematical formula* works to produce an image when that formula becomes the basis for a computer code. However, I will try to keep the details to a minimum and focus on some physical (pictorial) insights behind them, as much as I can. While many papers will be quoted below, more expository versions of the various derivations can be found in the lecture notes by Bleistein, Cohen, and Stockwell [1999]. We hope to publish these notes in the near future. Below, I will cite these notes only when they are the sole source for a particular derivation or result.

I have come to the point of view that Hagedoorn [1954] told us how to do Kirchhoff migration and inversion, but it took twenty years and more for us analyst-types to catch on [Schneider, 1978; Clayton and Stolt, 1981; Bleistein and Gray, 1985; Stolt and Weglein, 1985; Cohen and Hagin, 1985; and Bleistein, 1986]. At a summer workshop at Stanford, 1998, I presented a version of Hagedoorn's construction that I called modeling and migration on a shoestring. Indeed, I unlaced my shoe and carried out both processes on the blackboard, but with some particular mathematical points to be made from that exercise. Here, I will expand on that discussion to explain how it leads directly to Kirchhoff migration and inversion. (To me, the distinction in those designations is a matter of whether the objective is a reflector map only [migration], or a map plus some information about medium parameter variations across reflectors [inversion].)

Migration on a Shoestring.

I will begin here with a simple example extracted from the lecture notes that we use to teach this material at CSM [Bleistein, Cohen and Stockwell, 1999]. Figure 1 shows an earth model with a single reflecting surface in 2D. Above the model is a display of data traces that would be produced at 81 locations above the reflector, assuming a coincident source and receiver. The source, itself, was a bandlimited (delta-function-type) pulse. The wave speed used for this synthetically generated data was the constant, 2 km/sec. Thus, the geometrical optics rays are straight lines. The primary returns—specular reflections—occur at those points where the ray from source/receiver point is normal to the reflector. At

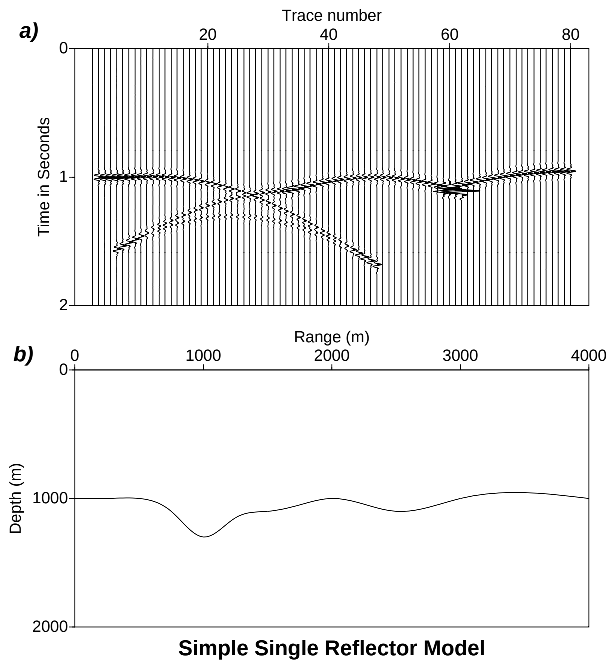

FIG. 1. a) A synthetic *zero-offset* seismic section and b) earth model. The synthetic was made with the program cshot.

these points, the incident and reflected rays make equal angles (zero) with the normal to the reflector.

For this simple example, if all we wanted to know was the arrival time of an event on a trace, we could use a much simpler computer, namely, a shoestring, to construct this plot. First, we have to give the vertical axis an interpretation as time, as well as depth. For a pulse to travel 1 km at 2 km/sec, takes one second. Hence, note that the depth of 1 km in the lower plot is the same length on the plot as the time, 1 sec on the upper plot.

Here is the modeling exercise.

> On the lower plot, pick a trace location—a source/receiver point— at the upper surface. With a compass, with that point as center, look for circles that are tangent to the physical model. (When the circles are tangent, the rays will be normal to the reflector.) When you find one, come back along that circle to the vertical position below the source/receiver point. I promise that the point you find will be the same as the center of the pulse on the upper plot.

So, for example, on the leftmost trace, a circle of radius "1 km" will be tangent to the model right below the source/receiver point at the upper surface. On the other hand, a few traces over we see that there are two "events" on a trace. One of them comes from a circle that is tangent to the model almost directly below the source/receiver point. The other comes from a larger circle (larger two-way traveltime) that is tangent to the model

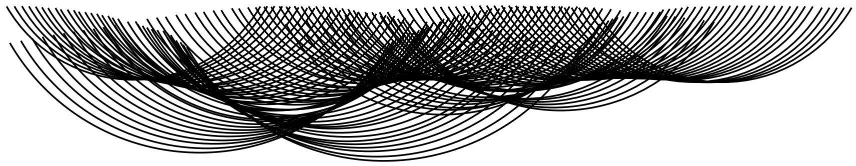

FIG. 2. An example of graphical migration. Plot done by Herman Jaramillo and Andreas Rueger when they took the inversion course at CWP.

somewhere beyond trace location 1000 m. For traces further to the right, that single second arrival splits into two arrivals, from one reflection point that moves to the left, deeper into the syncline below 1000 m, and from another that moves to the right up the flank of the syncline. On a larger plot, say on a blackboard, a shoestring, with a piece of chalk attached, works very well as a compass, so this is modeling on a shoestring—low cost computing, to be sure!

So, for the length scales that were used to depict time in Part a and depth in Part b, a compass or a shoestring could be used to "migrate" the reflection point in Part b to the arrival time on a trace in Part a.

Reversing the process is somewhat harder. Given a reflection "event" on a trace in Part a, all one can be sure of, is that it came from somewhere on a circle in the physical domain in Part b, having a radius equal to the one way travel distance consistent with the wave speed and traveltime of the event in Part a. However, if one draws all of those circles, then the *envelope* of the family of circles is a curve for which every point satisfies the appropriate specular reflection condition and produces a reflection event at precisely the right position in Part a. This process is illustrated in Figure 2. Here, we have taken the data in the time domain and graphically moved it—migrated it—back to its spatial position by looking at all possible candidate positions and then choosing the envelope as the coherent reflector image. A compass will do this construction on paper, but again, a shoestring works fine on a blackboard-sized image of the data. This graphical technique [Hagedoorn, 1954] precedes and anticipates Kirchhoff migration and inversion.

A mathematical insight

Let us look, now, at the mathematical process of generating the envelope of a family of curves. In doing so, we will use a notation that anticipates the general method of Kirchhoff migration and inversion. We denote by $\bar{x}$, the $x-$coordinate of the trace location, and we denote by $t(\bar{x})$, the observed traveltime of the peak on the trace. We think of the peak time as being "zero-time" for the source wavelet that propagated downward and was ultimately reflected back to the coincident receiver. Then the peak time of the return represents the two-way traveltime of the reflection response. From that response, we could only say that there was a reflection somewhere on an equi-travel-time curve, an *isochron*, in $\boldsymbol{x}$. This equation has the form

$$t(\bar{x}) - \tau(\boldsymbol{x}, \boldsymbol{x}_s(\bar{x}), \boldsymbol{x}_r(\bar{x})) = 0. \tag{1}$$

Here, τ represents the traveltime from the source, to the "scattering point" at depth, to the receiver.

We are being somewhat elaborate to describe this simple situation in which the source and receiver are coincident and have coordinates,

$$\boldsymbol{x}_s(\bar{x}) = \boldsymbol{x}_r(\bar{x}) = (\bar{x}, 0), \tag{2}$$

with the traveltime curves simply being circles with radius is $ct(\bar{x})/2$ and center, $(\bar{x}, 0)$,

$$t(\bar{x}) - \frac{2}{c}\sqrt{(x_1 - \bar{x})^2 + x_3^2} = 0. \tag{3}$$

However, we are suggesting a more general application of this simple idea by our notation in which $\boldsymbol{x}_s(\bar{x})$ and $\boldsymbol{x}_s(\bar{x})$ might be the coordinates of a common shot experiment, a common receiver experiment, a diffraction tomography experiment, or, whatever! Further, these points might not even lie on a horizontal acquisition line. Hence, keep (1) in mind, even though it seems like an elaborate notation for the simple example being treated here. Note, further, that for separated source and receiver, the traveltime would be a sum of solutions to the eikonal equation with a variable wavespeed and *mathematically* we could still contemplate determining the reflector by seeking out the envelope of the family of traveltime curves with respect to the parameter, $\bar{x}$, just as we did in the shoestring construction, above.

I know that the process of finding an envelope of a family of curves was one of those Calculus I topics where most everyone's eyes glazed over during the derivation of the method. So, I have included a short appendix to remind the reader why it works the way it does. For now, just take my word for it! To find the envelope of this family of curves, we

1. set the first derivative of the left side in (1) equal to zero;

2. use that equation to solve for $\bar{x}$ as a function of $\boldsymbol{x}$;

3. substitute back into (1) to obtain the equation of the envelope.

That's the rule!

There is another process in which we find a "critical value" of a parameter by setting the first derivative of a function equal to zero and then substituting back into the original function. That process is the *method of stationary phase*. Kirchhoff operators in the frequency domain have a phase function of the form, $-i\omega\tau(\boldsymbol{x}, \boldsymbol{x}_s(\bar{x}), \boldsymbol{x}_r(\bar{x}))$. Inversion will require integration in $\bar{x}$ (which amounts to summation over source/receiver pairs) and then integration over ω (which, the migrationists will tell you, amounts to sending the signal from $z = 0$ at all time, back to the depth of its origination in two-way traveltime at $t = 0$). When testing this operator on ray data or Kirchhoff-approximate data, we will find that the data includes a phase factor of the form, $i\omega t(\bar{x})$, with $t(\bar{x})$ being the model traveltime on the specular ray path from the source to the reflector to the receiver. Thus, we will be considering integrals of the form,

$$\beta(\boldsymbol{x}) = \int G(\omega, \bar{x}, \boldsymbol{x}) e^{i\omega\phi(\boldsymbol{x}, \boldsymbol{x}_s(\bar{x}), \boldsymbol{x}_r(\bar{x}))} d\omega d\bar{x}, \tag{4}$$

with

$$\phi(\boldsymbol{x}, \boldsymbol{x}_s(\bar{x}), \boldsymbol{x}_r(\bar{x})) = t(\bar{x}) - \tau(\boldsymbol{x}, \boldsymbol{x}_s(\bar{x}), \boldsymbol{x}_r(\bar{x})), \tag{5}$$

and G just representing all of the amplitude of the integrand resulting from the application of the operator to model data. Note that the phase, here, is just the function on the left side in equation (1).

To approximate the $\bar{x}-$integration by the method of stationary phase, we would

i set the first derivative of ϕ with respect to $\bar{x}$ equal to zero;

ii use that equation to solve for $\bar{x}$ as a function of $\boldsymbol{x}$;

iii substitute that value back into ϕ and G and apply the appropriate formula to obtain the stationary phase approximation to that integral.

Note that {i} and {ii} are the same steps that we take in defining an envelope. However, here, there is no *a priori* reason to set $\phi = 0$. Instead, let us think of setting $\phi =$ constant for a moment. For each choice of that constant, letting $\bar{x}$ vary generates a family of curves in the $\boldsymbol{x}$-domain. Then, we can think of the stationary value of $\bar{x}$ as determining the envelope of the family of curves for each constant value of ϕ. Again, the value $\phi = 0$ is special, because it is the actual reflector, but it has not as yet been distinguished in the stationary phase process. However, after the application of stationary phase, it will turn out that the resulting integrand has the form,

$$\beta(\boldsymbol{x}) = H(\boldsymbol{x}) \int F(\omega) e^{i\omega\phi} d\omega, \tag{6}$$

with H a "slowly" varying function over the wavelengths consistent with the support of $F(\omega)$ (the places where $F(\omega)$ is not equal to zero) and this latter function being the original source signature in the frequency domain.

In this theory, the sources are assumed to be impulses—bandlimited delta functions—that peak at argument zero. Thus, within a scale factor,

$$\beta(\boldsymbol{x}) \sim \delta_B(\phi). \tag{7}$$

Here, $\delta(\phi)$ is the Dirac delta function of argument ϕ; hence, its support is on the reflector, where $\phi = 0$. The subscript B connotes *bandlimited*, due to the finite extent in the frequency domain of the source signature (and further attenuation in the real problem being modeled here). The theory, then, will predict that if we apply an operator with this traveltime phase to bandlimited impulse response data as appears in Figure 1, the result will be a bandlimited delta function of an argument that is zero on the reflector. (From the fact that the $\boldsymbol{x}$-gradient of the traveltime is nonzero, we can even show that this function is not zero elsewhere in a neighborhood of the reflector.)

Consequently, the theory is the means by which we turn the shoestring construction into a viable mathematical method that relates the amplitude of the output to the reflection coefficient. Further, this asymptotic theory becomes a point of departure for generalizations to more accurate physical models and to situations in which simple ray theory breaks down.

Stationary phase: close enough.

Now, I have been told that there are people who do not believe in the accuracy of the method of stationary phase. For someone whose first career was in asymptotic expansions of integrals, that is a cutting remark! There is not enough space, here, for a longwinded discussion of the method of stationary phase. However, I would like to present an example that might shake the confidence of nonbelievers. In fact, I will choose an example that arises "naturally" in 2D wave propagation, namely, the Hankel function of the first kind, zero order, $H_0^1(\lambda)$. Within a scale factor, this is the Green's function or impulse response in the space-frequency domain, constant wavespeed medium, with $\lambda = \omega r/c$. Here, ω is the radial frequency, r is the distance between the source point and observation point, and c is the constant wavespeed. The leading order asymptotic expansion for this function is derived as a stationary phase contribution. For positive large argument, the result is

$$H_0^1(\lambda) \sim \sqrt{\frac{2}{\pi\lambda}} e^{i\{\lambda - \pi/4\}}.$$

However, in our application, λ will be proportional to the frequency, ω, which can be positive or negative. Consequently, we need a more general result, namely,

$$H_0^1(\lambda) \sim \sqrt{\frac{2}{\pi|\lambda|}} e^{i\{\lambda - \mathrm{sgn}(\omega)\pi/4\}}. \tag{8}$$

Note that this extension guarantees that this leading order asymptotic expansion transforms to its complex conjugate when λ is replaced by $-\lambda$; that is a known property of the Hankel function on the real axis. This is a result that is predicted to become progressively more accurate as $|\lambda| \to \infty$.

It is reasonable to ask how accurate this result is for finite values of λ. As guidance, I propose what I call *the Joe Keller rule of asymptotic expansions*: "Three is as close to infinity as one-third is to zero!" So, let us see. Figure 3 shows plots of the real and imaginary parts of the Hankel function and the leading order asymptotic expansion in (8). The asymptotic expansion tracks the exact function very well, too well to get a good feel for the error between the two from the plot. Therefore, let us examine the percentage error between the asymptotic expansion and the exact solution, which is displayed in Figure 4.

Now the inaccuracy of the asymptotic expansion for this example stands exposed for all to see. At the Joe Keller limit ($\lambda = 3$,) the error is about 4%. Beyond $\lambda = 12$, the error is less than 1% and is clearly heading towards zero with increasing λ. (We should only have data that is as good!) So, the right question to ask is, "What is a typical value of λ in geophysical applications?" In forward modeling,

$$\lambda = \frac{2\pi f L}{c}.$$

Here, f is the frequency in Hz, L is a "typical" length scale and c is the propagation speed. For zero-offset migration or inversion, double this value; for finite offset migration or inversion, multiply by $2\cos\theta$, with θ the opening angle between the specular rays from source and receiver at a point on a reflector.