2.5D Downward Continuation Using Data Mapping Theory

Steven D. Sheaffer & Norman Bleistein, Center for Wave Phenomena,
Colorado School of Mines, Golden, CO 80401

Summary

Unlike many previous implementations of Kirchhoff-type wavefield extrapolation (Berryhill, 1979; Bevc, 1995), we propose a procedure that is not only kinematically correct, but dynamically correct in heterogeneous media, in a model-consistent sense. Our approach is to specialize the general, "true-amplitude" data mapping platform of Bleistein & Jaramillo (1997) to the process of downward continuation. We apply stationary phase analysis to this general expression under the assumptions of downward continuation of both common-source and common-receiver geometries. The asymptotic analysis produces a stationary point for each set of input and output locations, and amplitudes that depend on parameters along the ray paths connecting these locations with the stationary point. In general inhomogeneous media, however, a large investment in external ray tracing is required to locate this point and to calculate the associated ray quantities. Alternatively, the procedure may be greatly simplified when assumptions about the wavespeed distribution can be made, and analytic results can be used in conjunction with, or in place of, ray tracing results. Here we present the form of the extrapolation integral for evaluation of amplitudes under the assumption of constant wavespeed, which allows this process to be performed entirely by analytic methods.

Introduction

The general data mapping expression of Bleistein & Jaramillo (1997) allows data collected with sources and receivers configured in some arbitrary input configuration, to be transformed to data equivalent to that collected in some specified output configuration. It is obtained by cascading an inversion formula and a modeling formula in 3D. Amplitude preservation requires that 3D wave propagation effects be accounted for, and therefore 3D integrals over data collected in a survey with two lateral dimensions. However, if the structure in the subsurface is assumed constant in one lateral direction, and data collected in the orthogonal direction, the problem can be reduced from 3D to 2.5D. This means that there will be no out-of-plane effects with respect to the vertical plane below the survey, and the dependence on the out-of-plane coordinate is integrated out of the 3D expression by stationary phase. This leaves an expression that depends only on the in-plane variables, but preserves 3D geometrical spreading effects. Therefore, it can be used to extrapolate 1D survey data with correct amplitudes, provided the survey lines are oriented normal to the assumed dominant structural axis. This is the 2.5D assumption, and we will derive our expressions under this assumption.

Following Bleistein & Jaramillo (1997), the general 2.5D data mapping integral is

$$
\begin{aligned}
u_O\left(\xi_O,\omega_O\right) \;\approx\; & \frac{\sqrt{|\omega_O|}}{4\pi^2} e^{-i\pi/4\,\mathrm{sgn}\,(\omega_O)} \int \sqrt{|\omega_I|} \\
& \cdot\, e^{i\pi/4\,\mathrm{sgn}\,(\omega_I)}\, u_I\left(\xi_I,\omega_I\right) \frac{a_O\left(\boldsymbol{x},\xi_O\right)}{a_I\left(\boldsymbol{x},\xi_I\right)} \\
& \cdot\, \frac{\left|\nabla_{\boldsymbol{x}}\tau_O\left(\boldsymbol{x},\xi_O\right)\right|}{\left|\nabla_{\boldsymbol{x}}\tau_I\left(\boldsymbol{x},\xi_I\right)\right|} \frac{\sqrt{\sigma_{IS}+\sigma_{IG}}}{\sqrt{\sigma_{OS}+\sigma_{OG}}} \frac{\sqrt{\sigma_{OS}\;\sigma_{OG}}}{\sqrt{\sigma_{IS}\;\sigma_{IG}}} \\
& \cdot\, \left|H\left(\boldsymbol{x},\xi_I\right)\right| e^{\left(i\omega_O\tau_O(\boldsymbol{x},\xi_O)-i\omega_I\tau_I(\boldsymbol{x},\xi_I)\right)} d\omega_I d\xi_I d^2x,
\end{aligned} \tag{1}
$$

where the subscripts I and O denote quantities associated with the input and output configurations, respectively. The subscripts S and G indicate source and geophone, with all vectors evaluated in the survey plane. Each a is the product of the Green's function amplitudes for the input and output paths through the depth point $\boldsymbol{x}$, evaluated in the survey plane. These are assumed to be ray theoretical Green's functions, and are not valid in the presence of a caustic. ξ_I and ξ_O are parameters describing a position on the recording and datuming surfaces, respectively, and H is the 2D Beylkin determinant for the problem. This is a function primarily dependent on source and receiver geometry, a full discussion of which can be found in Bleistein, et al. (1997). The symbol σ represents a parameter measured along the raypath and defined by,

$$
d\sigma \;=\; c^2(\boldsymbol{x})\, d\tau, \tag{2}
$$

where τ is time. The subscripts S and G indicate a parameter evaluated on the source-to-scatterer and scatterer-to-geophone paths, respectively.

We perform stationary phase analysis on this integral to produce a Kirchhoff-type equation for downward continuation of receivers given a fixed source (common-shot gather), as well as an analogous equation for the downward continuation of sources given a fixed receiver (common-reciever gather). A cascade of these two processes will completely extrapolate a set of data. Results are completely general with respect to velocity inhomogeneity, as well as to the shapes of the recording and datum surfaces. The final expression for 2.5D downward continuation of receivers for a fixed shot location is

$$u_O(\xi_O,t_O) \approx \frac{1}{\sqrt{2\pi}}\int \frac{\sqrt{c(\boldsymbol{x})}}{\sqrt{|\kappa_{OG}-\kappa_{IG}|}}\frac{\sqrt{|J(\boldsymbol{x}_{IG},\boldsymbol{x})|}}{\sqrt{|J(\boldsymbol{x}_{OG},\boldsymbol{x})|}} \cdot \frac{\sqrt{\sigma_S+\sigma_{IG}}}{\sqrt{\sigma_S+\sigma_{OG}}}\left|\frac{\partial \nabla_{\boldsymbol{x}}\tau_{IG}}{\partial \xi_I}\right| D_f(\xi_I,\tau_I(\boldsymbol{x},\xi_I))\,d\xi_I, \tag{3}$$

$$D_f(\xi_I,t) = \frac{1}{2\pi}\int \sqrt{|\omega_I|}\,u_I(\xi_I,\omega_I)\,e^{-i\omega_I t + i\pi/4\,\mathrm{sgn}(\omega_I)}d\omega_I.$$

where for a fixed source ξ_I and ξ_O parameterize the input and output receiver locations.

D_f is a frequency-domain filter on the input data. We obtained this result by assuming a fixed source and then performing stationary phase analysis on the d^2x integration in the original integral, to produce an expression that involves only an integration with respect to ξ_I, over the input surface.

The stationary value of $\boldsymbol{x}(\xi_I,\xi_O,t_O)$, is best described in terms of isochrons of the input and output source/receiver configurations. Here we use isochron to denote scattering surfaces which produce reflections of constant travel time for a given source and receiver. The values of t_O and ξ_O define an output isochron, $\tau_O(\boldsymbol{x},\xi_O)=t_O$. Given ξ_I, stationarity occurs at points where the isochron of the input configuration is tangent to this isochron of the output configuration. τ_I is the travel time on that input isochron and determines $\boldsymbol{x}$ as the point of tangency of the two isochrons. Given this value of $\boldsymbol{x}$, the weights in the integrand in (3) can be evaluated.

In the downward continuation of receivers, both the input and output configurations share the same ray from the fixed source to $\boldsymbol{x}$. Therefore, the above condition means that the scattered rays for the input and output receivers leave $\boldsymbol{x}$ at the same angle, and the ray from $\boldsymbol{x}$ to $\boldsymbol{x}_{OG}$ overlays the ray from $\boldsymbol{x}$ to $\boldsymbol{x}_{IG}$. See Figure 1. The output isochron and time $\tau_O = t_O$ is defined on the left-hand-side of the integral. The associated input isocrhon and time τ_I appearing in the phase is related to this by

$$\tau_I = \tau_O + \tau_{IO} = t_O + \tau_{IO} \tag{4}$$

where τ_{IO} is the travel time between the input and output receiver locations, or equivalently, between the recording and datuming surfaces.

The difference $\kappa_{OG}-\kappa_{IG}$ in the denominator is a difference of the curvatures of wavefronts traveling along this common ray between the stationary point and the output and input receiver locations, respectively. Since the original data mapping expression assumes no caustics, this difference is always positive for stationary

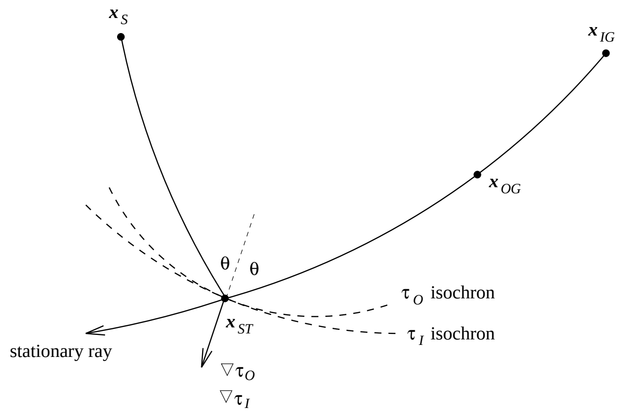

Figure 1. Geometry at stationarity.

points below the output datum. We exclude the singularity as well as any stationary points above the datum, since these are associated with energy scattered from features above the datum, and should not be included in the downward continued data. This is consistent with the limitations of the asymptotic analysis, which is not valid within a few wavelengths of the singularity.

For the downward continuation of sources given a fixed receiver (common-receiver gather), the equation is the same, but with the roles of the source and receiver locations interchanged.

Implementation in General Media

In evaluation of the integral, we know the desired output time, the input and output geometries, and the wavespeed model. To obtain the correct kinematics, we can ray trace between the recording and datuming surfaces to provide τ_{IO}, and therefore τ_I, giving the correct travel times and phase. However, evaluation of the amplitude factors requires determination of the isochrons and rays required to locate the stationary point, the relevant raypaths, and the Jacobians, curvatures, and parameters along the raypaths. Given what is known, this can only be accomplished in general, inhomogeonous media by extensive application of ray tracing. Since these factors differ for every component of the integration, as well as every output time, this process is extremely expensive, possibly prohibitively.

Constant Wavespeed Media

Implementation in general media depends on extensive ray tracing because nothing can be assumed about the shape of the isochrons and rays. In constant wavespeed, we know the relevant ray geometries, and can therefore determine analytic expressions for the location of the

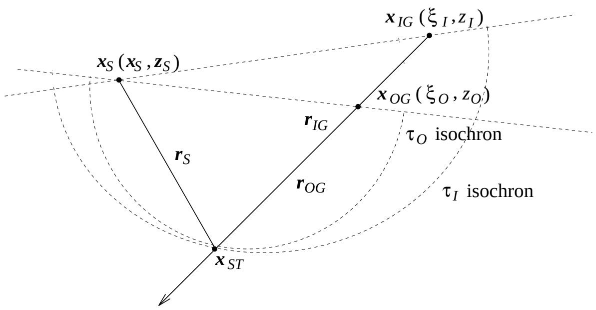

Figure 2. Geometry for receiver continuation in a constant-wavespeed medium.

stationary point and the relevant amplitude factors. Assuming arbitrary recording and datuming surface, let the source and receiver locations be given by

$$\boldsymbol{x}_{IG} = (\xi_I, z_I)\,, \quad \boldsymbol{x}_{OG} = (\xi_O, z_O)\,, \quad \boldsymbol{x}_S = (x_S, z_S)\,, \tag{5}$$

and define the stationary point as

$$\boldsymbol{x} \;=\; (\overline{x}_1, \overline{x}_3)\,. \tag{6}$$

As shown in Figure 2, rays are straight and isochrons are ellipses, tilted with respect to the coordinate system such that the source and the appropriate receiver location are at the foci.

In a constant-wavespeed medium, the σ-factors are simply rc, where r is the linear path length, and, at stationarity, the Jacobians are simply the path lengths between $\boldsymbol{x}$ and the receiver locations, or

$$|J(\boldsymbol{x}_{IG}, \boldsymbol{x})| \;=\; r_{IG} \;=\; \sqrt{(\overline{x}_1 - \xi_I)^2 + (\overline{x}_3 - z_I)^2} \tag{7}$$

$$|J(\boldsymbol{x}_{OG}, \boldsymbol{x})| \;=\; r_{OG} \;=\; \sqrt{(\overline{x}_1 - \xi_I)^2 + (\overline{x}_3 - z_O)^2}. \tag{8}$$

In homogeneous media wavefronts are spherical, so

$$\frac{1}{\kappa_{IG}} \;=\; r_{IG} \;=\; \frac{\sigma_{IG}}{c}, \qquad \frac{1}{\kappa_{OG}} \;=\; r_{OG} \;=\; \frac{\sigma_{OG}}{c}. \tag{9}$$

Finally, the gradient of the scattered traveltime is a vector pointing in the direction of r_{IG} with magnitude $1/c$. In constant wavepeed, it is given by

$$\left|\frac{\partial \nabla_{\boldsymbol{x}} \tau_{IG}}{\partial \xi_I}\right| = \frac{G(\overline{x}_1, \overline{x}_3)}{c\, {r_{IG}}^2}, \tag{10}$$

where G is a factor that contains the topographic variation in the recording surface, as

$$G(\overline{x}_1, \overline{x}_3) \;=\; \left[(\overline{x}_3 - z_I) - (\overline{x}_1 - \xi_I)\,\frac{\partial z_I}{\partial \xi_I}\right]\,. \tag{11}$$

These results allow the integral to be written

$$u_O(\xi_O, t_O) \;\approx\; \frac{1}{\sqrt{2\pi c}} \int d\xi_I\, \frac{G(\overline{x}_1, \overline{x}_3)}{r_{IG}} \cdot \frac{\sqrt{r_S + r_{IG}}}{\sqrt{r_S + r_{OG}}\,\sqrt{|r_{IG} - r_{OG}|}} D_f(\xi_I, \tau_I(\boldsymbol{x}, \xi_I))\,. \tag{12}$$

$$D_f(\xi_I, t) = \frac{1}{2\pi} \int \sqrt{|\omega_I|}\; u_I(\xi_I, \omega_I)\, e^{-i\omega_I t + i\pi/4\,\mathrm{sgn}\,(\omega_I)} d\omega_I.$$

The integral still contains functions of the stationary point, which requires that we determine the location of $\boldsymbol{x}$. As previously described, this point lies at the intersection of the ray through $\boldsymbol{x}_{IG}$ and $\boldsymbol{x}_{OG}$ and the output isochron $\tau_O = t_O$, or equivalently, with the input isochron $\tau_I = t_O + \tau_{IO}$, where τ_{IO} is independently determined.

Using the geometry shown in Figure 2, we can derive analytic expressions for the location of the stationary point by considering the tilted ellipse of the τ_O isochron and straight ray between the input and output locations. Given arbitrary recording and datuming surfaces, the procedure for each source-input-output receiver combination is:

(i) shift the origin in both coordinate directions to the source location, (x_S, z_S);
(ii) rotate the coordinate axes about the new origin so that the new horizontal axis is coincident with the major axis of the τ_O ellipse;
(iii) calculate the location of the stationary point and the required paths in the rotated coordinates.

The shift-rotation can be expressed as

$$x_S{}' = 0, \qquad z_S{}' = 0, \qquad z_O{}' = 0, \tag{13}$$

$$\varphi = \tan^{-1}\left(\frac{z_O - z_S}{\xi_O - x_S}\right), \tag{14}$$

$$\xi_O{}' = (\xi_O - x_S)\cos\varphi + (z_O - z_S)\sin\varphi, \tag{15}$$

$$\xi_I{}' = (\xi_I - x_S)\cos\varphi + (z_I - z_S)\sin\varphi, \tag{16}$$

$$z_I{}' = -(\xi_I - x_S)\sin\varphi + (z_I - z_S)\cos\varphi, \tag{17}$$

$$h' = \frac{\xi_O{}'}{2}, \tag{18}$$

Given this change in reference, the major axis of the ellipse representing the τ_O isochron is aligned with the x_1' axis. It is centered at $x_1' = h'$, as we have defined h' as the signed half-offset between the source and the output receiver location in the primed coordinate frame. From the general derivation, $\tau_O = t_O$, and using these facts the ellipse is defined by the equation

$$x_3'^2 = Q\left[(ct_O)^2 - 4\left(x_1' - h'\right)^2\right]. \tag{19}$$

$$Q = \left(\frac{(ct_O)^2 - 4h'^2}{4(ct_O)^2}\right). \tag{20}$$

The stationary ray is the line through $\boldsymbol{x}_{IG}$ and $\boldsymbol{x}_{OG}$, and is described by,

$$x_1' = \left(\frac{\xi_I' - \xi_O'}{z_I'}\right) x_3' + \xi_O'. \tag{21}$$

The stationary point is one of the two intersections of these two curves. For a straight ray that crosses the major axis of the ellipse, these points are distinguished by the fact that one must lie above the $x_3' = 0$ surface, the other below. Since our construction exists only in the $x_3' > 0$ halfspace, we solve (19) and (21) for x_3', then choose the positive solution. After some algebra, this produces

$$\overline{x}_3' = \sqrt{S + P^2} - P, \tag{22}$$

where

$$S = \left(\frac{Q\left((ct_O)^2 - \xi_O'^2\right) z_I'^2}{z_I'^2 + 4Q\left(\xi_I' - \xi_O'\right)^2}\right), \tag{23}$$

and

$$P = \left(\frac{2Q\left(\xi_I' - \xi_O'\right)\xi_O' z_I'}{z_I'^2 + 4Q\left(\xi_I' - \xi_O'\right)^2}\right). \tag{24}$$

with Q defined in (20). Equation (22) produces a value that is both positive and real, since, by the problem geometry, S is always positive. Now, $\overline{x}_1'$ is found using this result and either (19) or (21). Since rotation of the coordinate system does not change the path lengths between the source and receiver locations and the stationary point, they can be calculated in the primed coordinates, using

$$r_S = r_S' = \sqrt{\overline{x}_1'^2 + \overline{x}_3'^2}, \tag{25}$$

$$r_{OG} = r_{OG}' = \sqrt{\left(\overline{x}_1' - \xi_O'\right)^2 + \overline{x}_3'^2}, \tag{26}$$

$$r_{IG} = r_{IG}' = \sqrt{\left(\overline{x}_1' - \xi_I'\right)^2 + \left(\overline{x}_3' - z_I'\right)^2}. \tag{27}$$

The depth coordinates of the stationary point appear in the integral, so rotate the depth in the primed frame back to the unprimed frame, and undo the shift, via

$$\overline{x}_1 = \overline{x}_1' \cos\varphi - \overline{x}_3' \sin\varphi + x_S\ , \tag{28}$$

$$\overline{x}_3 = \overline{x}_1' \sin\varphi + \overline{x}_3' \cos\varphi + z_S. \tag{29}$$

The previous discussion is applicable to data collected on any arbitrary recording surface. If data is collected on a horizontal recording surface, however,

$$z_I = \frac{\partial z_I}{\partial \xi_I} = 0, \quad \text{and,} \quad G\left(\overline{x}_1, \overline{x}_3\right) = \overline{x}_3. \tag{30}$$

Location of the stationary point follows the same procedure as the general case.

The procedure for downward continuation of sources in constant wavespeed is analogous to the previous discussion, again under the interchange of the roles of sources and receivers.

Conclusions

We have developed a method for "true-amplitude" downward continuation of sources and receivers, based on data mapping theory. Due to the expense of performing this process in general, heterogeneous media, we have derived analytical expressions for evaluation of the amplitudes in a constant wavepseed media, as an example of how the process may be simplified in simple wavespeed models. We have also developed computer implementations of these methods, and synthetic data results will be shown in our presentation.

References

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