2.5D data mappings

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Summary

Data mapping is a procedure for transforming data from a given input source/receiver configuration and macro earth model to a different output source/receiver configuration and background model. In a companion paper, we described a "platform" for data mapping. The platform is an integral operator derived as a cascade of an inversion operator with a modeling operator. To apply this platform to a particular data mapping process, it is necessary to eliminate the integration over the modeling variables by analytic methods in order to obtain an operator that directly maps input data and physical model to output data and physical model. In this paper, 2.5D data mapping formulas for downward continuation of receiver (or sources), transformation to zero offset, and mapping from common offset to common shot are presented In general, the data mapping formalism maps Kirchhoff approximate model data in input variables to Kirchhoff data in output variables, with one exception: the method does not map the reflection coefficient. Kirchhoff data accurately models travel time geometrical spreading effects due to both source propagation and reflector curvature. In this sense, this is a "true amplitude" formalism.

Introduction

Data mapping is the process of transforming observed data from a given source/receiver configuration and model of the propagating medium to another configuration and model, under an assumed model of the propagation mechanism. (This is a generalization of the NMO/DMO process.) Our data mapping is a "true amplitude" process in the following sense. (i) Travel time and geometrical spreading effects of the input configuration are transformed to those effects of the output configuration. (ii) Reflector curvature geometrical spreading effects of the input configuration are transformed to those effects of the output configuration. However, the reflection coefficient of the input configuration is preserved in the output data. The formalism also provides a mechanism for determining both the input and the output geometrical optics incidence angles of the reflection process, thereby providing a basis for amplitude versus angle (AVA) analysis.

Two-and-one-half dimensional (2.5D) processing is a mathematical formalism that allows for the processing of a single line of data while allowing for three dimensional propagation of waves. The model for 2.5D processing assumes that data is gathered on a single line in the dominant dip direction; that is, there is no (or little) variation in the earth in the transverse direction orthogonal to the line of data as well as to the vertical plane through the acquisition line.

In Bleistein and Jaramillo [1998] and Jaramillo and Bleistein [1998], we describe a procedure to derive a general data mapping formula, both in 2.5D and 3D. The basic idea of the method is to cascade an inversion formula with a modeling formula. The combined formula maps a given data set to another. The result is an integral over the variables of the input data set to produce an earth model combined with an integral over the coordinates of the earth model to produce the output data set. The operator is a function of the input and output parameters and the earth model variables, as well. The idea, then, is to carry out the integration over the last set of (earth modeling) variables analytically (asymptotically) in order to obtain a weight that is a function of the input and output variables, only. This weight is then applied to the input data set to produce the output data set. This last analysis must be tailored to each specific data mapping implementation.

In this paper, we present the 2.5D data mapping formulas listed in the summary as well as the analysis tools necessary to derive these results.

The 2.5D data mapping platform formula

The 2.5D data mapping platform formula derived in Bleistein and Jaramillo [1998] is

$$
\begin{aligned}
u_O(\xi_O,\omega_O) \sim \frac{\sqrt{|\omega_O|}e^{-i\pi \operatorname{sgn}(\omega_O)/4}}{4\pi^2} \\
\cdot\int \sqrt{|\omega_I|}e^{i\pi \operatorname{sgn}(\omega_I)/4} u_I(\xi_I,\omega_I)\,d\omega_I\,d\xi_I \\
\cdot\int \frac{a_O(\mathbf{x},\xi_O)}{a_I(\mathbf{x},\xi_I)} \frac{|\nabla_x \tau_O(\mathbf{x},\xi_O)|}{|\nabla_x \tau_I(\mathbf{x},\xi_I)|} |H(\mathbf{x},\xi_I)| \qquad (1) \\
\cdot\frac{\sqrt{\sigma_{Is}+\sigma_{Ig}}}{\sqrt{\sigma_{Os}+\sigma_{Og}}} \frac{\sqrt{\sigma_{Os}\sigma_{Og}}}{\sqrt{\sigma_{Is}\sigma_{Ig}}} \\
\cdot e^{[i\omega_O\tau_O(\mathbf{x},\xi_O) - i\omega_I\tau_I(\mathbf{x},\xi_I)]} d^2x.
\end{aligned}
$$

In this equation, we use the subscripts, I and O to denote input and output variables, respectively, and we use the subscripts, s and g to denote source and receiver variables. Thus, the (two dimensional) positions of the input and output sources and receivers are defined through parameters, ξ_I and ξ_O, respectively, as

$$
\begin{aligned}
\mathbf{x}_s = \mathbf{x}_s(\xi_I), \qquad \mathbf{x}_g = \mathbf{x}_g(\xi_I), \\
\mathbf{y}_s = \mathbf{y}_s(\xi_O), \qquad \mathbf{y}_g = \mathbf{y}_g(\xi_O).
\end{aligned} \qquad (2)
$$

The functions, $u_I(\xi_I, \omega_I)$ and $u_O(\xi_O, \omega_O)$ are the input and output data of the mapping process. Furthermore,

$$\begin{aligned} \tau_I(\mathbf{x}, \xi_\mathbf{I}) &= \tau(\mathbf{x}, \mathbf{x}_s(\xi_\mathbf{I})) + \tau(\mathbf{x}, \mathbf{x}_g(\xi_\mathbf{I})); \\ a_I(\mathbf{x}, \cdot) &= A(\mathbf{x}, \mathbf{x}_s(\cdot)) \cdot A(\mathbf{x}, \mathbf{x}_g(\cdot)), \end{aligned} \tag{3}$$

with the separate travel times, $\tau(\mathbf{x}, \mathbf{x}_s(\xi_I))$, $\tau(\mathbf{x}, \mathbf{x}_g(\xi_I))$ and amplitudes, $A(\mathbf{x}, \mathbf{x}_s(\xi_I))$, $A(\mathbf{x}, \mathbf{x}_g(\xi_I))$ being solutions of appropriate eikonal and transport equations, respectively, with initial point, $\mathbf{x}_s$ or $\mathbf{x}_g$, and final point, $\mathbf{x}$. Similar expressions hold for the variables with subscript O. We allow for different propagation speeds in the eikonal equations (mode conversion) and transport equations. The amplitudes can also include products of transmission coefficients arising from interfaces above the surface, S_R. In addition, σ_{Is} (σ_{Ig}) is a running ray parameter along the ray from the source (receiver) to the scattering point, $\mathbf{x}$. It is the natural parameter that arises to describe out-of-plane geometrical spreading in the 2.5D case. It is related to travel time through the equation, $d\sigma/d\tau = c^2(\mathbf{x})$. Finally,

$$H(\mathbf{x}, \xi_I) = \det \begin{bmatrix} \nabla_x \tau(\mathbf{x}, \xi_I) \\ \dfrac{\partial}{\partial \xi_I} \nabla_x \tau(\mathbf{x}, \xi_I) \end{bmatrix} \tag{4}$$

is the 2D Beylkin [1985] determinant.

The objective, now, is to simplify this operator by approximating the integrations over the interior (depth) variables. Motivated by the work of Tygel et al [1998], we first make a change of variables to new coordinates, one of which is the input travel time, $\tau_I(\mathbf{x}, \xi_\mathbf{I})$, in (3), and the other is a running parameter along each curve of constant τ_I, that is, on the *isochrons* of τ_I. Let us call that second parameter, γ_I.

Result of stationary phase along the isochron

The first approximation that we consider is application of the method of stationary phase to the integral along the isochron. On this curve, τ_I is a constant and the phase to be analyzed is τ_O, as a function of the variable, γ_I and the constant, τ_I. The phase is stationary at that γ_I for which the isochrons of τ_I and τ_O are *tangent* to one another. The result of the method of stationary phase in γ_I applied to the integral in (1) is

$$\begin{aligned} u_O(\xi_O, \omega_O) \sim \frac{e^{i\pi \mathrm{sgn}(\omega_O)/4[\mathrm{sgn}(\kappa_O - \kappa_I) - 1]}}{[2\pi]^{3/2}} \\ \cdot \int \sqrt{|\omega_I|} e^{i\pi \mathrm{sgn}(\omega_I)/4} u_I(\xi_I, \omega_I) d\omega_I \, d\xi_I \\ \cdot \int \frac{a_O(\mathbf{x}, \xi_O)}{a_I(\mathbf{x}, \xi_I)} \frac{\sqrt{|\nabla_x \tau_O(\mathbf{x}, \xi_O)|}}{|\nabla_x \tau_I(\mathbf{x}, \xi_I)|^2} \frac{|H(\mathbf{x}, \xi_I)|}{\sqrt{|\kappa_O - \kappa_I|}} \\ \cdot \frac{\sqrt{\sigma_{Is} + \sigma_{Ig}}}{\sqrt{\sigma_{Os} + \sigma_{Og}}} \frac{\sqrt{\sigma_{Os}\sigma_{Og}}}{\sqrt{\sigma_{Is}\sigma_{Ig}}} dt_I e^{i\omega_O \tau_O(\mathbf{x}, \xi_O) - i\omega_I t_I}. \end{aligned} \tag{5}$$

In this equation, the new variables, κ_I and κ_O, arise from the evaluation of the second derivative of the phase at stationarity. These are the *curvatures* of the input and output isochrons, respectively, at the stationary point. We remark that these total isochron curvatures can be written in terms of the separate curvatures for the travel times from source or receiver, respectively, to the point at depth. That result is

$$\begin{aligned} \kappa_O - \kappa_I &= \cos\theta_O(\kappa_{Os} + \kappa_{Og}) + \sin^2\theta_O \nabla_x c \cdot \hat{\mathbf{n}}/c \\ &\quad - \cos\theta_I(\kappa_{Is} + \kappa_{Ig}) + \sin^2\theta_I \nabla_x c \cdot \hat{\mathbf{n}}/c, \end{aligned} \tag{6}$$

also derived independently by Tygel and presented in Tygel et al [1997].

The method of stationary phase *is invalid* when this curvature difference is zero, or even near zero. More accurately, the difference of the radii of curvatures of the isochrons must be "large" compared to the wavelength, say, at least three wavelengths, for the stationary phase evaluation to yield an adequate approximation of the integral. For example, for offset continuation through small offsets, or transformation to zero offset from a small initial offset, this result would not produce a good approximation of the operator represented by (1). We remark that the classic NMO/DMO sequence arises from exactly this type of asymptotic analysis. While that process is a kinematically accurate mapping from small offset to zero offset, it will be dynamically inaccurate, not producing the true amplitude prediced in the introduction.

We see, also, that the sign of the curvature difference affects the nature of u_O in the time domain. If the curvature difference is positive, there is no phase dependence in ω_O other than in u_O, itself. If the curvature difference is negative, u_O is multiplied by $-i\mathrm{sgn}(\omega_O)$ in the frequency domain. This is equivalent to the Hilbert transform of u_O in the time domain. Thus, we see that the relative curvature of the isochrons changes the character of the output data in the time domain.

Downward continuation of receivers (or sources)

For this case, it is easy to explain stationarity geometrically. Draw the ray from the input receiver through the output receiver and down to the isochron, $\tau_I = t_I$. Draw the ray from that intersection point to the common source point. The input and output travel times share the same travel time from source to isochron and then from isochron to the output receiver. In fact, they only differ by the travel time from the input receiver to the output receiver, which is a constant for fixed input and output geophone. Because the input and output rays overlay one another at the isochron, the two travel time gradients are identical and the isochrons are tangent. This also means that the opening angles, $2\theta_I$ and $2\theta_O$ between the rays from source and receiver at the iscochron, identical. That is,

$$|\nabla_x \tau_O(\mathbf{x}, \xi_O)| = |\nabla_x \tau_I(\mathbf{x}, \xi_I)| = 2\cos\theta\ /c. \tag{7}$$

Here, we no longer distinguish between input and output opening angles because they are the same. Consequently, the reflection coefficient at input incidence angle is also the reflection coefficient at output incidence angle; that is, the reflection coefficient *is preserved* in this mapping. Furthermore, for this case, the curvature difference can be shown to be positive; hence, we lose the phase factor from (5).

It is now straightforward to transform (5) to the time domain by multiplying by $\exp\{-i\omega_O t_O\}\ /2\pi$ and integrating on both sides. The only ω_O dependence on the right is now in the phase, producing a delta function in time, $\delta(t_O - t_I)$, thereby allowing us to carry out the time domain integration, as well. The result is,

$$U_O(\xi_O, t_O) \sim \frac{1}{[2\pi]^{3/2}} \int \frac{D_{1/2}(\xi_I, t_I)}{|\nabla_x \tau_I(\mathbf{x}, \xi_I)|^{3/2}} \frac{A(\mathbf{x}, \mathbf{x}_g(\xi_O))}{A(\mathbf{x}, \mathbf{x}_g(\xi_I))} \cdot \frac{\sqrt{\sigma_{Is} + \sigma_{Ig}}}{\sqrt{\sigma_{Os} + \sigma_{Og}}} \frac{\sqrt{\sigma_{Og}}}{\sqrt{\sigma_{Ig}}} \frac{|H(\mathbf{x}, \xi_I)|}{\sqrt{|\kappa_O - \kappa_I|}} d\xi_I. \tag{8}$$

Here,

$$D_{1/2}(\xi, t) = \frac{1}{2\pi} \int \sqrt{|\omega|} e^{i\pi \operatorname{sgn}(\omega_I)/4 - i\omega t} u_I(\xi, \omega) d\omega \tag{9}$$

is the Hilbert transform of the half-derivative of the input data. In the frequency domain, the product,

$$e^{i\pi/2 \operatorname{sgn}(\omega_I)} \cdot \sqrt{|\omega_I|} e^{-i\pi \operatorname{sgn}(\omega_I)/4} = i \operatorname{sgn}(\omega_I) \cdot \sqrt{-i\omega_I},$$

commutes, but their temporal counterparts, Hilbert transform and half-derivative, respectively, *do not.* The reason is that fractional derivatives can only be applied to causal functions and the Hilbert transform of a causal function is no longer causal. Hence, care is necessary in the order in which these operators are applied in the time domain.

In (8), $t_I = t_I(t_O, \xi_I, \xi_O)$, through the stationary phase condition that fixes the input and output travel time isochrons to be tangent. This type of mapping between input and output travel times is similar to the mapping of input to output travel times in DMO. Given t_O, ξ_I, ξ_O, we draw the isochron, $\tau_O = t_O$ and find the isochron of τ_I that is tangent to it. That determines t_I and we can proceed to evaluate the integrand. In Sheaffer and Bleistein [1998], the constant background implementation of this result is discussed in greater detail.

Time domain data mapping for other implementations

The technique we used—inverse Fourier transform from ω_O to t_O will always provide a mapping to output data in space time. However, when $\operatorname{sgn}(\kappa_O - \kappa_I) = -1$, the phase in (5) depends on ω_O. If we bring this dependence to the left side of the equation before taking the inverse Fourier transform, we obtain

$$D_0(\xi_O, t_O) \sim \frac{1}{[2\pi]^{3/2}} \int \frac{D_{1/2}(\xi_I, t_I)}{|\nabla_x \tau_I(\mathbf{x}, \xi_I)|^{3/2}} \frac{a_O(\mathbf{x}, \xi_O)}{a_I(\mathbf{x}, \xi_I)} \cdot \frac{\sqrt{\sigma_{Is} + \sigma_{Ig}}}{\sqrt{\sigma_{Os} + \sigma_{Og}}} \frac{\sqrt{\sigma_{Os}\sigma_{Og}}}{\sqrt{\sigma_{Is}\sigma_{Ig}}} \frac{|H(\mathbf{x}, \xi_I)|}{\sqrt{|\kappa_O - \kappa_I|}} d\xi_I. \tag{10}$$

In this equation,

$$D_O(\xi, t) = \frac{1}{2\pi} \int \sqrt{|\omega|} e^{i\pi \operatorname{sgn}(\omega_I)/4 [\operatorname{sgn}(\kappa_O - \kappa_I) - 1]} \cdot u_O(\xi, \omega)\, e^{-i\omega t} d\omega \tag{11}$$

could be either the data, $U_O(\xi, t)$, or its Hilbert transform, depending on $\operatorname{sgn}(\kappa_O - \kappa_I) - 1$. For example, when mapping from finite offset to zero offset, this difference is zero and data mapping produces a formula for the data, U_O, itself. When mapping from zero offset to finite offset, the output is the Hilbert transform of U_O. When mapping from common offset to common shot, D_O is the data itself when the output source and receiver are closer than the input common offset; D_O is the Hilbert transform of the data when the output source and receiver are further apart than the input source and receiver.

Stationary phase in t_I

We remark that for downward continuation of receivers (or sources) one can show that the phase at stationarity is linear in t_I, so that we could not apply the method of stationary phase to the integral in t_I. In other implementations, this is not the case and we can apply the method of stationary phase to the t_I integral in (5). We find that the condition of stationarity is that

$$\frac{\omega_O}{\omega_I} = \frac{\cos\theta_I}{\cos\theta_O}. \tag{12}$$

where, again, $2\theta_I$ and $2\theta_O$ are the angles between the incident rays from the source and receiver in the input or output configurations, evaluated at any subsurface point.

For a stationary point to exist, ω_I and ω_O must have the same sign. Furthermore, when mapping from a larger offet to a smaller offset, $\omega_I > \omega_O$; when mapping from a smaller offset to a larger offset, $\omega_I < \omega_O$. This further restricts the range of integration in ω_I. For the complementary range of integration, there is no stationary point and the contribution to the total integral is smaller, less than leading order in ω_O. Since this is a leading order asymptotic theory, we neglect those contributions.

Equation (12) can be viewed as defining a scaling between input and output frequencies. This is familiar from migration/inversion outputs, as well as from DMO processing, where the change in resolution is observed. This rescaling actually assures that the final reflector-normal direction image resolution after inversion or migration will be the same, whether the input or output data is processed.

We proceed to express the result of evaluating the right side of (5) by the method of stationary phase. The result is

$$u_O(\xi_O,\omega_O) \sim e^{i\pi \mathrm{sgn}(\omega_O)/4[\mathrm{sgn}(\kappa_O-\kappa_I)+\mathrm{sgn}(K)]}$$

$$\frac{1}{2\pi}\int_\Omega d\omega_I\, d\xi_I\, u_I(\xi_I,\omega_I)\frac{|H(\mathbf{x},\xi_I)|}{\sqrt{|(\kappa_O-\kappa_I)K|}}$$

$$\cdot\frac{a_O(\mathbf{x},\xi_O)}{a_I(\mathbf{x},\xi_I)}\frac{|\nabla_x\tau_O(\mathbf{x},\xi_O)|}{|\nabla_x\tau_I(\mathbf{x},\xi_I)|^{3/2}} \tag{13}$$

$$\cdot\frac{\sqrt{\sigma_{Is}+\sigma_{Ig}}}{\sqrt{\sigma_{Os}+\sigma_{Og}}}\frac{\sqrt{\sigma_{Os}\sigma_{Og}}}{\sqrt{\sigma_{Is}\sigma_{Ig}}}e^{i\omega_O t_O-i\omega_I t_I}.$$

In this equation, Ω is the restricted interval of integration imposed by stationarity. The travel times, t_O and t_I, are functions of $\xi_I,\omega_I,\xi_O,\omega_O$, defined by the two stationarity conditions, above. That is, first consider the points of tangency between isochrons for given input and output source and receiver locations. Among those, find those for which the ratio of cosines of incidence angles satisfies (12). That determines the travel times in this formula. Also, K is the second derivative of the phase with respect to t_I, evaluated at stationarity. After much calculation, to be described in a paper currently in preparation, we show that

$$K = -\frac{\cos^2\theta_O}{c}\nabla_x c\cdot\nabla_x\tau_O + \frac{\omega_O\sin^2\theta_O}{c}(\kappa_{Os}+\kappa_{Og}) \tag{14}$$

$$+\frac{\cos\theta_O\cos\theta_I}{c}\nabla_x c\cdot\nabla_x\tau_I - \frac{\cos\theta_O\sin^2\theta_I}{c\cos\theta_I}(\kappa_{Is}+\kappa_{Ig}).$$

In this equation, the κ's are the curvatures of the separate isochrons of the travel times from source or receiver to scattering point and for input or output variables, subscripted appropriately. These are actually easy to calculate; for example, in constant background media, they are just equal to the reciprocals of the travel distances.

In the familiar case of constant background DMO, the gradient of the travel time is zero and the output radius of curvature is smaller than the input; hence, the output curvatures are larger. In this case, $\kappa_O-\kappa_I$ is positive. On the other hand, $\theta_O=0$ and only the last term survives in the expression for K, above. In this case, the two sgn-functions appearing in the phase in (13) are of opposite sign and the filter outside the integral sign reduces to unity. Indeed, the frequency domain formula here reduces to the result of Bleistein et al [1998], for this case.

Conclusions

We have described here some results of 2.5D data mapping, in the frequency domain and in the time domain. For this first development, we have avoided caustics and multi-pathing of rays from point sources. We have singled out the application to downward continuation of receivers or sources because the asymptotic analysis for that case is different from other implementations. In other time domain implementations, we have noted the possibility that the output of this method is not the mapped time domain data, itself, but filtered data whose time domain interpretation is the Hilbert transform of the half-derivative of the output data. The choice depends on the relative curvature of the input and output isochrons. The filtering can be undone by post-processing. We have cited examples where this case does arise. Finally, we show that the frequency domain result can include a phase shift that depends on the curvatures of the isochrons. These are usually easy to analyze in the absence of multi-pathing.

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