Asymptotically True-amplitude One-way Wave Equations in t: modeling, migration and inversion

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Abstract

Currently used true-amplitude one-way wave equations in depth yield high quality solutions for forward modeling problems and inversion. A shortcoming of these equations is that they yield poor results *near* and fail *at* horizontal propagation, making the application to turned waves problematic, at the very least. One-way wave equations in time do not have this shortcoming; they are omni-directional in space. We introduce fairly simple forward and reverse time first order wave equations; they are essentially adjoints of one another. Spatial derivatives appear in these equations through a pseudo-differential operator—the square root of the Laplacian. With an appropriate definition of this operator, we have proved via ray theory that the solutions of one-way wave equations in time asymptotically approximate the solutions to the two-way wave equation to leading order for forward or reverse time propagation. For us "true-amplitude" is meant in this ray-theoretic (asymptotic) sense. The inverse series in powers of $i\omega$ in the frequency domain becomes a series in progressing waves in the time domain. The propagation of the leading order progressing wave is governed by the eikonal equation for the two-way wave equation and the slowly varying amplitude of this leading order progressing wave satisfies the same transport equation as for the two-way wave equation. This theory provides a solid theoretical base for the Explicit Marching algorithm for solving reverse time migration and anticipates an inversion—a true-amplitude reverse time migration. We present the equivalent initial value problem for Green's functions. For homogeneous media, the 3D Green's function derived by integral transform methods is complex-valued. Its real part is the propagating delta function that we expect and its imaginary part is the Hilbert transform of the real part, completing the so-called *analytic* Green's function. We confirm that the Kirchhoff approximation of asymptotic ray theory in frequency domain applies to progressing waves in time domain. A Green's identity relating solutions of the one-way wave equations

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and their adjoint is derived. This allows us to develop Kirchhoff integral representations from propagation of surface data into the Earth and the propagation of reflection data to the upper surface. Those identities plus identification of the adjoint operator for the forward modeling operator lead to migration and inversion formulas using our analytic Green's functions. Observed data at the upper surface must be extended to analytic data in order to apply the inversion theory.

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1 Introduction

Asymptotically true-amplitude one-way wave equations in depth yield high quality solutions for forward modeling problems and inversion. Zhang [1993] introduced two new one-way wave equations in depth and verified that the leading order their asymptotic solutions agreed with the asymptotic solutions of the two-way wave equation. This original work was for waves in two spatial dimensions and frequency. Zhang et al [2003] extended the theory to 3D, and based on the generalization, they proposed the theory of true-amplitude one-way wave equation migrations.

In theory, this separation into two one-way wave equations for up-going and down-going waves has a pathology in the neighborhood of horizontal propagation where the terminology, up-going and down-going, loses meaning.

Zhang et al [2007] derived a new one-way wave equation in time. All spatial directions are treated the same way. Thus, it can handle horizontal and turning waves routinely. Furthermore, this method provides a new way of doing reverse-time migration, that those authors call the "Explicit Marching" (EM) method. Unlike the conventional finite-difference methods, EM does not suffer from stability and numerical dispersion problems. In contrast to the one-way equations in depth, the pseudo-differential operator involved in EM is non-singular; the new equation can be numerically solved efficiently.

Here, we adapt the methodology of Zhang [1993] and Zhang et al [2003] to analyze the one-way wave equations in time. Though the equations we use look similar to those proposed by Gazdag [1981], and later were used by Baysal, et al [1983] for reverse-time migration, ours can deal with steep deeps (near horizontal propagation) and overhangs (turned waves). That is, these equations avoid the pathology of the spatial one-way wave equations in z at and near horizontal propagation: they have no favored direction in space.

A crucial difference between the method of Baysal, et al [1983] and the method proposed in Zhang et al [2007] and used here is that the real-valued space/time Green's function of the two-way wave equation used in Baysal, et al [1983] is necessarily replaced in Zhang et al [2007] and this paper by the *analytic* Green's function. The analytic Green's function has the real Green's function of the two-way wave equation as its real part and its Hilbert transform as the imaginary part.

1.1 Our methodology

As in the development of reverse time migration, we present the theory here in time rather than in frequency. Recall that an inverse power of $i\omega$ as a multiplier of a function in the frequency domain corresponds to integration of that function in the time domain. Thus, the successive terms in an inverse power series in $i\omega$ used in asymptotic ray theory—terms of the form, $f(\omega)\exp\{i\omega\tau(\boldsymbol{x})\}/(i\omega)^n$—correspond to multiple integrals of the basic function, say $F(\tau(\boldsymbol{x})-t)$, in the time domain. (Here, τ is the travel time function.) Hence the sequence of functions in the frequency domain transforms into increasingly smoother functions in the time domain.

We assume a relatively compact domain for the function F, such as a bandlimited delta function, so that with each term of the sequence of integrals of F in time a relatively smoother progressing wave. Hence, ray theory in the time domain is called the "progressing wave formalism"; see Lewis [1964]. Each progressing wave is multiplied by a more "slowly varying" spatial amplitude that needs to be determined. As in the frequency domain, the formalism determines the traveltime τ and these slowly varying amplitudes through a hierarchy of equations just as in ray theory in the frequency domain.

The spatial derivative in the one-way equations is the square root of the Laplace operator. We use the method of Zhang [1993] to write down an integral representation of this operator that avoids the square root. This is achieved through a formal application of pseudo-differential operator theory.

It is then possible to derive the hierarchy of equations for traveltime and amplitude familiar from ray theory. The leading order equation in the heirarchy leads to the correct eikonal equation. The next order equation in the hierarchy yields the transport equation for the leading order amplitude. We usually go no further than this leading order amplitude in solving problems asymptotically.

That first transport equation yields an amplitude that differs from the corresponding transport equation for the full wave equation only by a power of v, the wave speed. Thus, it is straightforward to write down the asymptotic solutions of the two-way wave equation in terms of the solutions of these one-way wave equations and *vice versa*. The calculations are tedious, but the method is straightforward.

One-way wave equations need initial data to generate well-posed problems for solution. This is a challenge for the Green's function where the initial data is prescribed by a balance between a second derivative with respect to time and a delta function in time as a factor of the source. We show that the "right" initial data for the Green's function for this one-way wave equation is actually a pseudo-differential operator in spacial coordinates acting on the spatial delta function. This echoes the corresponding result for the one-way wave equation in space; see Zhang et al [2003], where we consider "initial value" problems in z and prescribe "initial data" in z.

In the next section, we introduce the idea of a progressing wave formalism and present the asymptotic analysis for the two-way wave equation. This discussion provides a basis of comparison for the asymptotic analysis of the one-way wave equations to follow and also demonstrates the methodology of the progressing wave formalism for the reader who might be unfamiliar with this approach.

The following section introduces the one-way wave equations for forward and reverse time propagation. Here is where we use Zhang's [1993] method to express the square root operator as the integral of a quotient of second order operators.

The derived eikonal equation matches the eikonal equation for the two-way wave equation. The derived transport equation agrees with each of the transport equations derived previously for the two-way wave equation within a power of the wavespeed v, different for each of those two two-way wave equations. The latter equations arise from conservation laws for displacement or pressure, for example. The one-way equation is a mathematical device yielding analytic extensions of the solutions of the two-way wave equation. This is a *device* for approximating solutions of two-way wave equations for which there are no apparent physical conservation laws; we accept these solutions for what they are, absent a physical interpretation.

Next, we define the initial value problem for the Green's function and show that the solution for homogeneous media is the *analytic* Green's function, with the "right" real part for the two-way wave equation and imaginary part being the Hilbert transform of the real part.

We also develop a Kirchhoff approximation to generate data at a reflector in terms of the incident data and the reflection coefficient. We need this for modeling of observed data in the derivation of a Kirchhoff-type inversion formula.

Next, we introduce the adjoint operator for each of our one-way wave operators. With those in place, we can derive a Green's theorem from which we can derive representations of the downward continuation of observed surface data and for the upward propagation of our Kirchhoff-approximate data from a reflector to the upper surface.

A general feature of modeling with one-way wave equations is that (i) source information for the two-way wave equation provides initial data or final data for the one-way wave equations and (ii) boundary data for the two-way wave equation provides sources for the one-way wave equations. The latter echos the exploding reflector model for wave propagation.

We expect modeling and inversion outputs to exhibit the same level of quality and smoothness as is achieved by using the one-way wave equations in spatial coordinates. However this inversion is expected to have the additional advantage that there is no pathology for horizontal propagation or turning waves. Thus, this inversion should routinely image vertical flanks and the underside of salt domes just as reverse time migration does.

Finally, we derive common-shot inversion formulas in the frequency domain and in the time domain. The simplicity of form of our Green's functions in the frequency domain makes the derivation of an inversion formula in that domain a simple imitation of previous derivations by pseudo-inversion, as presented, for example, in Bleistein et al [2005]. Then return to the time domain and provide an inversion formula there. Both of these inversion formulas form an image by correlation of the downward continued data with the downward continued source.

The data used in this inversion must be consistent with forward modeling for this one-way wave equation. Acquired data is the solution of a two-way wave equation. As with the Green's function, we need to take those acquired data and extend them to *analytic* data by adding i times their Hilbert transform in time.

2 Progressing wave formalism for the two-way wave equation

Here, we present the derivation of the progressing wave formalism for the two-way wave equation

$$\mathcal{L}U = \frac{1}{\mathrm{v}^2}\left\{\frac{\partial^2 U}{\partial t^2} - (\mathrm{v}\nabla)^2 U\right\} = 0, \quad U = U(\boldsymbol{x}, t), \qquad (\mathrm{v}\nabla)^2 \equiv \mathrm{v}\nabla\cdot(\mathrm{v}\nabla). \tag{1}$$

Here, we artificially introduce v as the bulk modulus for a medium of density equal to one or the inverse of density with a bulk modulus equal to one. The former would be a wave equation for displacement; the latter would be a wave equation for pressure. See, for example, Chapman [2004]. We are not as concerned about the physics here as we are with the versatility of the progressing wave formalism for dealing with something more complex than the standard wave equation. However, we also state without proof the corresponding eikonal and transport equation for the simpler wave equation

$$\mathcal{L}\tilde{U} = \frac{1}{\mathrm{v}^2}\frac{\partial^2 \tilde{U}}{\partial t^2} - \nabla^2 \tilde{U} = 0. \tag{2}$$

Note that the previous wave equation (1) reduces to this one in piecewise constant media.

We develop a progressing wave formalism for asymptotic solution of these wave equations in a manner that is completely analogous to asymptotic ray theory.

To begin, we introduce a sequence of progressing wave functions $F_0[\tau(\boldsymbol{x}) - t], F_1[\tau(\boldsymbol{x}) - t], \ldots$, with the property that

$$F'_{n+1} = F_n, \quad n = 0, 1, 2, \ldots \ . \tag{3}$$

Here the prime $\{'\}$ means derivative with respect to total argument of the function. As defined, each wave function is therefore smoother (in the sense of having one more nonsingular derivative) than its predecessor as was the case for the example in the Introduction. We then assume that the solution U can be written as a series in these functions as follows.

$$U(\boldsymbol{x}, t) = A_0(\boldsymbol{x})F_0[\tau(\boldsymbol{x}) \mp t] + A_1(\boldsymbol{x})F_1[\tau(\boldsymbol{x}) \mp t] + \ldots \ . \tag{4}$$

We need save only these terms to determine the governing equations for τ and A_0, as we are performing a leading-order asymptotic analysis.

The representation of U in equation (4) is substituted into the wave equation (1). The series for the derivatives are as follows.

$$\begin{aligned}\frac{\partial U}{\partial t} &= \mp[A_0F'_0 + A_1F'_1] + \ldots \\ &= \mp[A_0F'_0 + A_1F_0] + \ldots \ .\end{aligned} \tag{5}$$

The second equality here uses the relationship between the progressing wave derivatives stated in equation (3). Similarly,

$$\frac{\partial^2 U}{\partial t^2} = A_0F''_0 + A_1F'_0] + \ldots \ . \tag{6}$$

We need nothing smoother than these two orders to derive the eikonal equation and the transport equation for A_0.

The slownesses,

$$\boldsymbol{p} = \nabla\tau = \begin{cases} (p_x, p_z), & \text{2D}, \\ (p_x, p_y, p_z), & \text{3D}, \end{cases} \tag{7}$$

are used below in calculating the spatial derivatives.

$$(\mathrm{v}\nabla)^2 = \mathrm{v}^2\nabla^2 + \mathrm{v}\nabla\mathrm{v}\cdot\nabla. \tag{8}$$

That is, $(\mathrm{v}\nabla)^2$ can be written as a sum of a second order differential operator plus a first order differential operator. Explicitly, these terms are

$$\mathrm{v}\nabla\mathrm{v}\cdot\nabla U = F_0'\mathrm{v}A_0\,(\nabla\mathrm{v})\cdot\boldsymbol{p} + \dots\ , \tag{9}$$

and

$$\nabla^2 U = [A_0F_0'' + A_1F_0']\boldsymbol{p}^2 + F_0'[2\boldsymbol{p}\cdot\nabla A_0 + A_0\nabla\cdot\boldsymbol{p}] + \dots\ . \tag{10}$$

In terms of these two operators,

$$(\mathrm{v}\nabla)^2 U = \mathrm{v}\left\{\mathrm{v}[A_0F_0'' + A_1F_0']\boldsymbol{p}^2 + F_0'[2\mathrm{v}\boldsymbol{p}\cdot\nabla A_0 + 2A_0\boldsymbol{p}\cdot\nabla\mathrm{v} + A_0\nabla\cdot\boldsymbol{p}] + \dots\right\}. \tag{11}$$

The series representation of the second time derivative of U in equation (6) and that of $(\mathrm{v}\nabla)^2 U$ in this last equation (11) are substituted into the wave equation (1). Collecting the coefficients of F_0'' and F_0' and setting them separately equal to zero leads to the following pair of equations.

$$A_0\left[(\mathrm{v}\boldsymbol{p})^2 - 1\right] = 0 \tag{12}$$

and

$$A_1\left[(\mathrm{v}\boldsymbol{p})^2 - 1\right] + \mathrm{v}[2\mathrm{v}\boldsymbol{p}\cdot\nabla A_0 + A_0\boldsymbol{p}\cdot\nabla\mathrm{v} + A_0\nabla\cdot\boldsymbol{p}] = 0. \tag{13}$$

For $A_0 \neq 0$, the first equation here leads to the eikonal equation, more familiarly written as

$$\boldsymbol{p}^2 = (\nabla\tau)^2 = \frac{1}{\mathrm{v}^2}. \tag{14}$$

In the second order equation (13), the multiplier of A_1 is now zero. Thus, this equation becomes

$$\mathrm{v}[2\mathrm{v}\boldsymbol{p}\cdot\nabla A_0 + A_0\boldsymbol{p}\cdot\nabla\mathrm{v} + A_0\nabla\cdot\boldsymbol{p}] = \frac{1}{A_0}\nabla\cdot\left[\mathrm{v}^2A_0^2\boldsymbol{p}\right] = 0, \tag{15}$$

leading to

$$\nabla\cdot\left[\mathrm{v}A_0^2\boldsymbol{p}\right] = 0. \tag{16}$$

This is the transport equation for the amplitude A_0 written in divergence form. This form, which is a statement of the conservation of energy flowing through the end caps of tube made of rays, has standard solutions which form the basis of ray amplitude theory.

The eikonal equation for the simpler wave equation (2) is the same as above. We call the new amplitude $\tilde{A}_0$. The transport equation for $\tilde{A}_0$ is the same as equation (15), except for lacking the term containing $\nabla\mathrm{v}$ in that equation. As a conseqence, the transport equation for $\tilde{A}_0$ is

$$\nabla\cdot\left[\tilde{A}_0^2\boldsymbol{p}\right] = 0. \tag{17}$$

The fact that the two transport equations preserve quantities in ray tubes that differ only by a power of v suggests that it is not necessary to deal with the more difficult wave equation (1) when addressing the one-way wave equation; that is confirmed in the next section.

3 One-way wave equation in time

We carry out the same asymptotic analysis as in the previous section for the one-way wave equations

$$\mathcal{L}_{\pm}W = \frac{1}{\mathrm{v}}\frac{\partial W}{\partial t} \pm \sqrt{\nabla^2}W = 0. \tag{18}$$

We will see below that the upper sign (+) corresponds to waves in which the traveltime function $\tau(\boldsymbol{x})$ increases on the wavefront with increasing time and the lower sign (-) corresponds to waves for which $\tau(\boldsymbol{x})$ increases on the wavefront with *decreasing* time; that is, forward and backwards propagation in time, respectively.

We need a formalism for defining the square root of the Laplacian here. Let us first introduce the straightforward correspondence,

$$\nabla \;\leftrightarrow\; i\boldsymbol{k}, \quad \boldsymbol{k} = \begin{cases} (k_x, k_z), & \text{2D}, \\ (k_x, k_y, k_z), & \text{3D}. \end{cases} \tag{19}$$

By using the correspondence in this equation we obtain the symbolic correspondence

$$\sqrt{\nabla^2} \;\leftrightarrow\; \sqrt{(i\boldsymbol{k})^2} = ik, \quad k = \sqrt{\boldsymbol{k}\cdot\boldsymbol{k}}. \tag{20}$$

Then, by imitating the methodology of Zhang et al [2003], we show in Appendix A—equation (A-6) that[1]

$$ik = i\,|k_z|\,[I(\boldsymbol{k}) + 1], \tag{21}$$

with

$$I(\boldsymbol{k}) = \frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1 - s^2\boldsymbol{k}_{\scriptscriptstyle T}^2}}{k_z^2 + s^2\boldsymbol{k}_{\scriptscriptstyle T}^2}ds, \qquad \boldsymbol{k}_{\scriptscriptstyle T} = \begin{cases} k_x, & \text{2D}, \\ (k_x, k_y), & \text{3D}. \end{cases} \tag{22}$$

Both the factor $i\,|k_z|$ in equation (21) and the integral $I(\boldsymbol{k})$ defined just above require interpretation as pseudo-differential operators. We interpret $i\,|k_z|$ as follows.

$$k_z > 0, \quad i\,|k_z| = ik_z \leftrightarrow \frac{\partial}{\partial z}; \quad k_z < 0, \quad i\,|k_z| = -ik_z \leftrightarrow -\frac{\partial}{\partial z}. \tag{23}$$

We remark that we could have as easily distinguished the x- or y-direction in defining these pseudo-differential operators. We would then have had an analogous interpretation for $i\,|k_x|$ or $i\,|k_y|$. We remind the reader that our method is merely a *device* for carrying out the necessary asymptotic analysis here. We will see below that the distinction of the z-direction plays no role in our final results of this section or beyond.

Now the differential equation (18) can be symbolically written as

$$\mathcal{L}_{\pm}W = \frac{1}{\mathrm{v}}\frac{\partial W}{\partial t} \pm i|k_z|\left\{W + \frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1 - s^2\boldsymbol{k}_{\scriptscriptstyle T}^2}\,W}{k_z^2 + s^2\boldsymbol{k}_{\scriptscriptstyle T}^2}ds\right\} = 0, \tag{24}$$

with the integral operator here subject to interpretation. If multiplication by functions of $i\boldsymbol{k}$ correspond to differentiation, then division by functions of $i\boldsymbol{k}$ should correspond to

[1] Although we define the gradient operator and the square root of the Laplacian operator below in 2D and 3D, only the analysis in 3D is be presented here.

integration, or equivalently, solving a differential equation or convolution with a Green's function for the differential operator in the denominator. The Green's function would be convolved with the function in the numerator. Equivalently, we solve the differential equation for an auxiliary function $W_{_T}(\boldsymbol{x}, t, s)$ as follows

$$W_{_T} = \frac{\boldsymbol{k}_{_T}^2 W}{k_z^2 + s^2 \boldsymbol{k}_{_T}^2} \quad \Longleftrightarrow \quad \left\{ k_z^2 + s^2 \boldsymbol{k}_{_T}^2 \right\} W_{_T} = \boldsymbol{k}_{_T}^2 W$$

$$\Longleftrightarrow \tag{25}$$

$$\frac{\partial^2 W}{\partial z^2} + s^2 \nabla_{_T}^2 W_{_T} = \nabla_{_T}^2 W, \qquad \nabla_{_T} = \begin{cases} \dfrac{\partial}{\partial x}, & \text{2D}, \\[2ex] \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y} \right), & \text{3D}. \end{cases}$$

This is exactly what is needed to interpret $I(\boldsymbol{k})W$ with I defined in equation (22) and ik in equation (21). By this device, we have interpreted the pseudo-differential operator $ikW \iff \sqrt{\nabla^2}W$ in terms of standard differential operators and an auxiliary function: determine ikW by solving for $W_{_T}(\boldsymbol{x}, t, s)$ in equation (25) as a function of W and s and then carry out the necessary integration in s to determine ikW.

We now have a compound of $\pm$ signs due to the signs in the differential equation (18) and the signs of $i|k_z|$ in equation (23). Therefore, below we separate $\mathcal{L}_\pm$ of equation (18) and allow the $\pm$ to correspond to the signs of $i\,|k_z|$ in equation (23). Then, the correspondence between this last expression and the square root of the Laplacian in equation (20) allows us to rewrite the one-way wave operators in equation (18) as

$$\begin{aligned} \mathcal{L}_+ W &= \frac{1}{\mathrm{v}} \frac{\partial W}{\partial t} \pm \frac{\partial W}{\partial z} \pm \frac{\partial}{\partial z} \frac{1}{\pi} \int_{-1}^{1} \sqrt{1 - s^2}\, W_{_T} ds = 0, \\ \mathcal{L}_- W &= \frac{1}{\mathrm{v}} \frac{\partial W}{\partial t} \mp \frac{\partial W}{\partial z} \mp \frac{\partial}{\partial z} \frac{1}{\pi} \int_{-1}^{1} \sqrt{1 - s^2}\, W_{_T} ds = 0 \end{aligned} \tag{26}$$

Here, whether the waves propagate forward in time ($\mathcal{L}_+$) or backward in time ($\mathcal{L}_-$)), the upper sign corresponds to downgoing waves and the lower sign corresponds to upgoing waves. Notice, however, that those signs are opposite in this pair of equations.

The procedure for asymptotic analysis of the one-way wave equations (18) is as follows.

1. First write down progressing wave series for both W and $W_{_T}$.
2. Use the spatial differential equation (25) to determine the coefficients of the series for $W_{_T}$ in terms of the coefficients of the series for W and functions of s. Substitute the series for $W_{_T}$ into the integral in the one-way wave equations (26) just above, and carry out the integrals with respect to s.

3. What results is a progressing wave series totally in terms of the traveltime and amplitudes of the progressing wave representation for W. The most singular part of this equation is the coefficient of F_0' and the next order will be the coefficient of F_0, itself. Setting those two coefficients equal to zero yields the eikonal equation and transport equation that we seek.

We step through this list in the following subsections.

3.1 Relationship between W and W_T

Here, we introduce a progressing wave series of the form of U in equation (4) and then use the partial differential equation (25) to relate the coefficients of the progressing wave expansion of W_T to the coefficients in the progressing wave formalism of W.

The two series that we use for W and W_T are

$$W(\boldsymbol{x},t) = B_0(\boldsymbol{x})F_0[\tau(\boldsymbol{x}) \mp t] + B_1(\boldsymbol{x})F_1[\tau(\boldsymbol{x}) \mp t] + \dots \quad , \tag{27}$$

and

$$W_T(\boldsymbol{x},t,s) = B_{0T}(\boldsymbol{x})F_0[\tau(\boldsymbol{x}) \mp t] + B_{1T}(\boldsymbol{x})F_1[\tau(\boldsymbol{x}) \mp t] + \dots \quad , \tag{28}$$

The relationships among the coefficients are derived in Appendix B. They are

$$B_{0T} = \frac{\boldsymbol{p}_T^2}{p_z^2 + s^2\boldsymbol{p}_T^2} B_0, \tag{29}$$

which is equation (B-7) of Appendix B, and

$$\begin{aligned} B_{1T} \quad = \quad & \frac{\boldsymbol{p}_T^2}{p_z^2 + s^2\boldsymbol{p}_T^2} B_1 \\ & + \frac{1}{B_0}\left\{ \frac{1}{p_z^2}\nabla_T \cdot \left(\frac{p_z^4 B_0^2 \boldsymbol{p}_T}{(p_z^2 + s^2\boldsymbol{p}_T^2)^2} \right) - \frac{1}{\boldsymbol{p}_T^2}\frac{\partial}{\partial z}\left(\frac{B_0^2 \boldsymbol{p}_T^4 p_z}{(p_z^2 + s^2\boldsymbol{p}_T^2)^2} \right) \right\}, \end{aligned} \tag{30}$$

which is equation (B-10) of Appendix B. Note that equation(29) is an algebraic equation in slownesses that matches the pseudo-differential operator equation (**??**) relating W_T to W. This should be expected: to leading order the differentiation of the traveltime function dominates the differentiation process just as it does in asymptotic ray theory in the frequency domain.

3.2 Progressing Wave Expansion for the One-Way Wave Operator

The equations (29) and (30) relating the coefficients in the progressing wave series for W and W_T are what we need to analyze the progressing wave series for the one-way differential equations (18).

Equation (5) provides the necessary time derivative with A's replaced by B's. In this manner The differential equation (26) leads to the following series to two orders in progressing wave functions.

$$
\begin{aligned}
\mathrm{v}\mathcal{L}_+ W &= -[B_0 F_0' + B_1 F_0] \pm \mathrm{v}[B_0 F_0' + B_1 F_0] p_z \pm \mathrm{v}\frac{\partial B_0}{\partial z} F_0 \\
&\qquad \pm \frac{\mathrm{v}}{\pi}\left\{ p_z F_0' \int_{-1}^{1} \sqrt{1-s^2}\, B_{0T} ds + p_z F_0 \int_{-1}^{1} \sqrt{1-s^2}\, B_{1T} ds \right. \\
&\qquad\qquad \left. + F_0 \frac{\partial}{\partial z} \int_{-1}^{1} \sqrt{1-s^2}\, B_{0T} ds \right\} = 0,
\end{aligned}
\tag{31}
$$

and

$$
\begin{aligned}
\mathrm{v}\mathcal{L}_- W &= -[B_0 F_0' + B_1 F_0] \mp \mathrm{v}[B_0 F_0' + B_1 F_0] p_z \mp \mathrm{v}\frac{\partial B_0}{\partial z} F_0 \\
&\qquad \mp \frac{\mathrm{v}}{\pi}\left\{ p_z F_0' \int_{-1}^{1} \sqrt{1-s^2}\, B_{0T} ds + p_z F_0 \int_{-1}^{1} \sqrt{1-s^2}\, B_{1T} ds \right. \\
&\qquad\qquad \left. + F_0 \frac{\partial}{\partial z} \int_{-1}^{1} \sqrt{1-s^2}\, B_{0T} ds \right\} = 0.
\end{aligned}
\tag{32}
$$

As expected, the most singular terms here are of order F_0'. Concentrating on those terms,

$$
\begin{aligned}
&\text{for } \mathcal{L}_+ : \quad F_0' \left\{ B_0[-1 \pm \mathrm{v}p_z] \pm \frac{\mathrm{v}p_z}{\pi} \int_{-1}^{1} \sqrt{1-s^2}\, B_{0T} ds \right\} = 0, \\
&\text{for } \mathcal{L}_- : \quad F_0' \left\{ B_0[-1 \mp \mathrm{v}p_z] \mp \frac{\mathrm{v}p_z}{\pi} \int_{-1}^{1} \sqrt{1-s^2}\, B_{0T} ds \right\} = 0.
\end{aligned}
\tag{33}
$$

We use equation (29) to rewrite this last equation totally in terms of B_0; that is,

$$
\begin{aligned}
&\text{for } \mathcal{L}_+ : \quad F_0' B_0 \left\{ -1 \pm \mathrm{v}p_z \pm \frac{\mathrm{v}p_z}{\pi} \int_{-1}^{1} \frac{\sqrt{1-s^2}\boldsymbol{p}_T^2}{p_z^2 + s^2 \boldsymbol{p}_T^2} ds \right\} = 0, \\
&\text{for } \mathcal{L}_- : \quad F_0' B_0 \left\{ -1 \mp \mathrm{v}p_z \mp \frac{\mathrm{v}p_z}{\pi} \int_{-1}^{1} \frac{\sqrt{1-s^2}\boldsymbol{p}_T^2}{p_z^2 + s^2 \boldsymbol{p}_T^2} ds \right\} = 0.
\end{aligned}
\tag{34}
$$

The integral here is expressible in terms of the integral $I(\boldsymbol{p})$, with $I(\boldsymbol{k})$ defined in equations (21) and (22). That is,

$$
\frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-s^2}\boldsymbol{p}_T^2}{p_z^2 + s^2 \boldsymbol{p}_T^2} ds = \frac{\sqrt{\boldsymbol{p}^2}}{|p_z|} - 1.
\tag{35}
$$

For waves propagating forward in time ($\mathcal{L}_+$), p_z is positive for downgoing waves and negative for upgoing waves; For waves propagating backward in time ($\mathcal{L}_-$), p_z is negative for downgoing waves and positive for upgoing waves. Thus, we interpret $|p_z|$ as follows.

$$\begin{aligned} &\text{For } \mathcal{L}_+ : \quad |p_z| = \pm p_z, \\ &\text{for } \mathcal{L}_- : \quad |p_z| = \mp p_z. \end{aligned} \tag{36}$$

By using this values for $|p_z|$ in the integral in the equation (35) and then substituting into the two leading order equations for $\mathcal{L}_\pm$ in equation (34) we conclude that in all cases

$$F_0' B_0 \left\{-1 + \mathrm{v}\sqrt{\boldsymbol{p}^2}\right\} = 0. \tag{37}$$

Now, for $B_0 \neq 0$, setting the coefficient of F_0' equal to zero leads to

$$\sqrt{\boldsymbol{p}^2} = 1/\mathrm{v}. \tag{38}$$

Because of the sign conventions we chose in the one-way wave equations (18) and in the progressing wave series (27) and (28), the single eikonal equation with positive sign arises here. However, in the arguments $\tau \mp t$, wavefronts move in the direction of $\nabla\tau$ for the upper sign and in the direction of $-\nabla\tau$ for the lower sign; this is forward or backward propagation in time. This eikonal equation is thus equivalent to the eikonal equation (14) for the two-way wave equation.

Let us now return to the operator equations (31) and (32). Because we have eliminated the term of order F_0' in those equations, the operator is now of order F_0 with error of order F_1. Collecting the terms of order F_0 in equation (31) leads to

$$\begin{aligned} &\mathcal{L}_+ : \quad F_0 \left\{ B_1[-1 \pm \mathrm{v}p_z] \pm \mathrm{v}\frac{\partial B_0}{\partial z} \pm \frac{\mathrm{v}p_z}{\pi} \int_{-1}^{1} \sqrt{1-s^2}\, B_{1T} ds \pm \frac{\mathrm{v}}{\pi}\frac{\partial}{\partial z}\left[\int_{-1}^{1} \sqrt{1-s^2}\, B_{0T} ds\right] \right\}, \\ &\mathcal{L}_- : \quad F_0 \left\{ B_1[-1 \mp \mathrm{v}p_z] \mp \mathrm{v}\frac{\partial B_0}{\partial z} \mp \frac{\mathrm{v}p_z}{\pi} \int_{-1}^{1} \sqrt{1-s^2}\, B_{1T} ds \mp \frac{\mathrm{v}}{\pi}\frac{\partial}{\partial z}\left[\int_{-1}^{1} \sqrt{1-s^2}\, B_{0T} ds\right] \right\}. \end{aligned} \tag{39}$$

Note that the dependence of B_{1T} on B_1 in equation (30) is exactly the same as the dependence of B_{0T} on B_0 in equation (29). Thus, further analysis of the terms involving B_{1T} and B_1 will lead to an ultimate expression, $B_1\left\{-1 + \mathrm{v}\sqrt{\boldsymbol{p}^2}\right\}$ with the second factor being zero from the eikonal equation, (38), above. This is completely analogous to the parallel procedure for the two-way wave equation. In particular, note in second equation of (14) in the expansion of the progressing wave, the coefficient of A_1 was exactly the eikonal difference that had already been set equal to zero. Consequently, there is no need to carry those terms any further.

Now we use the second line in equation (30) for B_{1T}. That is,

$$\text{for } \mathcal{L}_+ : \; W \;=\; \pm F_0 \left\{ \frac{\partial B_0}{\partial z} + \frac{1}{p_z B_0}\nabla_T \cdot \left(p_z^4 B_0^2 \boldsymbol{p}_T \frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-s^2}}{\left[p_z^2 + s^2\boldsymbol{p}_T^2\right]^2}\, ds \right) \right. \tag{40}$$

$$-\frac{p_z}{\boldsymbol{p}_T^2 B_0}\frac{\partial}{\partial z}\left(\boldsymbol{p}_T^2 p_z B_0^2 \frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-s^2}}{\left[p_z^2+s^2\boldsymbol{p}_T^2\right]^2}\,ds\right)+\frac{\partial}{\partial z}\left(B_0\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-s^2}\boldsymbol{p}_T^2}{p_z^2+s^2\boldsymbol{p}_T^2}ds\right)\Bigg\},$$

$$\text{for } \mathcal{L}_- :\ W = \mp F_0\left\{\frac{\partial B_0}{\partial z}+\frac{1}{p_z B_0}\nabla_T\cdot\left(p_z^4 B_0^2\boldsymbol{p}_T\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-s^2}}{\left[p_z^2+s^2\boldsymbol{p}_T^2\right]^2}\,ds\right)\right. \tag{41}$$

$$\left.-\frac{p_z}{\boldsymbol{p}_T^2 B_0}\frac{\partial}{\partial z}\left(\boldsymbol{p}_T^2 p_z B_0^2 \frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-s^2}}{\left[p_z^2+s^2\boldsymbol{p}_T^2\right]^2}\,ds\right)+\frac{\partial}{\partial z}\left(B_0\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-s^2}\boldsymbol{p}_T^2}{p_z^2+s^2\boldsymbol{p}_T^2}ds\right)\right\}.$$

The error in this approximation is of order F_1.

The integral in the last term here is given in equation (35), except that we now know that $\sqrt{\boldsymbol{p}^2}=1/\mathrm{v}$ from equation (38). The integral in the other two terms here is discussed in Appendix A. Its evaluation is

$$I_1(\boldsymbol{p})=\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-s^2}}{\left[p_z^2+s^2\boldsymbol{p}_T^2\right]^2}\,ds=\frac{1}{2\,|p_z^3|\,\sqrt{\boldsymbol{p}^2}}=\frac{\mathrm{v}}{2\,|p_z^3|}. \tag{42}$$

By substituting this expression for the middle integral in the previous equation and substituting the value of the last integral as given in equation (35), we find that

$$\begin{aligned}\text{for } \mathcal{L}_+ :\quad W &= \pm F_0\left\{\frac{1}{2B_0p_z}\nabla\cdot\left[p_z\mathrm{v}B_0^2\boldsymbol{p}_T\right]-\frac{p_z}{2B_0\boldsymbol{p}_T^2}\frac{\partial}{\partial z}\left[\frac{\mathrm{v}\boldsymbol{p}_T^4B_0^2}{p_z^2}\right]+\frac{\partial}{\partial z}\left[\frac{B_0}{\mathrm{v}p_z}\right]\right\},\\ \text{for } \mathcal{L}_- :\quad W &= \mp F_0\left\{\frac{1}{2B_0p_z}\nabla\cdot\left[p_z\mathrm{v}B_0^2\boldsymbol{p}_T\right]-\frac{p_z}{2B_0\boldsymbol{p}_T^2}\frac{\partial}{\partial z}\left[\frac{\mathrm{v}\boldsymbol{p}_T^4B_0^2}{p_z^2}\right]+\frac{\partial}{\partial z}\left[\frac{B_0}{\mathrm{v}p_z}\right]\right\}.\end{aligned} \tag{43}$$

Here, we have used the same logic of distinguishing the signs of $|p_z^3|$ as above.

Expanding out the terms here and setting the sum of terms in the braces equal to zero leads to

$$\mathrm{v}\boldsymbol{p}\cdot\nabla B_0+\frac{B_0}{2}\nabla\cdot(\mathrm{v}\boldsymbol{p})+\frac{B_0}{2p_z}\left[\mathrm{v}\boldsymbol{p}\cdot\nabla p_z+\frac{1}{\mathrm{v}^2}\frac{\partial\mathrm{v}}{\partial z}\right]=0. \tag{44}$$

The term in square brackets here is zero. To see this, consider the solution of the eikonal equation (38). The eikonal equation is solved by the method of characteristics, with the characteristic equations determining the rays. Those equations are [Bleistein, et al, 2001]

$$\frac{d\boldsymbol{x}}{ds}=\mathrm{v}\boldsymbol{p},\quad \frac{d\boldsymbol{p}}{ds}=\nabla\left\{\frac{1}{v}\right\}=-\frac{1}{\mathrm{v}^2}\nabla\mathrm{v}. \tag{45}$$

By examining the third component of the second equation here more closely,

$$\frac{\partial p_z}{ds}=\frac{d\boldsymbol{x}}{ds}\cdot\nabla p_z=\mathrm{v}\boldsymbol{p}\cdot\nabla p_z=-\frac{1}{\mathrm{v}^2}\nabla\mathrm{v}. \tag{46}$$

Here, we have used the chain rule and the first line of the previous equation to write the s derivative of p_z in terms of the gradient. Substitution of the last identity here into the term in square brackets in equation (43) confirms that this term is zero as claimed.

The first two terms in equation (44) for $\pm\mathcal{L}_{\pm}W$ combine into a single expression as follows.

$$\nabla \cdot \left[\mathrm{v} B_0^2 \boldsymbol{p}\right] = 0. \tag{47}$$

The expression in square brackets here is slightly different from the ones obtained for the two two-way wave equations. See the previous transport equations, (16) and (17). Thus,

$$A_0 = B_0, \quad \tilde{A}_0 = \sqrt{\mathrm{v}} B_0. \tag{48}$$

That is, the solution W of the one-way wave equation is

$$\mathcal{L}_{\pm}W = \frac{\partial W}{\partial t} \pm \mathrm{v}\sqrt{\nabla^2} W = 0, \tag{49}$$

is asymptotically equal to the solution U

$$U \sim W/\sqrt{\mathrm{v}}, \tag{50}$$

of the two-wave equation (1) and

$$\tilde{U} \sim \sqrt{\mathrm{v}} W \tag{51}$$

provides an asymptotic solution of the wave equation (2).

By using the progressing wave formalism, we have now confirmed that the asymptotic solution W of the one-way wave equations (18) in time are asymptotically equivalent to the two choices U and $\tilde{U}$ of two-way wave equations.

4 An approach to solving initial/boundary value problems for the one-way wave equations in time

The analytical scheme used in the previous section to derive the asymptotic solutions of the one-way wave equation (18) is not a viable solution technique in practice; solving for the auxiliary function W_T in order to find W is not efficient. We propose, instead, to use the pseudo-spectral method of solution as introduced by Kosloff and Baysal [1982]. This approach was suggested by Y. Zhang.

To apply this method, we introduce the forward spatial Fourier transform

$$\mathcal{F}[W(\boldsymbol{x},t)] = \tilde{W}(\boldsymbol{k},t) = \int_{-\infty}^{\infty} W(\boldsymbol{x},t)\exp\{-i\boldsymbol{k}\cdot\boldsymbol{x}\} d^3x, \tag{52}$$

and its inverse,

$$W(\boldsymbol{x},t) = \mathcal{F}^{-1}[\tilde{W}(\boldsymbol{k},t)] = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \tilde{W}(\boldsymbol{k},t)\exp\{i\boldsymbol{k}\cdot\boldsymbol{x}\}. \tag{53}$$

We now rewrite the one-way wave equations (18) as

$$\frac{1}{\mathrm{v}}\frac{\partial W}{\partial t} \pm \mathcal{F}^{-1}[ik\,\mathcal{F}[W(\boldsymbol{x},t)] = 0. \tag{54}$$

That is, we app;y the square root operator indirectly by computing the forward Fourier transform of W, multiplying by ik, and computing the inverse transform.

We are by no means suggesting that this is the only way to solve the one-way differential equation (18), but it is a classical approach to the solution. For a computationally more efficient approach, see Zhang et al [2007].

5 The Green's functions for the one-way wave equations

The Green's function for the two-way wave equation satisfies equation (1) with right side $-\delta(t)\delta(\boldsymbol{x})$ and initial data that $U \equiv 0$ for $t < 0$. In homogeneous media, $\mathrm{v} = \mathit{constant}$, the known solution of this equation in 3D is

$$G(\boldsymbol{x},t) = \frac{\delta(r/\mathrm{v} - t)}{4\pi r}, \tag{55}$$

with r being radial distance from the source point. Note that this Green's function is a one-term progressing wave series.

We propose to derive the Green's function for the one-way forward and reverse time wave equations by considering the following problem.

$$\begin{aligned} \mathcal{L}_\pm G_\pm(\boldsymbol{x},\boldsymbol{x}_0,t,t_0) &= \frac{1}{\mathrm{v}(\boldsymbol{x})}\frac{\partial G_\pm}{\partial t} \pm \sqrt{\nabla^2}G_\pm = \frac{1}{\mathrm{v}(\boldsymbol{x})}\frac{\partial G_\pm}{\partial t} \pm ikG_\pm = 0, \\ G_\pm(\boldsymbol{x},\boldsymbol{x}_0,t_0,t_0) &= \mp \mathrm{v}(\boldsymbol{x}_0)\frac{\delta(\boldsymbol{x}-\boldsymbol{x}_0)}{ik}, \qquad G_\pm \equiv 0, \quad \pm\,(t-t_0) < 0. \end{aligned} \tag{56}$$

Here, the operator $1/ik$ applied to the δ-function should be interpreted in a completely analogous manner as the operator ik in equation (54), above. The explicit calculation is carried out in Appendix C with the final result given by equation (C-4). We also observe from the form of the equation and initial conditions that the Green's functions are shift invariant and depend only on the temporal difference, $t - t_0$ since there is no temporal dependence in the differential equation, itself. Therefore, we always write

$$G_\pm(\boldsymbol{x},\boldsymbol{x}_0,t,t_0) \equiv G_\pm(\boldsymbol{x},\boldsymbol{x}_0,t-t_0). \tag{57}$$

We can simplify this problem by shifting the source point to the origin and considering instead,

$$\mathcal{L}_\pm G_\pm = \frac{1}{\mathrm{v}}\frac{\partial G_\pm}{\partial t} \pm \sqrt{\nabla^2}G_\pm = 0, \quad G_\pm(\boldsymbol{x},\boldsymbol{0},0) = \mp\frac{\mathrm{v}(\boldsymbol{0})}{ik}\delta(\boldsymbol{x}). \tag{58}$$

The initial data $G_\pm(\boldsymbol{x},\boldsymbol{0},0)$ has point support at $\boldsymbol{x} = \boldsymbol{0}$. Thus, for the leading order asymptotic solution, we should expect that the solution for constant v will have the proper "initial weight" to match the solution for variable v. So consider the problem of equation (56) for constant v and proceed to solve by employing Fourier transforms where the dual of the operator $\sqrt{\nabla^2}$ is just ik, as suggested in the previous section.

We define

$$\gamma_{\pm}(\boldsymbol{k}, \omega) = \pm \int_0^{\pm\infty} dt \int_{-\infty}^{\infty} dx\, dy\, dz G_{\pm}(\boldsymbol{x}, \boldsymbol{0}, t) \exp\{-i\boldsymbol{k} \cdot \boldsymbol{x} + i\omega t\}. \tag{59}$$

The choice of lower limit in the temporal transform provides a *causal* solution (upper sign) or *anti-causal* solution (lower sign) in different half-planes of complex frequency. That is, the frequency domain transform is defined initially only for $\pm\Im[\omega]$ being above or below all singularities of $\gamma_{\pm}$ in the compex ω-domain, respectively, for the upper or lower signs. Extension of the functions $\gamma_{\pm}$ beyond these initial domains of definition are achieved by analytic continuation.

To proceed, compute the temporal Fourier transform of the time derivative of $G_{\pm}$ as follows.

$$\begin{aligned} \int_0^{\pm\infty} dt \frac{\partial G_{\pm}(\boldsymbol{x}, \boldsymbol{0}, t)}{\partial t} \exp\{i\omega t\} &= -G_{\pm}(\boldsymbol{x}, \boldsymbol{0}, 0) - i\omega \int_0^{\pm\infty} dt G_{\pm}(\boldsymbol{x}, \boldsymbol{0}, t) \exp\{i\omega t\} \\ &= \pm \frac{\mathrm{v}(\boldsymbol{0})}{ik} \delta(\boldsymbol{x}) - i\omega \int_0^{\pm\infty} dt G_{\pm}(\boldsymbol{x}, \boldsymbol{0}, t) \exp\{i\omega t\}. \end{aligned} \tag{60}$$

With this representation of the transform of the temporal derivative we can complete the Fourier transform of the differential equation (56) for $G_{\pm}$:

$$\pm \left[-i\omega \pm i\mathrm{v}k\right] \gamma_{\pm} = -\frac{\mathrm{v}}{ik}, \tag{61}$$

for which the solution is

$$\gamma_{\pm} = \mp \frac{\mathrm{v}}{k\left[\omega \mp \mathrm{v}k\right]}. \tag{62}$$

We calculate the inverse Fourier transforms of $\gamma_{\pm}$ in Appendix D, stating the result for initial time zero and source location at $\boldsymbol{0}$ in equation (D-6). By restoring the temporal and spatial shifts to $(t_0, \boldsymbol{x}_0)$, we obtain

$$G_{\pm}(\boldsymbol{x}, \boldsymbol{x}_0, t - t_0) \sim \frac{1}{4\pi r} \left\{ \delta(r/\mathrm{v} \mp (t - t_0)) \pm \frac{i}{\pi(r/\mathrm{v} \mp (t - t_0))} \right\}, \quad r = |\boldsymbol{x} - \boldsymbol{x}_0|. \tag{63}$$

Here we have restored the temporal and spatial shifts of the original differential equations (56) for the Green's functions. The real parts of $G_{\pm}$ are the causal/anti-causal Green's functions of the two-way wave equation. The imaginary parts here yield the analytic solutions that we claimed as being typical of the solution of one-way wave equations in time. The complex values arise from the implementation of the operator ik. This operator yields analytic solutions whenever it is applied.

These Green's functions have the structure of the progressing wave $W(\boldsymbol{x}, t)$ of equation (27), modulo the additional shifts, $(\boldsymbol{x}_0, t_0)$. We conclude that for variable wave speed $\mathrm{v}(\boldsymbol{x})$, the asymptotic Green's functions are

$$G_\pm(\boldsymbol{x},\boldsymbol{x}_0,t-t_0) \sim B(\boldsymbol{x},\boldsymbol{x}_0)\left\{\delta\Big(\tau(\boldsymbol{x},\boldsymbol{x}_0)\mp(t-t_0)\Big) \pm \frac{i}{\pi}\cdot\frac{1}{\tau(\boldsymbol{x},\boldsymbol{x}_0)\mp(t-t_0)}\right\},$$
$$\pm(t-t_0)>0, \quad \lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\{|\boldsymbol{x}-\boldsymbol{x}_0|B(\boldsymbol{x},\boldsymbol{x}_0)\} = \frac{1}{4\pi}. \tag{64}$$

6 Adjoint operators and some consequences

The use of the square root of the Laplacian complicates the derivation of an adjoint operator and its consequences. Here, we propose an adjoint and show how it leads to integral representations of upward and downward propagation of wave fields. We use the symbol $\mathcal{L}_\pm^\dagger$ for adjoint and reserve * for complex conjugate. Below, complex conjugate will be applied to progressively larger expressions and the over-bar notation would become intractable for the reader. The operators $\mathcal{L}_\pm W$ were defined in equation (18). Formally, we know that the adjoint of a first order operator such as $\partial/\partial t$ is just its negative, modulo initial and final data. Surprisingly, the adjoint of the operator ikW will prove to be just itself. Boundary data for the two-way wave equation are sources for the one-way wave equation, so that no boundary terms arise from the domain integration of this operator.

We propose, then, that

$$\mathcal{L}_\pm^\dagger W = -\mathcal{L}_\mp W = -\left\{\frac{1}{\mathrm{v}(\boldsymbol{x})}\frac{\partial W}{\partial t} \mp [ikW]\right\}. \tag{65}$$

Here and below, products in which one of the terms is ikW arise. It is important to note to which function in the product the multiplication by ik applies: it requires applying Fourier transform to that function, multiplying the function by ik and applying inverse Fourier transform. Hence, we use the careful notation where that term is placed in square brackets.

Let us now consider

$$\begin{aligned} I &= \int d^3x' \int_{t_-}^{t_+} dt' \left\{U\mathcal{L}_\pm W - W\mathcal{L}_\pm^\dagger U\right\}(\boldsymbol{x}',t') \\ &= I_1 + I_2, \end{aligned} \tag{66}$$

with

$$\begin{aligned} I_1 &= \int \frac{d^3x'}{\mathrm{v}(\boldsymbol{x}')} \int_{t_-}^{t_+} dt' \left\{U\frac{\partial W}{\partial t} + W\frac{\partial U}{\partial t}\right\} \\ I_2 &= \pm\int d^3x' \int_{t_-}^{t_+} dt' \left\{U[ikW] - W[ikU]\right\}. \end{aligned} \tag{67}$$

The integral I_1 is an exact differential in time. Therefore, we simplify this integral by carrying out the integration in t as follows.

$$I_1 = \int \frac{d^3x'}{\mathrm{v}(\boldsymbol{x}')} U(\boldsymbol{x}', t') W(\boldsymbol{x}', t') \Big|_{t_-}^{t_+}. \tag{68}$$

By choosing one of these functions to have initial or final data equal to a Dirac delta function in space, this integral yields the other function evaluated at $(\boldsymbol{x}, t_\pm)$. This is one ingredient in integral representations of solutions in terms of Green's functions.

Now let us consider the integral I_2 in equaiton (67). In particular, let us consider the first term and only its spatial integral.

$$I_3 = \int d^3x' U(\boldsymbol{x}')[ikW(\boldsymbol{x}')]. \tag{69}$$

For our purposes here, the dependence on t' is unimportant and we have dropped it for brevity. The difficulty we face is that the operator ikW is defined in terms of forward and inverse Fourier transforms in equation (54).

Thus,

$$\begin{aligned}
I_3 &= \frac{1}{(2\pi)^3} \int d^3x' U(\boldsymbol{x}') \int ik\, dk^3 \int d^3x'' W(\boldsymbol{x}'') \exp\{i\boldsymbol{k} \cdot (\boldsymbol{x}' - \boldsymbol{x}'')\} \\
&= \frac{1}{(2\pi)^3} \int d^3x' U(\boldsymbol{x}') \int ik\, dk^3 \left\{ \int d^3x'' W^*(\boldsymbol{x}'') \exp\{i\boldsymbol{k} \cdot (\boldsymbol{x}'' - \boldsymbol{x}')\} \right\}^* \\
&= -\frac{1}{(2\pi)^3} \int d^3x' U(\boldsymbol{x}') \left\{ \int ik\, dk^3 \int d^3x'' W^*(\boldsymbol{x}'') \exp\{i\boldsymbol{k} \cdot (\boldsymbol{x}'' - \boldsymbol{x}')\} \right\}^* \\
&= -\frac{1}{(2\pi)^3} \left\{ \int d^3x' U^*(\boldsymbol{x}') \int ik\, dk^3 \int d^3x'' W^*(\boldsymbol{x}'') \exp\{i\boldsymbol{k} \cdot (\boldsymbol{x}'' - \boldsymbol{x}')\} \right\}^* \\
&= -\frac{1}{(2\pi)^3} \left\{ \int d^3x'' W^*(\boldsymbol{x}'')[ikU^*(\boldsymbol{x}'')] \right\}^* \\
&= \frac{1}{(2\pi)^3} \int d^3x'' W(\boldsymbol{x}'')[ikU(\boldsymbol{x}'')]
\end{aligned} \tag{70}$$

In each step here, we have expended the braces around complex conjugation further to the left while correspondingly conjugating the appropriate functions in the interior to leave the value unchanged.

Given what we started out with for I_3 in equation (69) and what we ended up with in equation (70), we conclude that the integral I_2 in equation (67) is equal to zero!

Now, we equate the expression for I_1 in equation (68) and the definition of I in equation (66) to find that

$$\int d^3x' \int_{t_-}^{t_+} dt' \left\{ U\mathcal{L}_\pm W - W\mathcal{L}_\pm^\dagger U \right\} (\boldsymbol{x}', t') = \int \frac{d^3x'}{\mathrm{v}(\boldsymbol{x}')} U(\boldsymbol{x}', t') W(\boldsymbol{x}', t') \Big|_{t_-}^{t_+}. \tag{71}$$

This is the basic identity that we use to downward continue data from the upper surface and to upward continue data from a reflector at depth.

6.1 Symmetries of the Green's functions

There are certain symmetries of the Green's functions that is needed in the discussion below. We present those here.

6.1.1 Symmetry in temporal variables

From the differential equations (56) for $G_\pm(\boldsymbol{x}, \boldsymbol{x}_0, t - t_0)$, it should be clear that, in fact,

$$G_\pm(\boldsymbol{x}, \boldsymbol{x}_0, t, t_0) \equiv G_\pm(\boldsymbol{x}, \boldsymbol{x}_0, t - t_0). \tag{72}$$

Thus, let us introduce

$$\tilde{G}_\pm(\boldsymbol{x}, \boldsymbol{x}_0, t - t_0) = G_\pm(\boldsymbol{x}, \boldsymbol{x}_0, t_0 - t), \tag{73}$$

and observe that

$$\frac{\partial}{\partial t}\tilde{G}_\pm(\boldsymbol{x}, \boldsymbol{x}_0, t - t_0) = -\frac{\partial}{\partial t}G_\pm(\boldsymbol{x}, \boldsymbol{x}_0, t_0 - t). \tag{74}$$

Returning to the defining equation (56) for the Green's functions, we can now see that

$$\begin{aligned} 0 &= \mathcal{L}_\pm G_\pm(\boldsymbol{x}, \boldsymbol{x}_0, t - t_0) = \frac{1}{\mathrm{v}}\frac{\partial G_\pm(\boldsymbol{x}, \boldsymbol{x}_0, t - t_0)}{\partial t} \pm \sqrt{\nabla^2}G_\pm(\boldsymbol{x}, \boldsymbol{x}_0, t - t_0) \\ &= -\frac{1}{\mathrm{v}}\frac{\partial G_\pm(\boldsymbol{x}, \boldsymbol{x}_0, t_0, t)}{\partial t} \pm \sqrt{\nabla^2}G_\pm \\ &= -\left[\frac{1}{\mathrm{v}}\frac{\partial \tilde{G}_\pm(\boldsymbol{x}, \boldsymbol{x}_0, t - t_0)}{\partial t} \mp \sqrt{\nabla^2}G_\pm(\boldsymbol{x}, \boldsymbol{x}_0, t - t_0)\right] = -\mathcal{L}_\mp \tilde{G}_\pm(\boldsymbol{x}, \boldsymbol{x}_0, t - t_0). \end{aligned} \tag{75}$$

Thus, $\tilde{G}_\pm(\boldsymbol{x}, \boldsymbol{x}_0, t - t_0)$ satisfy the differential equations (56) of $G_\mp((\boldsymbol{x}, \boldsymbol{x}_0, t - t_0)$ with the data for the former at t_0 of opposite sign to the data of the latter at t_0. We conclude then that

$$\tilde{G}_\pm(\boldsymbol{x}, \boldsymbol{x}_0, t - t_0) = G_\pm(\boldsymbol{x}, \boldsymbol{x}_0, t_0, t) = -G_\mp(\boldsymbol{x}, \boldsymbol{x}_0, t, t_0). \tag{76}$$

This symmetry is used below.

6.1.2 Symmetry in spatial variables

Let us suppose that U and W satisfy the following equation.

$$\begin{aligned} \mathcal{L}_\pm(\boldsymbol{x}', \boldsymbol{x})U_\pm(\boldsymbol{x}', \boldsymbol{x}, t', t_-) &= 0, \quad U_\pm(\boldsymbol{x}', \boldsymbol{x}, t_-, t_-) = \pm\mathrm{v}(\boldsymbol{x})\delta(\boldsymbol{x}' - \boldsymbol{x}), \\ \mathcal{L}^\dagger_\pm(\boldsymbol{x}', \boldsymbol{x})W_\pm(\boldsymbol{x}', \boldsymbol{y}, t', t_+) &= -\mathcal{L}_\mp(\boldsymbol{x}', \boldsymbol{x})W_\pm(\boldsymbol{x}', \boldsymbol{y}, t', t_+) = 0, \\ & \qquad W_\pm(\boldsymbol{x}', \boldsymbol{y}, t_+, t_+) = \pm\mathrm{v}(\boldsymbol{y})\delta(\boldsymbol{x}' - \boldsymbol{y}) \end{aligned} \tag{77}$$

Apply our Green's identity, equation (71) to these two functions to obtain

$$0 = \int \frac{d^3x'}{\mathrm{v}(\boldsymbol{x}')}\left[U_\pm(\boldsymbol{x}', \boldsymbol{x}, t_+, t_-)\mathrm{v}(\boldsymbol{y})\delta(\boldsymbol{x}' - \boldsymbol{y}) - W_\pm(\boldsymbol{x}', \boldsymbol{y}, t_-, t_+)\mathrm{v}(\boldsymbol{x})\delta(\boldsymbol{x}' - \boldsymbol{x})\right]. \tag{78}$$

Now use the δ-functions to evaluate the integral and conclude that

$$U_\pm(\boldsymbol{y},\boldsymbol{x},t_+,t_-) = W_\pm(\boldsymbol{x},\boldsymbol{y},t_-,t_+). \tag{79}$$

By comparing the problems for $U_\pm$ and $W_\pm$ here with the differential equations (56), we conclude that

$$U_\pm(\boldsymbol{y},\boldsymbol{x},t_+,t_-) = -ikG_\pm(\boldsymbol{y},\boldsymbol{x},t_+-t_-), \quad W_\pm(\boldsymbol{x},\boldsymbol{y},t_-,t_+) = ikG_\mp(\boldsymbol{x},\boldsymbol{y},t_--t_+). \tag{80}$$

Now we apply the temporal symmetry of equation (76) to conclude that

$$ikG_\pm(\boldsymbol{y},\boldsymbol{x},t_+-t_-) = ikG_\pm(\boldsymbol{x},\boldsymbol{y},t_+-t_-). \tag{81}$$

We see here that the spatial symmetry applies to $ikG_\pm$ and not to the Green's functions themselves.

However, let us now use the differential equations (56) for $G_\pm$ to rewrite this last symmetry as follows

$$\frac{1}{\mathrm{v}(\boldsymbol{y})}\frac{\partial G_\pm(\boldsymbol{y},\boldsymbol{x},t-t_-)}{\partial t} = \frac{1}{\mathrm{v}(\boldsymbol{x})}\frac{\partial G_\pm(\boldsymbol{x},\boldsymbol{y},t-t_-)}{\partial t}. \tag{82}$$

We have now generated a list of symmetries among the Green's functions that will allow us to proceed below to derive modeling formulas for downward continuation of data, upward propagation of reflection data and a pseudo inverse of the modeling operator for observed data at the upper surface. That pseudo-inverse will provide a true-amplitude migration of the observed data for a reflectivity function—a reflector map and an estimate of a specular reflection coefficient at each point of the reflector.

6.1.3 Complex conjugates

Let us first rewrite the differential equation and problem for $G_\pm$ in equation (58) as follows.

$$\frac{1}{\mathrm{v}}\frac{\partial G_\pm}{\partial t} \pm ikG_\pm = 0, \quad G_\pm(\boldsymbol{x},\boldsymbol{0},0) = \mp\frac{\mathrm{v}(\boldsymbol{0})}{ik}\delta(\boldsymbol{x}). \tag{83}$$

Next, we take complex conjugates in this equation:

$$\frac{1}{\mathrm{v}}\frac{\partial G^*_\pm}{\partial t} \mp ikG^*_\pm = 0, \quad G^*_\pm(\boldsymbol{x},\boldsymbol{0},0) = \pm\frac{\mathrm{v}(\boldsymbol{0})}{ik}\delta(\boldsymbol{x}). \tag{84}$$

By interchanging signs here,

$$\frac{1}{\mathrm{v}}\frac{\partial G^*_\mp}{\partial t} \pm ikG^*_\mp = 0, \quad G^*_\mp(\boldsymbol{x},\boldsymbol{0},0) = \mp\frac{\mathrm{v}(\boldsymbol{0})}{ik}\delta(\boldsymbol{x}). \tag{85}$$

Thus, we see that G^*_+ is the same solution of the anti-causal problem here as is G_- and, similarly, G^*_- is the same solution of the causal problem as is G_+. That is,

$$G^*_\pm(\boldsymbol{x},\boldsymbol{x}_0,t) = G_\mp(\boldsymbol{x},\boldsymbol{x}_0,t), \quad \mp t > 0. \tag{86}$$

6.2 Downward continuation

Suppose now that we are given observed data for U at $z = 0$. For the one-way wave equation, these data propagated forward in time to get to $z = 0$. Thus we want to propagate the data *backwards* in time to observe where it was located in depth at earlier times.

$$\begin{aligned} \mathcal{L}_-(\boldsymbol{x}', t')W(\boldsymbol{x}', t') &= \delta(z')D(x', y', t'), \quad W(\boldsymbol{x}', \infty) = 0, \\ \mathcal{L}_-^\dagger(\boldsymbol{x}', t')U(\boldsymbol{x}', \boldsymbol{x}, t', t) &= -\mathcal{L}_+U(\boldsymbol{x}', \boldsymbol{x}, t', t) = 0, \quad U(\boldsymbol{x}', \boldsymbol{x}, t, t) = \mathrm{v}(\boldsymbol{x})\delta(\boldsymbol{x}' - \boldsymbol{x}). \end{aligned} \tag{87}$$

We see here W is the response to an "exploding reflector" at $z' = 0$ and

$$U(\boldsymbol{x}', \boldsymbol{x}, t', t) = ikG_+(\boldsymbol{x}', \boldsymbol{x}, t' - t) = -\frac{1}{\mathrm{v}(\boldsymbol{x}')}\frac{\partial G_+(\boldsymbol{x}', \boldsymbol{x}, t' - t)}{\partial t'}, \tag{88}$$

with G_- being the Green's function as defined in equation (56). Furthermore, the propagation is backwards in time because $t' > t$. By using this information in the Green's identity of equation (71), we find that

$$\begin{aligned} W(\boldsymbol{x}, t) &= \int\limits_{z'=0} dx'dy' \int\limits_t^\infty dt' D(x', y', t')\frac{1}{\mathrm{v}(\boldsymbol{x}')}\frac{\partial G_+(\boldsymbol{x}', \boldsymbol{x}, t' - t)}{\partial t'} \\ &= -\int\limits_{z'=0} dx'dy' \int\limits_t^\infty dt' D(x', y', t')\frac{1}{\mathrm{v}(\boldsymbol{x}')}\frac{\partial G_-(\boldsymbol{x}', \boldsymbol{x}, t - t')}{\partial t'}. \end{aligned} \tag{89}$$

In the second line here, we used the temporal symmetry of the Green's functions in equation (76). The first line emphasizes that the solution at time t depends on the data for all times greater than t. The second form reminds us that this is really a back propagation in time from times t' to earlier time t.

Although we do not normally think of the downward propagation of observed data as conforming to the exploding reflector model, it really does. Here, it becomes explicit in that the problem for W in equation (87) uses the observed data as source at $z' = 0$.

6.3 Upward propagation of reflection data

Now, suppose that we have reflection data on a surface S, such as we derived in Section 7. The surface would be described by a function of two parameters, $\boldsymbol{x} = \boldsymbol{x}_0(\sigma_1, \sigma_2)$.

We introduce $\gamma(\boldsymbol{x})$, the *singular function of the surface*. This is a Dirac delta function of signed normal distance from the surface at each point. As with any other distribution, we define this one by its "action" on nice functions:

$$\int_{\mathcal{D}} dV f(\boldsymbol{x})\gamma(\boldsymbol{x}) = \int_S dS f(\boldsymbol{x}), \tag{90}$$

as long as the domain $\mathcal{D}$ includes the surface S.

Let us now set up the appropriate two problems for the Green's identity in equation (71).

$$\begin{aligned} \mathcal{L}_+(\boldsymbol{x}', t')W(\boldsymbol{x}', t') &= \gamma(\boldsymbol{x}')D(\boldsymbol{x}', t'), \quad W(\boldsymbol{x}', 0) = 0, \\ \mathcal{L}_+^\dagger(\boldsymbol{x}', t')U(\boldsymbol{x}', \boldsymbol{x}, t', t) &= -\mathcal{L}_- U(\boldsymbol{x}', \boldsymbol{x}, t', t) = 0, \quad U(\boldsymbol{x}', \boldsymbol{x}, t, t) = \mathrm{v}(\boldsymbol{x})\delta(\boldsymbol{x}' - \boldsymbol{x}). \end{aligned} \tag{91}$$

As above,

$$U(\boldsymbol{x}', \boldsymbol{x}, t', t) = -ikG_-(\boldsymbol{x}', \boldsymbol{x}, t' - t) = -\frac{1}{\mathrm{v}(\boldsymbol{x}')}\frac{\partial G_-(\boldsymbol{x}', \boldsymbol{x}, t' - t)}{\partial t'}. \tag{92}$$

In analogy with the previous section, we apply the Green's identity of equation (71), to conclude that

$$\begin{aligned} W(\boldsymbol{x}, t) &= -\int_S dS' \int_0^t dt' D(x', y', t') \frac{1}{\mathrm{v}(\boldsymbol{x}')}\frac{\partial G_-(\boldsymbol{x}', \boldsymbol{x}, t' - t)}{\partial t'} \\ &= \int_S dS' \int_0^t dt' D(x', y', t') \frac{1}{\mathrm{v}(\boldsymbol{x}')}\frac{\partial G_+(\boldsymbol{x}', \boldsymbol{x}, t - t')}{\partial t'}. \end{aligned} \tag{93}$$

In the last line here, we used the temporal symmetry of the Green's functions in equation

The solution propagates forward in time from the data values on the surface S. As in the previous example, we use the differential equation (56) to replace ikG_+ by the time derivative of G_+ divided by v in the last line.

In summary, we have derived a Green's identity in equation (71). We used it to derive propagator integrals of Kirchhoff-type to downward continue surface data backwards in time and downwards into $z > 0$ and then to upward continue reflection data prescribed on a surface at depth.

7 Reflection and the Kirchhoff approximation

When an acoustic wave is incident on an interface, it gives rise to reflected and transmitted waves. For elastic waves, there are additional mode-converted waves initiated at an interface. Here, we confirm that the standard procedure used in asymptotic ray theory for acoustic waves does not change when we do the same analysis in the time domain. Implicit in this example is the fact that the same will be true for elastic waves with their initiation of mode-converted waves.

For the two-way wave equation, the derived data for the reflected and transmitted waves provide input to a Green's function integral representation of the propagation of the reflected and transmitted waves away from the interface, by providing leading order approximations of these waves themselves and their normal derivatives, as well. Theory tells us that this is an *over-determination* of prescribed data, but the leading order wave field produced by this method is sufficiently accurate to be useful in modeling and inversion.

The original Kirchhoff approximation was introduced for the specific boundary conditions $u = 0$ or $\partial u/\partial n = 0$ at the interface. In Bleistein [1984], we simply used these asymptotic ray theory data to extend the Kirchhoff approximation to interfaces. We have not been able to find an earlier reference to this extension although we believe it was in the "folklore" of users of the Kirchhoff approximation well before that.

Let us suppose that there is an interface S defined by

$$S: \quad \boldsymbol{x} = \boldsymbol{x}_0(\boldsymbol{\sigma}), \quad \boldsymbol{\sigma} = (\sigma_1, \sigma_2) \tag{94}$$

and that a wave,

$$U_I(\boldsymbol{x}, t) = B_I F_I(\tau_I(\boldsymbol{x}) \mp t), \tag{95}$$

is incident on S. This wave gives rise to reflected and transmitted waves near S, U_R and U_T, respectively:

$$U_R(\boldsymbol{x}, t) = B_R F_R(\tau_R(\boldsymbol{x}) \mp t), \quad U_T(\boldsymbol{x}, t) = B_T F_T(\tau_T(\boldsymbol{x}) \mp t). \tag{96}$$

The total solution on the two sides of S are

$$U = U_I + U_R, \quad \text{and} \quad U = U_T. \tag{97}$$

Our objective is to derive initial data on S for the propagation of these waves away from S. Standardly, we impose two continuity conditions on S that the total solutions and their normal derivatives be continuous on S; that is

$$U_I + U_R = U_T, \quad \frac{\partial U_I}{\partial n} + \frac{\partial U_R}{\partial n} = \frac{\partial U_T}{\partial n}, \quad \text{on } S. \tag{98}$$

We have no hope of satisfying these equations unless we require that

$$F_I(\tau_I(\boldsymbol{x}_0) \mp t) = F_R(\tau_R(\boldsymbol{x}_0) \mp t) = F_T(\tau_T(\boldsymbol{x}_0) \mp t), \qquad \text{on } S, \tag{99}$$

and further that

$$\tau_I(\boldsymbol{x}_0) = \tau_R(\boldsymbol{x}_0) = \tau_T(\boldsymbol{x}_0) \qquad \text{on } S. \tag{100}$$

This is exactly equivalent to matching of the phases of the complex wave forms in the frequency domain. Here, it is necessary because of the localization of the progressing waves in space at a given time.

This last equation provides initial data for the travel times τ_R and τ_T on S. Furthermore, we can differentiate these last equations with respect to σ_1 and σ_2. Those equations tell us that the tangential component of the travel time gradients must also be equal; that is Snell's law:

$$\nabla\tau_I \cdot \frac{\partial \boldsymbol{x}_0(\boldsymbol{\sigma})}{\partial \sigma_i} = \nabla\tau_R \cdot \frac{\partial \boldsymbol{x}_0(\boldsymbol{\sigma})}{\partial \sigma_i} = \nabla\tau_T \cdot \frac{\partial \boldsymbol{x}_0(\boldsymbol{\sigma})}{\partial \sigma_i}, \quad i = 1, 2. \tag{101}$$

More succinctly, we write this equation as

$$\nabla_{\text{tang}}\tau_I = \nabla_{\text{tang}}\tau_R = \nabla_{\text{tang}}\tau_T, \tag{102}$$

with the subscript "tang" denoting the tangential part of the gradient. We find the normal component of the gradient by using the eikonal equation, (37). However, we have to distinguish v on the two sides of the surface S; denote them by v_- as the wave speed on the same side of S as the incident wave and v_+ on the transmitted side. Then we conclude that

$$\begin{aligned}\frac{\partial\tau_{_R}}{\partial n} &= -\mathrm{sgn}\left[\frac{\partial\tau_{_I}}{\partial n}\right]\sqrt{\frac{1}{\mathrm{v}_-^2}-[\nabla_{\mathrm{tang}}\tau_{_I}]^2},\\ \frac{\partial\tau_{_T}}{\partial n} &= \mathrm{sgn}\left[\frac{\partial\tau_{_I}}{\partial n}\right]\sqrt{\frac{1}{\mathrm{v}_+^2}-[\nabla_{\mathrm{tang}}\tau_{_I}]^2}, \quad \text{on } S.\end{aligned} \tag{103}$$

Now we return to the continuity conditions in equation (98) to determine how $B_{_R}$ and $B_{_T}$ are related to $B_{_I}$. Using the representations of the wave functions themselves in equations (95) and (96), we can now strip away the progressing waves in the first equation and its derivative with respect to argument in the second equation, leaving the normal derivatives of the travel times in the resulting equation. This leads to the standard equations,

$$B_{_I}+B_{_R}=B_{_T}, \quad \frac{\partial\tau_{_I}}{\partial n}\left[B_{_I}-B_{_R}\right]=\frac{\partial\tau_{_T}}{\partial n}B_{_T}, \tag{104}$$

and solutions,

$$\begin{gathered} B_{_R}=RB_{_I}, \qquad B_{_T}=TB_{_I}, \quad \text{on } S,\\ R=\frac{\dfrac{\partial\tau_{_I}}{\partial n}-\dfrac{\partial\tau_{_T}}{\partial n}}{\dfrac{\partial\tau_{_I}}{\partial n}+\dfrac{\partial\tau_{_T}}{\partial n}}, \qquad T=\frac{2\dfrac{\partial\tau_{_T}}{\partial n}}{\dfrac{\partial\tau_{_I}}{\partial n}+\dfrac{\partial\tau_{_T}}{\partial n}}.\end{gathered} \tag{105}$$

These are the standard reflection and transmission coefficients derived in asymptotic ray theory in the frequency domain.

In summary, of the reflected and transmitted waves on the surface S are given by

$$U_{_R}(\boldsymbol{x},t)=RB_{_I}(\boldsymbol{x})F_{_I}(\tau_{_I}(\boldsymbol{x})\mp t), \quad U_{_T}(\boldsymbol{x},t)=TB_{_I}(\boldsymbol{x})F_{_I}(\tau_{_I}(\boldsymbol{x})\mp t), \quad \boldsymbol{x} \text{ on } S, \tag{106}$$

with the reflection coefficient R and the transmission coefficient T defined in equation (105) above.

8 A pseudo-inverse of modeling for common-shot Inversion

In this section, we consider the modeling identity for reflected data in equation (93). We assume that we have a common-shot data set at $\boldsymbol{x}\equiv\boldsymbol{x}_r=(x_r,y_r,0)$. In that wave field representation, we use the Kirchhoff approximation for $U_{_R}$ in equation (106) with $B_{_I}F_{_I}$ being the Green's function $G_+(\boldsymbol{x},\boldsymbol{x}_s,t)$. In this manner, we arrive at

$$\begin{gathered} W(\boldsymbol{x}_r,t)=\int d^3x'\int_0^t dt'\frac{\mathcal{R}_0(\boldsymbol{x}',\boldsymbol{x}_s)}{\mathrm{v}(\boldsymbol{x}')}G_+(\boldsymbol{x}',\boldsymbol{x}_s,t')\frac{\partial G_+(\boldsymbol{x}',\boldsymbol{x}_r,t-t')}{\partial t'},\\ \mathcal{R}_0(\boldsymbol{x}',\boldsymbol{x}_s)=R(\boldsymbol{x}',\boldsymbol{x}_s)\gamma(\boldsymbol{x}')\end{gathered} \tag{107}$$

Here, $\mathcal{R}_0$ is the reflectivity function that we seek in the inverse problem and $R(\boldsymbol{x}', \boldsymbol{x}_s)$ is the ray-theoretic reflection coefficient R as given in equation (105). The function $\mathcal{R}_0$ corresponds to the reflectivity β of Bleistein et al [2001]. This notation is consistent with the notation in Bleistein et al [2005]. When we divide the integrand by $|\nabla[\tau(\boldsymbol{x}, \boldsymbol{x}_s) + \tau(\boldsymbol{x}, \boldsymbol{x}_r)]|$ we obtain the reflectivity β_1 of Bleistein et al [2001], denoted by $\mathcal{R}$ in Bleistein et al [2005].

We apply the temporal Fourier transform to both sides of this equation, using lower case letters in the frequency domain to correspond to the capital letters for the functions here in the time domain. We provide explanations of each line of this multi-line equation directly below.

$$\begin{aligned}
w(\boldsymbol{x}_r, \omega) &= \int_0^\infty dt \int_0^t dt' \int d^3x' \frac{\mathcal{R}_0(\boldsymbol{x}', \boldsymbol{x}_s)}{\mathrm{v}(\boldsymbol{x}')} G_+(\boldsymbol{x}', \boldsymbol{x}_s, t') \frac{\partial G_+(\boldsymbol{x}', \boldsymbol{x}_r, t-t')}{\partial t'} \exp\{i\omega t\} \\
&= -\int_0^\infty \exp\{i\omega t'\} dt' \int_{t'}^\infty \exp\{i\omega(t-t')\} dt \int d^3x' \\
&\qquad \cdot \frac{\mathcal{R}_0(\boldsymbol{x}', \boldsymbol{x}_s)}{\mathrm{v}(\boldsymbol{x}')} G_+(\boldsymbol{x}', \boldsymbol{x}_s, t') \frac{\partial G_+(\boldsymbol{x}', \boldsymbol{x}_r, t-t')}{\partial t}, \\
&= -\int_0^\infty \exp\{i\omega t'\} dt' \int_0^\infty \exp\{i\omega\sigma)\} d\sigma \int d^3x' \\
&\qquad \cdot \frac{\mathcal{R}_0(\boldsymbol{x}', \boldsymbol{x}_s)}{\mathrm{v}(\boldsymbol{x}')} G_+(\boldsymbol{x}', \boldsymbol{x}_s, t') \frac{\partial G_+(\boldsymbol{x}', \boldsymbol{x}_r, \sigma)}{\partial \sigma}, \\
&= \int d^3x' \frac{\mathcal{R}_0(\boldsymbol{x}', \boldsymbol{x}_s)}{\mathrm{v}(\boldsymbol{x}')} i\omega g_+(\boldsymbol{x}', \boldsymbol{x}_s, \omega) g_+(\boldsymbol{x}', \boldsymbol{x}_r, \omega)].
\end{aligned} \tag{108}$$

Here, the right sides are obtained as follows.

1. The first equality just introduces the definition of the Fourier transform on the right hand side.

2. In the second equality, the orders of integration in t and t' are interchanged, a differentiation with respect to t' has been rewritten as a differentiation with respect to t and the single exponential has been rewritten as a product of two exponentials.

3. In the third equality, the origin in the t-integral has been shifted from t' to zero with the change of variable of integration, $t - t' = \sigma$.

4. In the fourth equality, we recognize the temporal integrals as a product of the Fourier transforms of the two Green's functions of the previous line with the differentiation with respect to σ leading to a multiplier of $-i\omega$.

The final result here has the form of a linear integral operator on the reflectivity $\mathcal{R}_0$; that is,

$$w(\boldsymbol{x}_r, \omega) = K[\mathcal{R}_0(\boldsymbol{x}')]. \tag{109}$$

If we knew the pseudo-inverse of this operator, then we could obtain an approximate inverse in the form

$$\mathcal{R}_0(\boldsymbol{x}) = K^{-1}[w(\boldsymbol{x}_r, \omega)]. \tag{110}$$

Formally,

$$K^{-1} = \frac{K^\dagger}{||K^\dagger K||}. \tag{111}$$

Here, the adjoint $K^\dagger$ is an integral over the variables $(\boldsymbol{x}_r, \omega)$ and the norm $||K^\dagger K||$ is an integral over $(\boldsymbol{x}_r, \omega)$ and the three elements of $\boldsymbol{x'}$. Thus, our next task is to identify $K^\dagger$.

We find the operator $K^\dagger$ by considering the inner product between two functions, one $w(\boldsymbol{x}_r, \omega)$ and the other $u(\boldsymbol{x'})$ and then seek the operator for which,

$$\Big\langle w, K[u] \Big\rangle = \Big\langle K^\dagger[w], u \Big\rangle, \tag{112}$$

with

$$\Big\langle w, K[u] \Big\rangle = \int d^2x_r d\omega \int d^3x' w^*(\boldsymbol{x}_r, \omega) u(\boldsymbol{x'}) g_+(\boldsymbol{x'}, \boldsymbol{x}_s, \omega) \frac{i\omega}{\mathrm{v}(\boldsymbol{x'})} g_+(\boldsymbol{x'}, \boldsymbol{x}_r, \omega)]. \tag{113}$$

It is fairly straightforward to recast this integral with the kernel of the operator attached to the function w as follows

$$\begin{aligned} \Big\langle w, K[u] \Big\rangle &= -\int d^2x_r d\omega \int d^3x' \Big[w(\boldsymbol{x}_r, \omega) g_+^*(\boldsymbol{x'}, \boldsymbol{x}_s, \omega) \frac{i\omega}{\mathrm{v}(\boldsymbol{x'})} g_+^*(\boldsymbol{x'}, \boldsymbol{x}_r, \omega) \Big]^* u(\boldsymbol{x'}) \\ &= \Big\langle K^\dagger[w], u \Big\rangle, \end{aligned} \tag{114}$$

with

$$\begin{aligned} K^\dagger[w](\boldsymbol{x}) &= -\int d^2x_r \frac{i\omega d\omega}{\mathrm{v}(\boldsymbol{x})} w(\boldsymbol{x}_r, \omega) g_+^*(\boldsymbol{x}, \boldsymbol{x}_s, \omega) g_+^*(\boldsymbol{x}, \boldsymbol{x}_r, \omega) \\ &= -\int d^2x_r \frac{i\omega d\omega}{\mathrm{v}(\boldsymbol{x})} w(\boldsymbol{x}_r, \omega) g_-(\boldsymbol{x}, \boldsymbol{x}_s, \omega) g_-(\boldsymbol{x}, \boldsymbol{x}_r, \omega). \end{aligned} \tag{115}$$

Here, the second equality is asymptotically true and derived in Appendix E. In particular, see equation (E-8).

8.1 Migration operator in time deduced from $K^\dagger[w]$ in the frequency domain.

We remark that if $w(\boldsymbol{x}, \omega)$ is the observed data on the upper surface, then this last integral is a correlation-type migration. This formula can be expressed in the time domain, as well. To do that, we need a notation for the temporal Fourier transform. We have already used $\mathcal{F}$ to denote the spatial Fourier transform, so let us use $\boldsymbol{\mathcal{F}}$ to denote the temporal Fourier transform of an anti-causal function. That is,

$$\boldsymbol{\mathcal{F}}[U(\ldots, t)] \;=\; u(\ldots, \omega)$$

$$
\begin{aligned}
&= \int_{-\infty}^{0} U(\ldots,t)\exp\{i\omega t\}\,dt, \\
\boldsymbol{\mathcal{F}}^{-1}[u(\ldots,\omega)] &= U(\ldots,t) = \frac{1}{2\pi}\int_{\Gamma} u(\ldots,\omega)\exp\{-i\omega t\}\,d\omega.
\end{aligned}
\tag{116}
$$

Here, Γ is a line parallel to the $\Re[\omega]$-axis, below all singularities of the function $u(\ldots,\omega)$.

We begin our analysis here with the last line of equation (115) for $K^{\dagger}[w](\boldsymbol{x})$ as a frequency domain integral and being introducing the definitions of various functions in terms of their temporal transforms as defined in equation (116). We provide explanations of each line of this multi-line equation directly below.

$$
\begin{aligned}
K^{\dagger}[W](\boldsymbol{x},t) &= -\int d^2x_r i\omega d\omega \frac{w(\boldsymbol{x}_r,\omega)}{\mathrm{v}(\boldsymbol{x})}\int_{-\infty}^{0} dt G_{-}(\boldsymbol{x},\boldsymbol{x}_s,t)\exp\{i\omega t\} g_{-}(\boldsymbol{x},\boldsymbol{x}_r,\omega) \\
&= \int d^2x_r d\omega \frac{w(\boldsymbol{x}_r,\omega)}{\mathrm{v}(\boldsymbol{x})}\int_{-\infty}^{0} dt G_{-}(\boldsymbol{x},\boldsymbol{x}_s,t)\exp\{i\omega t\}\int_{-\infty}^{0} dt' \frac{\partial G_{-}(\boldsymbol{x},\boldsymbol{x}_r,t')\exp\{i\omega t'\}}{\partial t'} \\
&= 2\pi\int \frac{d^2x_r}{\mathrm{v}(\boldsymbol{x})}\int_{-\infty}^{0} dt G_{-}(\boldsymbol{x},\boldsymbol{x}_s,t)\int_{-\infty}^{0} dt' \frac{\partial G_{-}(\boldsymbol{x},\boldsymbol{x}_r,t')}{\partial t'} W(\boldsymbol{x}_r,-(t+t')) \\
&= 2\pi\int \frac{d^2x_r}{\mathrm{v}(\boldsymbol{x})}\int_{-\infty}^{0} dt G_{-}(\boldsymbol{x},\boldsymbol{x}_s,t)\int_{-t}^{\infty} du \frac{\partial G_{-}(\boldsymbol{x},\boldsymbol{x}_r,-(u+t))}{\partial u} W(\boldsymbol{x}_r,u) \\
&= 2\pi\int \frac{d^2x_r}{\mathrm{v}(\boldsymbol{x})}\int_{0}^{\infty} dt G_{-}(\boldsymbol{x},\boldsymbol{x}_s,-t)\int_{t}^{\infty} \frac{du}{\mathrm{v}(\boldsymbol{x}_r)} \frac{\partial G_{-}(\boldsymbol{x},\boldsymbol{x}_r,t-u)}{\partial u}\mathrm{v}(\boldsymbol{x}_r) W(\boldsymbol{x}_r,u) \\
&= 2\pi\int_{0}^{\infty} dt G_{-}(\boldsymbol{x},\boldsymbol{x}_s,-t) W(\boldsymbol{x},t)
\end{aligned}
\tag{117}
$$

The substitutions in each line here proceed as follow.

1. In line one, $g_{-}(\boldsymbol{x},\boldsymbol{x}_s,\omega)$ has been replaced by its definition in terms of $G_{-}(\boldsymbol{x},\boldsymbol{x}_s,t)$
2. In line two, the same was done with $g_{-}(\boldsymbol{x},\boldsymbol{x}_r,\omega)$, except that we also absorbed the factor of $i\omega$ into this transform by introducing a $-\partial/\partial t'$.
3. In line three the inverse transform of w was introduced.
4. In line four, the change of variable of integration from t' to u was introduced, with $t+t'=-u$.
5. In line five, the variable t was replaced by the variable $-t$ with appropriate changes in the limits of integration.
6. In line six, we identified the downward propagation of the data $\mathrm{v}(\boldsymbol{x}_r)W(\boldsymbol{x}_r,u)$ to $\mathrm{v}(\boldsymbol{x})W(\boldsymbol{x},t)$ by using the derived formula in equation (89). In this form, we see the

migration formed by a correlation of the downward propagated data with the backward propagate point source.

8.2 Asymptotic analysis of $||K^\dagger K||$

To transform our migrations into inversions, we need to calculate $||K^\dagger K||$. Actually, "true-amplitude" processing only makes sense when an image is formed by a single arrival at the image point. Furthermore, the interpretation of the output amplitude at the image point in terms of reflectivity is based on leading order asymptotic theory—asymptotic ray theory—for single arrivals. Hence, for normalization purposes, we may use leading order asymptotic analysis to estimate the norm $||K^\dagger K||$ that we seek.

The operator K is implicit in the representation of $w(\boldsymbol{x}_r,\omega)$ in equation (108). Similarly, the operator $K^\dagger$ is implicit in the representation of $K^\dagger[w]$ in equation (115). The cascade of these two operators is

$$||K^\dagger K|| = \int d^2x_r d\omega \frac{\omega}{\mathrm{v}(\boldsymbol{x})} g_-(\boldsymbol{x},\boldsymbol{x}_s,\omega) g_-(\boldsymbol{x},\boldsymbol{x}_r,\omega) \int d^3x' \frac{\omega}{\mathrm{v}(\boldsymbol{x}')} g_+(\boldsymbol{x}',\boldsymbol{x}_s,\omega) g_+(\boldsymbol{x}',\boldsymbol{x}_r,\omega). \tag{118}$$

As noted above, we can use leading order asymptotic approximations here for the Green's functions. Those leading order approximations are given in equation (E-8). When we use those values here with appropriate substitution of arguments, we obtain the following integral representation for $||K^\dagger K||$.

$$\begin{aligned} ||K^\dagger K|| &= 16 \int \frac{\omega^2 d^2x_r d\omega d^3x'}{\mathrm{v}(\boldsymbol{x})\mathrm{v}(\boldsymbol{x}')} \\ &\qquad \cdot B(\boldsymbol{x},\boldsymbol{x}_s) B(\boldsymbol{x},\boldsymbol{x}_r) B(\boldsymbol{x}',\boldsymbol{x}_s) B(\boldsymbol{x}',\boldsymbol{x}_r) \exp\{i\omega\Phi(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{x}_s,\boldsymbol{x}_r)\}, \qquad (119) \\ \Phi(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{x}_s,\boldsymbol{x}_r) &= \tau(\boldsymbol{x}',\boldsymbol{x}_s) + \tau(\boldsymbol{x}',\boldsymbol{x}_r) - [\tau(\boldsymbol{x},\boldsymbol{x}_s) + \tau(\boldsymbol{x},\boldsymbol{x}_r)]. \end{aligned}$$

We derive the asymptotic expansion of this integral in Appendix F. The result given in equation (F-18) is

$$\|(K^\dagger K)^{-1}\| \sim \frac{\cos\beta_r \cos^2\theta}{4\pi \mathrm{v}(\boldsymbol{x}_r) B^2(\boldsymbol{x},\boldsymbol{x}_s)} \tag{120}$$

8.2.1 Common shot inversion

Note that the inversions and migrations that we have written above are integrals over $\boldsymbol{x}_r$ and ω or t. It is easy now to introduce $\|(K^\dagger K)^{-1}\|$ from equation (120) into the migration in frequency domain in equation (115) or migration in the time domain in equation (117) in order to turn either of those formulas into an inversion. We claim then that the following formulas provide inversions of frequency and time domain data.

$$\begin{aligned} \mathcal{R}_0(\boldsymbol{x},\boldsymbol{x}_s) &= \frac{1}{4\pi \mathrm{v}(\boldsymbol{x})|B_0(\boldsymbol{x},\boldsymbol{x}_s)|^2} \int d^2x_r \frac{\cos\beta_r \cos^2\theta}{\mathrm{v}(\boldsymbol{x}_r)} \\ &\qquad \cdot \int i\omega\, d\omega w(\boldsymbol{x}_r,\omega) g_-(\boldsymbol{x},\boldsymbol{x}_s,\omega) g_-(\boldsymbol{x},\boldsymbol{x}_r,\omega), \end{aligned} \tag{121}$$

and

$$\mathcal{R}_0(\boldsymbol{x},\boldsymbol{x}_s) = \frac{1}{2\mathrm{v}(\boldsymbol{x})|B_0(\boldsymbol{x},\boldsymbol{x}_s)|^2}\int d^2x_r \frac{\cos\beta_r\cos^2\theta}{\mathrm{v}(\boldsymbol{x}_r)} \cdot \int_0^\infty dt G_-(\boldsymbol{x},\boldsymbol{x}_s,-t)\int_t^\infty du \frac{\partial G_-(\boldsymbol{x},\boldsymbol{x}_r,t-u)}{\partial u} W(\boldsymbol{x}_r,u). \tag{122}$$

Note that for this time domain formula we did not use the last representation of equation (117) because the factor $\cos^2\theta$ is a function of $\boldsymbol{x}_r$. Instead, we introduced $\|(K^\dagger K)^{-1}\|$ into the fifth line of that equation, absent the two factors of $\mathrm{v}(\boldsymbol{x}_r)$.

In the frequency domain formula, equation (121), the integration over $\boldsymbol{x}_r$ provides the downward continuation of the observed data and the integral over frequency forms the image. In the time domain, the integral over $\boldsymbol{x}_r$ and the integral over u provide the downward continuation of the data and the integral over t forms the image.

The data W must be consistent with forward modeling for this one way wave equation. Acquired data is the solution of a two-way wave equation. As with the Green's function, we need to take those acquired data and extend them to *analytic* data by adding i times their Hilbert transform in time.

The scaling factor $\mathrm{v}(\boldsymbol{x})|B_0(\boldsymbol{x},\boldsymbol{x}_s)|^2$ is an asymptotic expansion of $||K^\dagger K||$ under the assumption that the incident wave does not have a caustic near the image point $\boldsymbol{x}$. The interpretation of the reflectivity $\mathcal{R}_0$ in terms of a reflection coefficient as in equation (107), has not been shown to be valid near a caustic. Thus, we can content ourselves with simply regularizing this denominator so that it does not become infinite at the caustic.

Here is one recipe for such regularization. Using ray theory in ray-centered coordinates, one can show that

$$\frac{1}{|B_0(\boldsymbol{x},\boldsymbol{x}_s)|^2} = (4\pi)^2|\det[\mathbf{Q}]|, \tag{123}$$

with $\mathbf{Q}$ the standard notation of dynamic ray theory in ray-centered coordinates. It is this determinant that goes to zero at caustics. On the other hand, in homogeneous media, $|\det[\mathbf{Q}]| = |\boldsymbol{x}-\boldsymbol{x}_s|^2$. Thus we propose the replacement

$$\frac{1}{|B_0(\boldsymbol{x},\boldsymbol{x}_s)|^2} \Longrightarrow (4\pi)^2\left[\det[\mathbf{Q}]| + i\epsilon|\boldsymbol{x}-\boldsymbol{x}_s|^2\right], \tag{124}$$

for some appropriately chosen "small" ϵ. Now, away from caustics, this factor has phase near zero and we simply ignore the imaginary part to estimate the reflection coefficient. Near a caustic, the phase of this new term moves away from zero and the estimate of a reflection coefficient is unreliable. However, the output does not have a zero at the caustic and we are not evaluating $1/|B_0(\boldsymbol{x},\boldsymbol{x}_s)|^2$ which becomes infinite at the caustic.

A KMAH index can also be incorporated into the frequency domain formula to account for phase shifts at caustics. The phase shifts would introduce multipliers of powers of i in the time domain, progressively interchanging the real and imaginary parts of the Green's functions. with -1 factors introduced with each pair of phase shifts

9 Summary and conclusions

We have shown that a simple pair of one-way first order wave equations have the right asymptotic properties to serve as replacements for the usual second order two-way wave equation. These equations use t as the distinguished variable in contrast to the usual use of z in this role in the literature. Where the one-way wave equations in z fail at horizontal propagation, the one-way equations in time do not; all directions of propagation are treated alike by these equations. An analytic implementation of the method was presented to calculate the Green's function for homogeneous media. This method produces an *analytic* Green's function with its real part providing the Green's function for the two-way wave equation and its imaginary part being the Hilbert transform of the real part. This also happens for the one-way wave equations in z. Downward propagation of data prescribed at $z = 0$ was also demonstrated by using the complex conjugate of the anti-causal Green's function as propagator.

We derived a Green's identity to connect the solution of a one-way equation to the solution of the adjoint equation. That allowed us to derive integral formulas to describe downward propagation of observed data from $z = 0$ and upward propagation of reflection data from a surface at depth. The latter would use Kirchhoff-approximate data that was also derived here.

Finally, we derived the pseudo-inverse of the forward modeling formula for Kirchhoff-approximate reflected data. That pseudo-inversion operator provides an inversion of the data in the same sense as Kirchhoff inversion.

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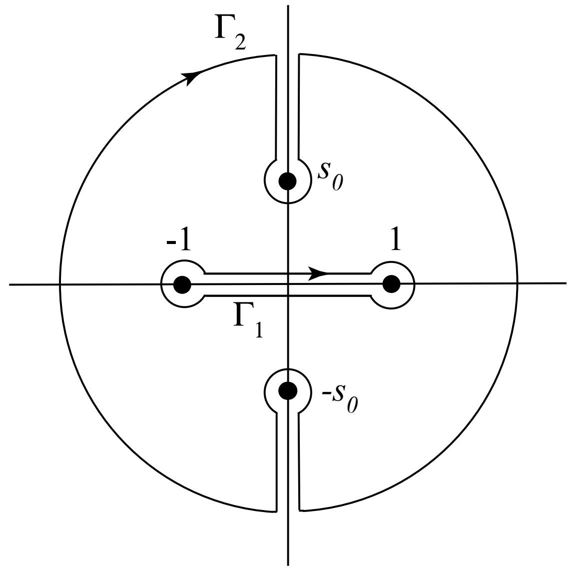

Figure A-1: The complex s-plane with the contours, Γ_1 and Γ_2. The integrand has branch points at $s = \pm 1$ and poles at the points $\pm s_0$ with $s_0 = i\,|k_z|\,/\sqrt{\boldsymbol{k}_T^2}$. When the vertical pieces of contour come together, the integrals along them cancel, leaving only the integrals around the poles and the integral on the large circle whose radius becomes infinite.

A Analysis of the integrals $I(\boldsymbol{k})$ and $I_1(\boldsymbol{k})$

The purpose of this appendix is to calculate the two integrals $I(\boldsymbol{k})$ of equation (22) and $I_1(\boldsymbol{p})$ of equation (42).

We discuss I first, repeated here:

$$I(\boldsymbol{k}) = \frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-s^2}\boldsymbol{k}_T^2}{k_z^2 + s^2\boldsymbol{k}_T^2} ds, \qquad \boldsymbol{k}_T = \begin{cases} k_x, & \text{2D}, \\ (k_x, k_y), & \text{3D}. \end{cases} \tag{A-1}$$

The integrand here has branch points at $s = \pm 1$ and poles at $\pm s_0$, with

$$s_0 = i\frac{|k_z|}{\sqrt{\boldsymbol{k}_T^2}}. \tag{A-2}$$

Furthermore, $\sqrt{1-s^2}$ is positive and real on the initial interval $(-1, 1)$.

The interval of integration can be replaced by the contour Γ_1 of Figure A-1 by standard contour integration techniques; this introduces a factor of 1/2; that is,

$$I(\boldsymbol{k}) = \frac{1}{2\pi}\int_{\Gamma_1} \frac{\sqrt{1-s^2}\boldsymbol{k}_T^2}{k_z^2 + s^2\boldsymbol{k}_T^2} ds. \tag{A-3}$$

The contour Γ_1 can be replaced by the contour Γ_2. The vertical segments of the contour are shown as separate to make their orientation clearer. However, in practice we bring them together. Since the integrand is single valued around the poles, these integrals on the vertical pieces cancel, both above the real axis and below. We are then left with integrals around poles $\pm s_0$ which are evaluated as a sum of residues at those poles. In addition, we have the integral on the large circle whose radius is allowed to approach infinity.

It is fairly straightforward to calculate the sum of the residues, so that

$$I(\boldsymbol{k}) = \frac{k}{|k_z|} - \frac{1}{2\pi} \int\limits_{|s|=R} \frac{\sqrt{1 - s^2 \boldsymbol{k}_T^2}}{k_z^2 + s^2 \boldsymbol{k}_T^2} ds. \tag{A-4}$$

Here, we have changed the sign on the integral so that its orientation is positive; that is, *counterclockwise*. As $R \to \infty$, we need only retain the leading powers of s in the numerator and the denominator; that is,

$$\frac{1}{2\pi} \int\limits_{|s|=R} \frac{\sqrt{1 - s^2 \boldsymbol{k}_T^2}}{k_z^2 + s^2 \boldsymbol{k}_T^2} ds = \frac{1}{2\pi} \int\limits_{|s|=R\to\infty} \frac{-is}{s^2} ds = -\frac{1}{2\pi} \int\limits_{|s|=R\to\infty} \frac{ids}{s} = 1. \tag{A-5}$$

Here, the first equality just effects the large-$|s|$ assumption. That is, it extends to $s > 1$ as $-i\sqrt{s^2 - 1}$ by standard complex analysis, which we have approximated as $-is$ for large $|s|$. The second equality simplifies the quotient. The final equality applies Cauchy's residue formula, in which case we see that this integral on a circle of increasingly large radius has a finite limit, namely, one.

When this evaluation of the integral appearing in equation (A-4) for $I(\boldsymbol{k})$, the final evaluation of I is

$$I(\boldsymbol{k}) = \frac{k}{|k_z|} - 1. \tag{A-6}$$

This is equivalent to equation (21) for I.

Now consider I_1 as defined in equation (42) and repeated here.

$$I_1(\boldsymbol{p}) = \frac{1}{\pi} \int\limits_{-1}^{1} \frac{\sqrt{1 - s^2}}{\left[p_z^2 + s^2 \boldsymbol{p}_T^2\right]^2} ds. \tag{A-7}$$

The analysis of this integral uses the same contour deformations as above. However, now, the poles are second order and the integral on the circle of radius R approaches zero as $R \to \infty$. The reason is that now the denominator is $\mathrm{O}(|s|^4)$ for large $|s|$ while the numerator is again of order $|s|$. The calculation of the sum of residues in this case leads to the stated result in equation (42).

B Determining B_{0T}, B_{1T} in terms of B_0, B_1

In this appendix, we present the details of the determination of the coefficients B_{0T} and B_{1T} in terms of B_0 and B_1. To this end, the series of equations (27) and (28) need to be substituted into the partial differential equation (25) that relates the two functions W and W_T. The series for the derivatives are as follows. For the transverse Laplacians,

$$\nabla_T^2 W = [B_0 F_0'' + B_1 F_0'] \boldsymbol{p}_T^2 + F_0'[2\boldsymbol{p}_T \cdot \nabla_T B_0 + +B_0 \nabla_T \cdot \boldsymbol{p}_T] + \dots\ , \tag{B-1}$$

and

$$\nabla_T^2 W_T = [B_{0T} F_0'' + B_{1T} F_0']\boldsymbol{p}_T^2 + F_0'[2\boldsymbol{p}_T \cdot \nabla_T B_{0T} + B_{0T} \nabla_T \cdot \boldsymbol{p}_T] + \dots \ . \tag{B-2}$$

Similarly, for the second derivative with respect to z,

$$\frac{\partial^2 W_T}{\partial z^2} = [B_0 F_0'' + B_1 F_0'] p_z^2 + F_0' \left[2p_z \frac{\partial B_0}{\partial z} + B_0 \frac{\partial p_z}{\partial z} \right] + \dots \ , \tag{B-3}$$

and

$$\frac{\partial^2 W_T}{\partial z^2} = [B_{0T} F_0'' + B_{1T} F_0'] p_z^2 + F_0' \left[2p_z \frac{\partial B_{0T}}{\partial z} + B_{0T} \frac{\partial p_z}{\partial z} \right] + \dots \ . \tag{B-4}$$

Here,

$$\boldsymbol{p}_T = \begin{cases} p_x, & \text{2D}, \\ (p_x, p_y), & \text{3D}. \end{cases} \tag{B-5}$$

The series in equations (B-1), (B-2) and (B-4) are substituted into equation (25) and the coefficients of F_0'' and F_0' on the two sides of the equation are set equal. This leads to the following pair of equations.

$$\begin{aligned} \left[p_z^2 + s^2 \boldsymbol{p}_T^2 \right] B_{0T} &= \boldsymbol{p}_T^2 B_0, \\ \left[p_z^2 + s^2 \boldsymbol{p}_T^2 \right] B_{1T} &= \boldsymbol{p}_T^2 B_1 + 2\boldsymbol{p}_T \cdot \nabla_T B_0 + B_0 \nabla_T \cdot \boldsymbol{p}_T \\ &- \left[2p_z \frac{\partial B_{0T}}{\partial z} + B_{0T} \frac{\partial p_z}{\partial z} + s^2 \left[2\boldsymbol{p}_T \cdot \nabla_T B_{0T} + B_{0T} \nabla_T \cdot \boldsymbol{p}_T \right] \right]. \end{aligned} \tag{B-6}$$

The first equation here has solution

$$B_{0T} = \frac{\boldsymbol{p}_T^2}{p_z^2 + s^2 \boldsymbol{p}_T^2} B_0. \tag{B-7}$$

We observe that the second equation has transverse derivatives multiplying the combination

$$B_0 - s^2 B_{0T} = \frac{p_z^2}{p_z^2 + s^2 \boldsymbol{p}_T^2} B_0. \tag{B-8}$$

By using this identity and the solution for B_{0T} of equation (29) in the second equation in (B-6), we can rewrite that equation as

$$\begin{aligned} \left[p_z^2 + s^2 \boldsymbol{p}_T^2 \right] B_{1T} &= \boldsymbol{p}_T^2 B_1 + 2\boldsymbol{p}_T \cdot \nabla_T \left(\frac{p_z^2 B_0}{p_z^2 + s^2 \boldsymbol{p}_T^2} \right) + \frac{p_z^2 B_0}{p_z^2 + s^2 \boldsymbol{p}_T^2} \nabla_T \cdot \boldsymbol{p}_T \\ &-2p_z \frac{\partial}{\partial z} \left(\frac{\boldsymbol{p}_T^2 B_0}{p_z^2 + s^2 \boldsymbol{p}_T^2} \right) - \frac{\boldsymbol{p}_T^2 B_0}{p_z^2 + s^2 \boldsymbol{p}_T^2} \frac{\partial p_z}{\partial z}. \end{aligned} \tag{B-9}$$

The solution of this equation is

$$
\begin{aligned}
B_{1T} &= \frac{\boldsymbol{p}_T^2}{p_z^2 + s^2\boldsymbol{p}_T^2} B_1 + \frac{1}{(p_z^2 + s^2\boldsymbol{p}_T^2)} \left\{ 2\boldsymbol{p}_T \cdot \nabla_T \left(\frac{p_z^2 B_0}{p_z^2 + s^2\boldsymbol{p}_T^2} \right) + \frac{p_z^2 B_0}{p_z^2 + s^2\boldsymbol{p}_T^2} \nabla_T \cdot \boldsymbol{p}_T \right. \\
&\qquad \left. -2p_z \frac{\partial}{\partial z} \left(\frac{B_0 \boldsymbol{p}_T^2}{p_z^2 + s^2\boldsymbol{p}_T^2} \right) - \frac{\boldsymbol{p}_T^2 B_0}{p_z^2 + s^2\boldsymbol{p}_T^2} \frac{\partial p_z}{\partial z} \right\}. \\
&= \frac{\boldsymbol{p}_T^2}{p_z^2 + s^2\boldsymbol{p}_T^2} B_1 \qquad \text{(B-10)} \\
&\qquad + \frac{1}{B_0} \left\{ \frac{1}{p_z^2} \nabla_T \cdot \left(\frac{p_z^4 B_0^2 \boldsymbol{p}_T}{(p_z^2 + s^2\boldsymbol{p}_T^2)^2} \right) - \frac{1}{\boldsymbol{p}_T^2} \frac{\partial}{\partial z} \left(\frac{B_0^2 \boldsymbol{p}_T^4 p_z}{(p_z^2 + s^2\boldsymbol{p}_T^2)^2} \right) \right\}.
\end{aligned}
$$

In the last line here, we have isolated the s-dependence in a single factor under a 2D divergence and a similar single factor under the z-derivative—very nearly a divergence.

C The distribution $\delta(\boldsymbol{x} - \boldsymbol{\xi})/ik$

In equation (56) we introduced the distribution $\delta(\boldsymbol{x} - \boldsymbol{\xi})/ik$ as the final data for an anti-causal Green's function. Earlier, in equation (56), we introduced the same distribution with $\boldsymbol{\xi} = \boldsymbol{0}$ in defining the Green's function for the causal one-way Green's function.

Here, we reinterpret this distribution by applying the same spectral method as we used to define multiplication by ik in equations (52) and (53).

Thus, we write

$$
\begin{aligned}
\frac{\delta(\boldsymbol{x} - \boldsymbol{\xi})}{ik} &= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} d^3k \frac{1}{ik} \int d^3x' \delta(\boldsymbol{x}' - \boldsymbol{\xi}) \exp\{i\boldsymbol{k} \cdot (\boldsymbol{x} - \boldsymbol{x}')\} \\
&= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} d^3k \frac{1}{ik} \exp\{i\boldsymbol{k} \cdot (\boldsymbol{x} - \boldsymbol{\xi})\} \qquad \text{(C-1)} \\
&= -\frac{i}{(2\pi)^3} \int_0^{\infty} k dk \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \exp\{ikr\cos\theta\}, \quad r = |\boldsymbol{x} - \boldsymbol{\xi}|.
\end{aligned}
$$

Here, in the second line, we used the δ function to evaluate the integral over $\boldsymbol{x}'$. In the the third line, we transformed to polar coordinates in $\boldsymbol{k}$ with the polar axis parallel to the vector $\boldsymbol{x} - \boldsymbol{\xi}$. In this last line, we see that the integrand is independent of the angle ϕ, so that the ϕ-integration only introduces a factor of 2π. Further, the integral in θ is exact, leading to

$$
\frac{\delta(\boldsymbol{x} - \boldsymbol{\xi})}{ik} = \frac{1}{(2\pi)^2 r} \int_0^{\infty} dk \left[\exp\{-ikr\} - \exp\{ikr\} \right]. \qquad \text{(C-2)}
$$

These two integrals are known distributions. They can be derived from the identities in equation (A.7) of Bleistein et al [2001], namely that

$$
\frac{1}{2\pi} \int_0^{\infty} dk \exp\{\pm ikr\} = \frac{1}{2} \left[\delta(r) \pm \frac{i}{\pi r} \right]. \qquad \text{(C-3)}
$$

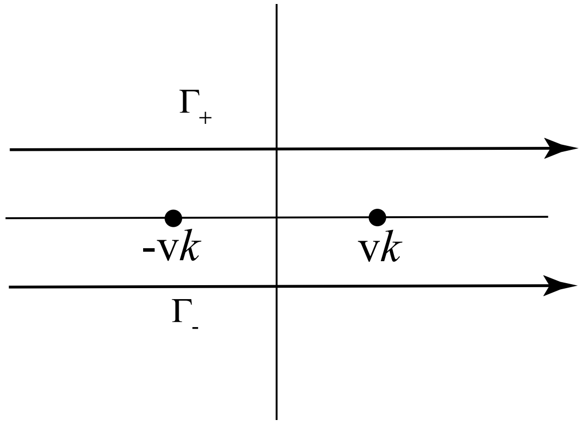

Figure D-1: The contours, $\Gamma_{\pm}$, for the frequency domain integration of $\gamma_{\pm}$. Also shown are the poles on the $\Re\{\omega\}$-axis at $\pm \mathrm{v}k$ for $\gamma_{\pm}$, respectively.

We add the two together with opposite signs as indicated in the previous equation to find that

$$\frac{\delta(\boldsymbol{x}-\boldsymbol{\xi})}{ik} = \frac{1}{2\pi^2 ir}. \tag{C-4}$$

D Inverting the Fourier transform, $\gamma_{\pm}$, for the Green's functions in homogeneous media

Here we derive the inverse Fourier transforms of the functions $\gamma_{\pm}$ in equation (62). To begin, we write the inverse transform as an integral in ω and $\boldsymbol{k}$ as follows.

$$G_{\pm}(\boldsymbol{x},\boldsymbol{0},t) = -\frac{\mathrm{v}}{(2\pi)^4}\int_{\Gamma_{\pm}} d\omega \int_{-\infty}^{\infty} dk_1 dk_2 dk_3 \frac{\exp\{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t\}}{k\,[\omega \mp \mathrm{v}k]} \tag{D-1}$$

In this equation, the contours $\Gamma_{\pm}$ are infinite lines parallel to the $\Re[\omega]$-axis above (+) or below (-) all singularities of the integrand, namely, any line above (+) or below (-) the real ω-axis, respectively, for the causal and anti-causal solutions. See Figure **??**, where the contours $\Gamma_{\pm}$ are depicted as well as the poles $\pm\mathrm{v}k$ of the functions γ_{pm}, respectively.

In the next sequence, we rewrite the k-domain integrals in equation (D-1) in polar coordinates, then cancel a common factor of k in numerator. As a last step in this sequence, we integrate in ϕ, thereby cancelling a factor of 2π since the integrand is independent of ϕ.

$$\begin{aligned}
G_{\pm}(\boldsymbol{x},\boldsymbol{0},t) &= -\frac{\mathrm{v}}{(2\pi)^4}\int_{\Gamma_{\pm}} d\omega \int_0^{\infty} k^2 dk \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \frac{\exp\{ikr\cos\theta - i\omega t\}}{k\,[\omega \mp \mathrm{v}k]} \qquad \text{(D-2)}\\
&= -\frac{\mathrm{v}}{(2\pi)^4}\int_{\Gamma_{\pm}} d\omega \int_0^{\infty} k dk \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \frac{\exp\{ikr\cos\theta - i\omega t\}}{[\omega \mp \mathrm{v}k]}\\
&= -\frac{\mathrm{v}}{(2\pi)^3}\int_{\Gamma_{\pm}} d\omega \int_0^{\infty} k dk \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} \frac{\exp\{ikr\cos\theta - i\omega t\}}{[\omega \mp \mathrm{v}k]}.
\end{aligned}$$

Next, the integration in θ can readily be calculated because of the factor of $\sin\theta$ in the amplitude. Therefore,

$$G_\pm(\boldsymbol{x},\mathbf{0},t) = -\frac{\mathrm{v}}{i(2\pi)^3 r}\int_{\Gamma_\pm} d\omega \int_0^\infty dk \frac{\exp\{ikr - i\omega t\} - \exp\{-ikr - i\omega t\}}{\omega \mp \mathrm{v}k}. \tag{D-3}$$

The contour Γ_+ lies above the pole of the integrand at $\omega = \mathrm{v}k$; similarly, the contour Γ_- lies below the pole of the integrand at $\omega = -\mathrm{k}$. Let us consider G_+ for the moment. For $t < 0$ we can close the contour of integration Γ_+ by a semicircle in the upper half plane whose radius is allowed to approach infinity. The integral around the closed path is equal to the integral along Γ_+, but the integral around the closed path encloses no singularities of the integrand. Therefore, by Cauchy's theorem, $G_+ = 0$ for $t < 0$. We characterize this below by a multiplier of the Heaviside function, $H(t)$. Similarly, for $t > 0$ we can apply the same arguments to the integral on the contour Γ_- and conclude that $G_- = 0$ for $t > 0$. We can handle both of these observations at the same time with the multiplier $H(\pm t)$. For the complimentary temporal domains, we close the contours along the opposite semicircles, now enclosing a pole in each case. Thus, we compute the causal and anti-causal Green's functions as residues of the integrands, as follows.

$$G_\pm(\boldsymbol{x},\mathbf{0},t) = \pm H(\pm t)\frac{\mathrm{v}}{(2\pi)^2 r}\int_0^\infty dk\left[\exp\{ik(r \mp \mathrm{v}t)\} - \exp\{-ik(r \pm \mathrm{v}t)\}\right]. \tag{D-4}$$

This integral is a difference of well-known distributions. They can be determined using the basic discussion of the Fourier transform of the Heaviside function in Section A.7 of Bleistein et al [2001], among other places. They are stated in this paper as equation (C-3). Both are complex-valued with the imaginary part being the Hilbert transform of the real part, which is half of a Dirac delta function. Together, they provide half the "analytic delta function."

By using the identities of equation (C-3) in the two integrals of equation (D-4) for $G_\pm$ we find that

$$\begin{aligned} G_\pm(\boldsymbol{x},\mathbf{0},t) &= \pm H(\pm t)\frac{\mathrm{v}}{4\pi r}\left\{\delta(r \mp \mathrm{v}t) - \delta(r \pm \mathrm{v}t) + \frac{i}{\pi}\left[\frac{1}{r \mp \mathrm{v}t} + \frac{1}{r \pm \mathrm{v}t}\right]\right\} \\ &= \pm H(\pm t)\frac{1}{4\pi r}\left\{\delta(r/\mathrm{v} \mp t) - \delta(r/\mathrm{v} \pm t) + \frac{i}{\pi}\left[\frac{1}{r/\mathrm{v} \mp t} + \frac{1}{r/\mathrm{v} \pm t}\right]\right\}. \end{aligned} \tag{D-5}$$

Here, the imaginary parts are to be interpreted as Cauchy principal values in any integrations. In the second line, we have extracted a factor of $1/\mathrm{v}$ from the term in brackets.

We only care about the leading order or most singular terms here for our asymptotic analysis in the progressing wave formalism. Therefore we only keep the terms that are singular for $\pm t > 0$; that is,

$$G_\pm(\boldsymbol{x},\mathbf{0},t) \sim H(\pm t)\frac{1}{4\pi r}\left\{\delta(r/\mathrm{v} \mp t) \pm \frac{i}{\pi(r/\mathrm{v} \mp t)}\right\}. \tag{D-6}$$

In equation (63) in the text, we state this result with the initial time shifted from zero back to t_0 and the support of the source delta function shifted from $\mathbf{0}$ to $\boldsymbol{x}_0$.

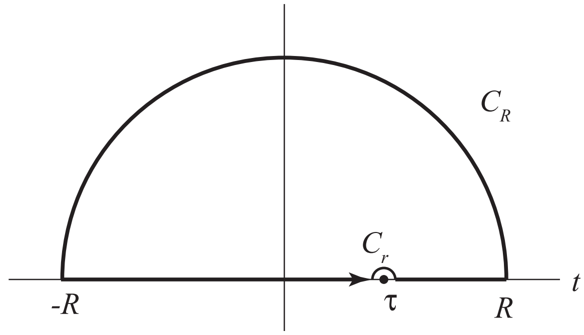

Figure E-1: The oriented contour of integration in the complex t-plane for determination of the Cauchy principal value integrals. R and r are the radii of the semi-circles.

E The temporal Fourier transforms of the asymptotic Green's functions.

In this appendix, we derive the temporal Fourier transforms $g_\pm$ of the asymptotic Green's functions $G_\pm$ in equation (64). That is, we consider the pair of integrals

$$g_\pm(\boldsymbol{x}, \boldsymbol{x}_0, \omega) = B \int_0^{\pm\infty} \left\{ \delta\Big(\tau \mp t\Big) + \frac{i}{\pi} \cdot \frac{1}{\tau \mp t} \right\} \exp\{i\omega t\} dt. \quad \pm \Im(\omega) > 0. \tag{E-1}$$

Here, the spatial dependencies are unimportant, so we have suppressed them. Furthermore, the Fourier transform of the imaginary parts in curly brackets are to be interpreted as Cauchy principal value integrals.

In the representation of g_- in equation (E-1), let us replace t by $-t$, leading to the following:

$$g_\pm(\boldsymbol{x}, \boldsymbol{x}_0, \omega) = \pm B \int_0^{\infty} \left\{ \delta(\tau - t) + \frac{i}{\pi} \cdot \frac{1}{\tau - t} \right\} \exp\{\pm i\omega t\} dt. \quad \pm \Im(\omega) > 0. \tag{E-2}$$

First, let us consider the two integrals with the delta functions:

$$\pm B \int_0^{\infty} \delta(\tau - t) \exp\{\pm i\omega t\} dt = \pm B \exp\{\pm i\omega\tau\}, \quad \pm \Im(\omega) > 0. \tag{E-3}$$

Of course, the analytic continuation of these two functions to the entire complex ω-plane are just the explicit exponentials.

Next, we consider the principal value integrals of the second terms here. We introduce the integral on the counter-clockwise contour of Figure E-1 in the complex t-plane. Call the whole contour C. On that contour, C_R and C_r are semi-circles of radius R and r, respectively. Let us consider the following limit on the integral on the contour C.

$$\lim_{\substack{R\to\infty \\ r\to 0}} \int_C \frac{i}{\pi} \cdot \frac{1}{\tau - t} \exp\{\pm i\omega t\} dt \quad = \quad \frac{i}{\pi} \int_{-\infty}^{0} \cdot \frac{1}{\tau - t} \exp\{\pm i\omega t\} dt + \frac{i}{\pi} \fint_0^{\infty} \frac{1}{\tau - t} \exp\{\pm i\omega t\} dt$$

$$+\lim_{R\to\infty}\frac{i}{\pi}\int_{C_R}\frac{1}{\tau-t}\exp\{\pm i\omega t\}dt \tag{E-4}$$

$$+\mathrm{Res}\left\{\frac{1}{\tau-t}\exp\{\pm i\omega t\}\right\}, \quad \pm\Im(\omega)>0.$$

Here, $⨍$ denotes the Cauchy principal value that we seek and "Res" denotes the residue of the integrand at the pole at $t=\tau$. We remark that, in fact, this integral over the contour C for any choice of R and r is zero by Cauchy's theorem, because the integrand is analytic inside the contour.

We first have to deal with the integral on the path C_R. Note that $\exp\{\pm i\omega t\}$ has a negative real part for $\pm\Im(\omega)>0$. Thus, by standard analysis, the integral over the contour C_R approaches zero as $R\to\infty$.

Next, we consider

$$\frac{i}{\pi}\int_{-\infty}^{0}\cdot\frac{1}{\tau-t}\exp\{\pm i\omega t\}dt = \mp\frac{1}{\pi\omega}+\mathrm{O}\left(\frac{1}{\omega^2}\right) \tag{E-5}$$

by using integration by parts. Even to leading order, this term is smaller than order one in ω and hence does not contribute to the leading order asymptotic expansion of this Fourier transform.

Finally, we observe that

$$\mathrm{Res}\left\{\frac{1}{\tau-t}\exp\{\pm i\omega t\}\right\} = -\exp\{i\omega\tau\}. \tag{E-6}$$

We combine all of our observations here below equation (E-4) and solve for the principal value integral there to conclude that

$$\frac{i}{\pi}⨍_0^{\infty}\frac{1}{\tau-t}\exp\{\pm i\omega t\}dt = \exp\{i\omega\tau\}. \tag{E-7}$$

That is, the Cauchy principal value integrals have the same value as the Fourier transforms of the delta functions, but only to leading order asymptotically.

We now combine the transforms in equation (E-6) with the transforms of the delta functions in equation (E-3) and substitute into equation (E-2) for $g_\pm$ to conclude that

$$g_\pm(\boldsymbol{x},\boldsymbol{x}_0,\omega) = \pm 2B(\boldsymbol{x},\boldsymbol{x}_0)\exp\{\pm i\omega\tau(\boldsymbol{x},\boldsymbol{x}_0)\}. \tag{E-8}$$

We remark here that if we consider the effect of caustics, then we would introduce a phase shift,

$$g_\pm(\boldsymbol{x},\boldsymbol{x}_0,\omega) = \pm 2B(\boldsymbol{x},\boldsymbol{x}_0)\exp\{\pm i\omega\tau(\boldsymbol{x},\boldsymbol{x}_0)\pm iK\pi/2\}. \tag{E-9}$$

Here, K is the KMAH index, counting the number of caustics passed on the ray from $\boldsymbol{x}_0$ to $\boldsymbol{x}$.

F Asymptotic analysis of $||K^\dagger K||$

In this appendix, we derive the asymptotic expansion of norm $||K^\dagger K||$ as defined by equatoin (119). For this derivation, we assume that there are no caustics near the image point $\boldsymbol{x}$ or near the source and receiver points $\boldsymbol{x}_s$ and $\boldsymbol{x}_r$. When there are caustics nearby, then the "true-amplitude" claim is no longer true, although the output still provides an image of the reflector. Thus, in such regions, we will content ourselves with a regularization of the asymptotic approximation derived here.

As a first step, note from the representation in equation (119) that the phase function Φ is zero when $\boldsymbol{x}' = \boldsymbol{x}$. That tells us that this is a dominant critical point in the asymptotic expansion of the integral over $\boldsymbol{x}'$. Therefore, let us approximate Φ in the neighborhood of $\boldsymbol{x}$ by the first terms of its Taylor series in $\boldsymbol{x}'$ as follows.

$$\begin{gathered}\Phi(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{x}_s, \boldsymbol{x}_r) \sim [\boldsymbol{p}_s(\boldsymbol{x}) + \boldsymbol{p}_r(\boldsymbol{x})] \cdot (\boldsymbol{x}' - \boldsymbol{x}), \\ \boldsymbol{p}_s(\boldsymbol{x}) = \nabla\tau(\boldsymbol{x}, \boldsymbol{x}_s), \quad \boldsymbol{p}_r(\boldsymbol{x}) = \nabla\tau(\boldsymbol{x}, \boldsymbol{x}_r).\end{gathered} \tag{F-1}$$

Next, we introduce a change of variables of integration from $(\boldsymbol{x}_r, \omega)$ to $\boldsymbol{k}$, defined implicitly by

$$\boldsymbol{k} = \omega[\boldsymbol{p}_s(\boldsymbol{x}) + \boldsymbol{p}_r(\boldsymbol{x})]. \tag{F-2}$$

We approximate the integral for $||K^\dagger K||$ in equation (119) using these last two equations and setting $\boldsymbol{x}' = \boldsymbol{x}$ in the amplitude. That is,

$$\begin{aligned}||K^\dagger K|| \ \sim \ 16 \int & \frac{\omega^2 d^3k d^3x'}{\mathrm{v}^2(\boldsymbol{x})} \frac{\partial(\omega, \boldsymbol{x}_r)}{\partial(\boldsymbol{k})} \\ & \cdot B^2(\boldsymbol{x}, \boldsymbol{x}_s) B^2(\boldsymbol{x}, \boldsymbol{x}_r) \exp\{i\boldsymbol{k} \cdot (\boldsymbol{x}' - \boldsymbol{x})\}.\end{aligned} \tag{F-3}$$

The rightmost factor here is the Jacobian of transformation from the variables $(\omega, \boldsymbol{x}_r)$ to $\boldsymbol{k}$. Of course, that Jacobian is also evaluated at $\boldsymbol{x}' = \boldsymbol{x}$. In fact, we will show below that ω^2 times the Jacobian is independent of ω. Hence the entire amplitude of the integrand is independent of ω. Therefore, we treat is as "slowly varying" with respect to the high frequency content of the phase and treat it as a constant. In this case, the $\boldsymbol{k}$ integration yields $(2\pi)^3$ times $\delta(\boldsymbol{x}' - \boldsymbol{x})$, which, in turn, allows us to compute the $\boldsymbol{x}'$ integral. The result is that

$$||K^\dagger K|| \sim 128\pi^3 \frac{\omega^2}{\mathrm{v}^2(\boldsymbol{x})} \frac{\partial(\omega, \boldsymbol{x}_r)}{\partial(\boldsymbol{k})} B^2(\boldsymbol{x}, \boldsymbol{x}_s) B^2(\boldsymbol{x}, \boldsymbol{x}_r). \tag{F-4}$$

We need to simplify the Jacobian appearing in this last equation. This is related to the Beylkin determinant for common-shot inversion which we want to write in terms of the amplitude $B^2(\boldsymbol{x}, \boldsymbol{x}_r)$. We start by considering the inverse of that Jacobian and use the definition of $\boldsymbol{k}$ in equation (F-2) to find that

$$\frac{\partial(\boldsymbol{k})}{\partial(\omega, \boldsymbol{x}_r)} = \omega^2 \det[\boldsymbol{h}]; \quad \boldsymbol{h} = \begin{bmatrix} \boldsymbol{p}_s(\boldsymbol{x}) + \boldsymbol{p}_r(\boldsymbol{x}) \\ \\ \dfrac{\partial \boldsymbol{p}_r(\boldsymbol{x})}{\partial x_{r1}} \\ \\ \dfrac{\partial \boldsymbol{p}_r(\boldsymbol{x})}{\partial x_{r2}} \end{bmatrix}. \tag{F-5}$$

In the literature, $\det[\boldsymbol{h}]$ is known as the Beylkin determinant—in this case, for common-shot inversion. Note from this equation that

$$\frac{1}{\omega^2} \frac{\partial(\boldsymbol{k})}{\partial(\omega, \boldsymbol{x}_r)}$$

is independent of ω, as claimed.

To compute $h = \det[\boldsymbol{h}]$, we first consider the product of matrices

$$\boldsymbol{H} = \boldsymbol{h} \begin{bmatrix} \dfrac{1}{\mathrm{v}(\boldsymbol{x})} & 0 & p_{r1} \\ \\ 0 & \dfrac{1}{\mathrm{v}(\boldsymbol{x})} & p_{r2} \\ \\ 0 & 0 & p_{r3} \end{bmatrix} = \begin{bmatrix} \dfrac{p_{s1} + p_{r1}}{\mathrm{v}(\boldsymbol{x})} & \dfrac{p_{s2} + p_{r2}}{\mathrm{v}(\boldsymbol{x})} & \boldsymbol{p}_r \cdot (\boldsymbol{p}_s + \boldsymbol{p}_r) \\ \\ \dfrac{1}{\mathrm{v}(\boldsymbol{x})} \dfrac{\partial p_{r1}}{\partial x_{r1}} & \dfrac{1}{\mathrm{v}(\boldsymbol{x})} \dfrac{\partial p_{r2}}{\partial x_{r1}} & \boldsymbol{p}_r(\boldsymbol{x}) \cdot \dfrac{\partial \boldsymbol{p}_r}{\partial x_{r1}} \\ \\ \dfrac{1}{\mathrm{v}(\boldsymbol{x})} \dfrac{\partial p_{r1}}{\partial x_{r2}} & \dfrac{1}{\mathrm{v}(\boldsymbol{x})} \dfrac{\partial p_{r2}}{\partial x_{r2}} & \boldsymbol{p}_r(\boldsymbol{x}) \cdot \dfrac{\partial \boldsymbol{p}_r}{\partial x_{r2}} \end{bmatrix}. \tag{F-6}$$

In this matrix

$$\boldsymbol{p}_r \cdot (\boldsymbol{p}_s + \boldsymbol{p}_r) = \frac{1}{\mathrm{v}^2(\boldsymbol{x})}[\cos 2\theta + 1] = \frac{2}{\mathrm{v}^2(\boldsymbol{x})} \cos^2 \theta, \tag{F-7}$$

with 2θ being the opening angle between the rays from the source and the receiver at the point $\boldsymbol{x}$. Furthermore,

$$\boldsymbol{p}_r(\boldsymbol{x}) \cdot \frac{\partial \boldsymbol{p}_r}{\partial x_{rj}} = \frac{1}{2} \frac{\partial \boldsymbol{p}_r^2}{\partial x_{rj}} = \frac{1}{2} \frac{\partial}{\partial x_{rj}} \left[\frac{1}{\mathrm{v}^2(\boldsymbol{x})} \right] = 0, \quad j = 1, 2. \tag{F-8}$$

By using these last two results in equation (F-6), taking determinants on both sides and solving for $\det[\boldsymbol{h}]$, we find

$$\det[\boldsymbol{h}] = \frac{2\cos^2 \theta}{\mathrm{v}(\boldsymbol{x}) \cos \beta_x} \det \begin{bmatrix} \dfrac{\partial p_{r1}}{\partial x_{r1}} & \dfrac{\partial p_{r2}}{\partial x_{r1}} \\ \\ \dfrac{\partial p_{r1}}{\partial x_{r2}} & \dfrac{\partial p_{r2}}{\partial x_{r2}} \end{bmatrix}. \tag{F-9}$$

Here, β_x is the dip angle with the vertical of the ray arriving from $\boldsymbol{x}_r$ at $\boldsymbol{x}$. It arises from the determinant of the matrix multiplying $\boldsymbol{h}$ in equation (F-6).

Now we need to interpret the matrix on the right side of equation (F-9) for det$[\boldsymbol{h}]$. To this end, we first interchange the orders of differentiation in that equation as follows.

$$\frac{\partial p_{rj}}{\partial x_{rk}} = \frac{\partial^2 \tau(\boldsymbol{x}, \boldsymbol{x}_r)}{\partial \boldsymbol{x}_j \partial \boldsymbol{x}_{rk}} = \frac{\partial^2 \tau(\boldsymbol{x}, \boldsymbol{x}_r)}{\partial \boldsymbol{x}_{rk} \partial \boldsymbol{x}_j} = -\frac{\partial p^0_{rj}}{\partial x_k}, \quad j, k = 1, 2 \tag{F-10}$$

Here, p^0_{rj} is the initial value of a slowness component at $\boldsymbol{x}_r$ on the ray to $\boldsymbol{x}$. The minus sign arises from the fact that the $\boldsymbol{x}_r$ gradient at the upper surface points upward and the initial direction of the ray points downward.

Now, the matrix on the right side of equation (F-9) becomes

$$\begin{bmatrix} \dfrac{\partial p_{r1}}{\partial x_{r1}} & \dfrac{\partial p_{r2}}{\partial x_{r1}} \\ \\ \dfrac{\partial p_{r1}}{\partial x_{r2}} & \dfrac{\partial p_{r2}}{\partial x_{r2}} \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial p^0_{r1}}{\partial x_1} & -\dfrac{\partial p^0_{r2}}{\partial x_1} \\ \\ -\dfrac{\partial p^0_{r1}}{\partial x_2} & -\dfrac{\partial p^0_{r2}}{\partial x_2} \end{bmatrix} \tag{F-11}$$

It is easier for us to calculate the determinant of the inverse of this matrix, which is given by

$$\det \begin{bmatrix} -\dfrac{\partial p^0_{r1}}{\partial x_1} & -\dfrac{\partial p^0_{r2}}{\partial x_1} \\ \\ -\dfrac{\partial p^0_{r1}}{\partial x_2} & -\dfrac{\partial p^0_{r2}}{\partial x_2} \end{bmatrix}^{-1} = \det \begin{bmatrix} \dfrac{\partial x_1}{\partial p^0_{r1}} & \dfrac{\partial x_1}{\partial p^0_{r2}} \\ \\ \dfrac{\partial x_2}{\partial p^0_{r1}} & \dfrac{\partial x_2}{\partial p^0_{r2}} \end{bmatrix} = P_3 \tag{F-12}$$

Suppose we consider the Jacobian of the rays from $\boldsymbol{x}_r$ to $\boldsymbol{x}$ with parameters (p^0_{r1}, p^0_{r2}) labeling the rays and arc length s as the running parameter along the ray. That Jacobian is given by

$$J = \frac{\partial \boldsymbol{x}}{\partial s} \cdot \frac{\partial \boldsymbol{x}}{\partial p^0_{r1}} \times \frac{\partial \boldsymbol{x}}{\partial p^0_{r1}}. \tag{F-13}$$

The product, $J dp^0_{r1} dp^0_{r2}$ defines the cross section of the ray tube from the source at the point $\boldsymbol{x}$. See Figure F-1. On the other hand, $P_3 dp^0_{r1} dp^0_{r2}$ is the area of the the intersection of the ray tube with the horizontal plane at $\boldsymbol{x}$ as shown in the figure. In fact, these two cross sectional areas are related through a factor of $\cos\beta_x$ as follows.

$$P_3 \cos\beta_x = J; \quad P_3 = J/\cos\beta_x. \tag{F-14}$$

Recall that P_3 is the determinant of the inverse of the matrix on the right hand side of equation (F-9), which was our last expression for det$[\boldsymbol{h}]$. Thus, we use this last expression for P_3 to rewrite det$[\boldsymbol{h}]$ as

$$\det[\boldsymbol{h}] = \frac{2\cos^2\theta}{\mathrm{v}(\boldsymbol{x}) J}. \tag{F-15}$$

It is this ray Jacobian that we can now relate to the amplitude $B_0(\boldsymbol{x}, \boldsymbol{x}_r)$. In fact, with the initial parameters, (p^0_{r1}, p^0_{r2}) and running parameter being arc length s, it is fairly straightforward to show that

$$B_0^2(\boldsymbol{x}, \boldsymbol{x}_r) = \frac{1}{16\pi^2} \frac{\mathrm{v}(\boldsymbol{x})\mathrm{v}(\boldsymbol{x}_r)}{J\cos\beta_r}. \tag{F-16}$$

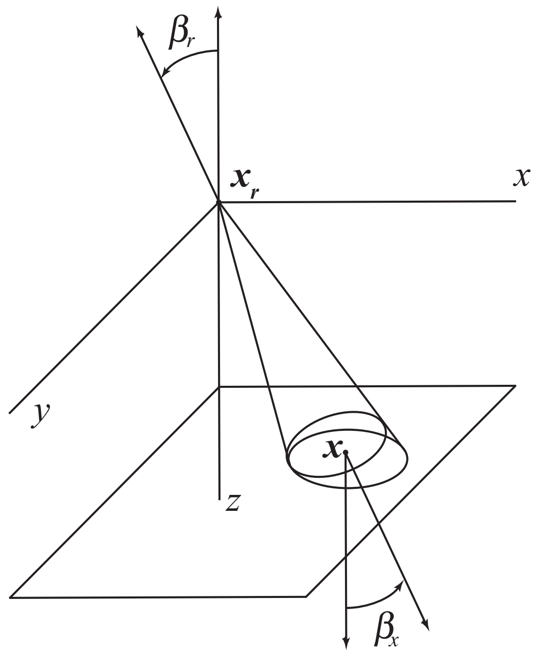

Figure F-1: Cross sectional area of a ray tube from the surface point $\boldsymbol{x}_r$ at the image point $\boldsymbol{x}$. β_r is the dip angle with respect to the vertical of the central ray of the ray tube at the upper surface; β_x is the dip angle with respect to the vertical of the central ray of the ray tube at $\boldsymbol{x}$. The latter is the projection angle of the cross section of the horizontal intersection of the ray tube onto the orthogonal cross section of the ray tube.

Here, β_r is the initial dip angle with the vertical of the ray from $\boldsymbol{x}_r$ to $\boldsymbol{x}$. We solve here for J and substitute into equation (F-15) to find that

$$\det[\boldsymbol{h}] = 32\pi^2 \cos^2\theta \frac{B^2(\boldsymbol{x}, \boldsymbol{x}_r)\cos\beta_r}{\mathrm{v}^2(\boldsymbol{x})\mathrm{v}(\boldsymbol{x}_r)}. \tag{F-17}$$

By comparing equation (F-4) and (F-5), we see that we need to replace ω^2 times the Jacobian in the former equation by the reciprocal of $\det[\boldsymbol{h}]$, which we just determined in the previous equation. We then find that

$$||K^\dagger K|| \sim \frac{4\pi \mathrm{v}(\boldsymbol{x}_r) B^2(\boldsymbol{x}, \boldsymbol{x}_s)}{\cos\beta_r \cos^2\theta}. \tag{F-18}$$