Analysis of the Neighborhood of a Smooth Caustic for True-Amplitude One-Way Wave Equations

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Abstract

In earlier papers it was shown that true amplitude one-way wave equations provide “true amplitude” in the sense that the \textit{WKBJ} or ray-theoretic solutions agree with the corresponding solutions of the full (two-way) wave equation. In the neighborhood of smooth caustics—smooth envelopes of rays—these ray-theoretic solutions break down with the predicted amplitudes of the solutions becoming infinite. For the two-way wave equation, that breakdown is well understood. The exact solution of the wave equation remains finite and it is only the \textit{WKBJ} formalism that fails. Asymptotic solutions have been developed using higher functions (Airy functions). These solutions remain finite in a neighborhood of smooth caustics, adequately describing the exact solution, even in the limit where the observation point is on the caustic. A corresponding theory for the one-way wave equations is not yet available. However, numerical examples indicate that the solutions of the one-way wave equations remain finite and smooth near caustics. Furthermore, the mechanism for deriving the uniformly valid \textit{WKBJ} solutions near smooth caustics strongly suggests that the uniform theory as derived for the two-way wave equation should be extendable to the one-way wave equation. Absent that extension, we present here a specific example in which we can analytically derive a uniformly valid asymptotic expansion near a smooth caustic via a standard extension of the method of stationary phase designed to deal with the neighborhood of smooth caustics.

1 Introduction

Caustics are envelopes of rays—curves in 2D or surfaces in 3D. They can be smooth, such as in the wake of a boat or the bright spots on the floor of a pool in sunlight, or they can be cusped, such as at the focal point of a less-than-prefect (real-world) lens. The ridge of the wake, the bright spots on the pool floor or the ability of a lens to burn paper are all indicators of higher amplitude of the solution of the appropriate wave equation. In fact, the asymptotic order in “high frequency” changes in the neighborhood of a cusp, which is closely related to the failure of the ray theoretic solution to adequately characterize that neighborhood.

Figure 1 is an example of a caustic in a $v(z)$ medium. The data were created to assure that the upward propagating rays were direct \textit{away} from the $z$-axis in a medium in which the wave speed increases with depth. For some of the rays, the plot extends to their turning point; for others, the turning points are beyond the transverse range of the plot. The theory

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A family of rays in a $v(z)$ medium forming a caustic in the subsurface.

guarantees that the caustic will lie above the turning point. The surface data at $z = 0$ that generated this ray plot could have arisen from a reflector lying above the depicted turned rays.

The envelope of the rays—the caustic—is clearly visible in the plot. It has two branches that meet on the $z$-axis. Near the caustic, on one “upper” or “lit” side of a branch, there are two rays passing through each point that touch that branch of the caustic nearby. See the enlargement in Figure 2 for an enhanced view of the neighborhood of the caustic.

If we think of travel time progressing in the upward direction on each ray trajectory, one of the rays has not yet arrived at the caustic; the other one has passed beyond the caustic. The ray pair coalesces to a single ray as the observation point moves to the caustic.

There is also a third ray, progressing to the complementary branch of the caustic with increasing time. In the enlargement in Figure 2, this is the ray that is nearly orthogonal to the caustic. As the observation point moves in toward the cusp of the caustic on the $z$-axis, all three rays tend to coalesce into the one vertical ray along the $z$-axis.

The density of rays is an indicator of increased amplitude in the near-caustic region, with even greater increase at the cusp, which is an imperfect focal point of the rays. On the caustic, ray theory predicts infinite amplitude because the ray Jacobian appearing in the denominator of the amplitude formula is equal to zero on the caustic. This is a manifestation of the breakdown of ray theory and not a characterization of the exact solution of the wave equation at the caustic.

The objective of this paper is to describe the asymptotic solution of the one-way wave equation in the neighborhood of the smooth part of the caustic (away from the cusp) for a prescribed data set at $z = 0$. Those data have a ray family as indicated in Figure 1.

The analysis proceeds as follows. We prescribe data in 2D at $z = 0$ for the one-way wave equation in $(x, z, \omega)$ that supports upward traveling waves. The solution to this problem is derived by Fourier transform in $x$, followed by exact solution of the resulting ordinary
differential equation in $z$. The analytical solution in $(x, z, \omega)$ is then written as an inverse Fourier transform in the dual variable to $x$, namely, $\omega p_1$.

We then apply the stationary phase to this inverse transform. We find that there are two nearby stationary points when the observation point is near to the lit side of a caustic. Each stationary point describes a ray. The stationary points are the $x$-components of the slowness along the ray; note that in a $v(z)$ medium, this slowness is a constant along a ray and only the vertical component of slowness varies with depth. For the specific example of the figure, there is a third stationary point well-separated from the first two, until the observation point moves towards the cusp. The two nearby stationary points coalesce when the observation point is on the smooth part of the caustic, leading to a single higher (second) order stationary point; hence the different asymptotic order in the (dimensionless) high frequency.

One can see from either figure that the third ray—the one that is tangent to the complementary branch of the caustic—has a transverse slowness value/stationary point well-separated from the pair of this discussion. As the observation point moves to the cusp, all three slowness values/stationary points coalesce. Now the order of the stationary point is higher (third) still and the asymptotic solution changes order in high frequency yet again.

So, one can be content with three results derived by the method of stationary phase to describe the wave form. One is a representation valid when the observation point is bounded away from the caustic. For the given example, the leading order asymptotic solution is expressed as a sum of contributions from the three distinct simple stationary points. Two are attached to rays which, in the upward propagating direction, have not yet reached the
caustic. The third is attached to a ray that has passed tangent to the caustic and proceeded beyond the caustic. The stationary phase formula for this case introduces a factor of $1/|\omega|^{1/2}$ in each of the three contributions to the asymptotic expansion of the wave field for this case of three “simple” stationary points.

The second representation occurs when the observation point is on the caustic; two stationary points–transverse slownesses–have coalesced to yield a single higher order stationary point. The wave field is then described as a sum of two contributions, one from this higher order stationary point and one from the still-simple stationary point attached to the ray that is propagating towards the complementary branch of the caustic. The stationary phase formula for the coalesced pair of stationary points introduces a factor of $1/|\omega|^{1/3}$, while the multiplier for the simple stationary point remains $1/|\omega|^{1/2}$.

The third representation occurs right at the cusp, where all three stationary points–transverse slownesses–have coalesced to yield a single third order stationary point. Here, the stationary phase formula introduces a multiplier of $1/|\omega|^{1/4}$.

We will not deal here with the neighborhood of the cusp. Thus, we will be concerned with the transition that occurs in the first two representations described above. There is an alternative to these separate representations, namely, an asymptotic expansion that remains uniformly valid for observation points in a region around the smooth part of the caustic, reducing to the first representation when the observation point is bounded away from the caustic and reducing to the second representation when the stationary point is on the caustic. In this discussion, we completely disregard the contribution from the third ray with stationary point bounded away from the two nearby stationary points that are allowed to coalesce.

These various cases of transition in order of stationary point have been well-studied. For uniform asymptotic expansions with two nearby stationary points, see, for example, Chester, et al, [1957], Friedman [1959], and Bleistein and Handelsman [1986]. The representation involves the Airy function and its derivative, which are the simplest functions that can effect this transition as the observation point “moves” in a region around the smooth part of the caustic. We will show here how the Fourier representation of the wave field fits the requirements to assure such a uniform expansion for the solution of our one-way wave equation.

There is also a parallel study of the extension of ray theory to the neighborhood of caustics [Kravtsov, 1964a, 1964b; Ludwig, 1966]. Using the asymptotic theory for nearby stationary points as a point of departure, this theory assumes a wave form described in terms of the Airy function and its derivative. The ties between the argument of the Airy functions, attendant phase shift and amplitudes, and the travel time and amplitude of the \textit{WKBJ} solution form are revealed in the analysis. The caustic is a curve or surface where the Jacobian of ray theory is zero, making the \textit{WKBJ} approximation invalid there. This can be seen again from the figures; the cross-section of a ray tube becomes zero on the caustic. The Jacobian is a characterization of that cross section for differential changes in the ray parameter.

On the other hand, the amplitude of the Airy function and its derivative remain finite on the caustic; the argument of these functions is zero there and these special functions cannot be replaced by their asymptotic expansions. Away from the caustics, asymptotic expansion of the Airy functions restores the \textit{WKBJ} or ray theoretic wave form.
There is, of course, a further generalization that characterizes the region of the cusp, but that is beyond the scope and interest of the present discussion. See, for example, Bleistein [1967] and Ursell [1972]. That analysis is for the case of more than two coalescing stationary points.

We will show that the inverse transform representation of the solution mentioned above has exactly the features of coalescing stationary points as described in the cited references on integrals with two nearby stationary points. Thus, we can write down the representation of the uniform asymptotic expansion in terms of the Airy function and its derivative by applying the established theory.

2 Fourier Conventions and Some of Their Consequences.

Here, we describe the Fourier conventions to be used in the analysis of our sample problem, below. We will consider problems in two spatial dimensions, \((x, z)\) and time \(t\). Thus, the solution of the one-way wave equation is expressed as \(W(x, z, t)\). We will not bother with tilde’s and hat’s for Fourier transform; the variables in question should be read from the context. Thus, for example, we write

\[
W(k_1, z, \omega) = \int_{-\infty}^{\infty} dx \int_0^\infty W(x, z, t) \exp\{-i(\omega t - k_1 x)\}, \tag{1}
\]

and then

\[
W(x, z, t) = \frac{1}{(2\pi)^2} \int_{\Gamma} d\omega \int_{-\infty}^{\infty} dk_1 W(k_1, z, \omega) \exp\{i(\omega t - k_1 x)\}. \tag{2}
\]

Here, for causality, \(\Gamma\) is a contour below all singularities of \(W\) in the complex \(\omega\)-plane; typically, we arrive at an integral along the real axis only by analytical continuation achieved by moving the contour \(\Gamma\) up to the real axis from below.

The structure in this last equation suggests that \(W\) might be thought of as a superposition of simple waveforms such as

\[
W_\omega(x, z, t) = A(x, z, t) \exp\{i(\omega - \tau(x, z))\}. \tag{3}
\]

With this sign convention,

\[
\nabla \tau = (p_x, p_z) \tag{4}
\]

points in the direction of increasing time along the ray. We will need some care to define the travel time appropriately to adhere to this sign convention below.

We will not have need to return to the time domain. Thus, of interest to us will be the function \(W(k_1, z, \omega)\) and its partial inverse transform,

\[
W(x, z, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_1 W(k_1, z, \omega) \exp\{-ik_1 x\}. \tag{5}
\]

Further, we prefer to work in a spatial Fourier variable that has the dimension of slowness, rather than the dimension of inverse length associated with \(k_1\). To this end, we introduce the Fourier variable \(p_1\) by setting

\[
k_1 = \omega p_1. \tag{6}
\]
(It will turn out later that the stationary value of $p_1$ is, indeed, $p_x$, but that remains yet to be seen.)

This new Fourier variable modifies the partial inverse transform for $W$ in (5) to

$$W(x, z, \omega) = \frac{|\omega|}{2\pi} \int_{-\infty}^{\infty} dp_1 W(\omega p_1, z, \omega) \exp\{-i\omega p_1 x\}. \quad (7)$$

The introduction of the factor $|\omega|$ naturally arises when one examines the limits in the two separate cases, $\omega > 0$ and $\omega < 0$. Of course, this trick fails for $\omega = 0$. However, below we will be interested in “high frequency” asymptotics for which $|\omega|$ is bounded away from zero.

We now have the Fourier conventions in place for the discussion below.

### 3 The Differential Equation for $W(p_1, z, \omega)$ and its Exact Solution

Here we present the problem to be analyzed asymptotically in later sections. That is, we state the differential equation in the Fourier domain, write down its exact solution modulo “final data” at $z = 0$ and then determine those data for a specific data choice in the $(x, z)$-domain.

We start from a wave equation in 2D with a depth dependent $[v(z)]$ wave speed. Thus, after applying Fourier transform in time and $x$, we consider a function $W(k_1, z, \omega)$ that satisfies the one-way upward propagating wave equation,

$$\frac{dW}{dz} - i k_3 W + \frac{1}{2k_3} \frac{dk_3}{dz} = 0, \quad (8)$$

as found in Zhang et al [2003], among other sources by some or all of those authors. Here,

$$k_3 = \frac{\omega}{v(z)} \sqrt{1 - \frac{(vk_1)^2}{\omega^2}}. \quad (9)$$

The change of Fourier variable from $k_1$ to $p_1$ defined by (6) recasts the last two equations as

$$\frac{dW}{dz} - i \omega p_3 W + \frac{1}{2p_3} \frac{dp_3}{dz} = 0, \quad (10)$$

and

$$p_3 = p_3(p_1, z) = \sqrt{p^2(z) - p_1^2}, \quad p(z) = \frac{1}{v(z)}, \quad (11)$$

with $p(z)$ being the slowness of the medium.

As noted by G. Zhang [1993], this equation has an exact solution in WKBJ form. Indeed, this was the motivation for introducing this equation for a fully heterogeneous medium–$v = v(x, z)$ or $v = v(x, y, z)$–with $k_1$ or $(k_1, k_2)$ representing derivatives and the various functions in the differential equation (10) being pseudo-differential operators. For the $v(z)$ medium considered here, the exact solution is

$$W(p_1, z, \omega) = W(p_1, 0, \omega) \frac{\sqrt{p_3(p_1, 0)}}{\sqrt{p_3(p_1, z)}} \exp\left\{i\omega \int_0^z p_3(p_1, z')dz'\right\}. \quad (12)$$
We need only to determine $W(p_1, 0, \omega)$ to complete the solution. We tend to think of this as boundary data for a problem in the spatial domain. However, space and time are indistinguishable in 1D. If we think, instead, in terms of the phase defining a travel time\footnote{We reserve $\tau$ for the full travel time as a function of $(x, z)$, below.}

\[ \sigma = \sigma_0 - \int_0^z p_3(p_1, z')dz', \]

then, we can view the missing data as providing “final values” for the equivalent temporal problem. Not surprisingly, the change to this independent variable leads to the solution,

\[ W(p_1, z(\sigma, p_1), \omega) = W(p_1, 0, \omega) \left( \frac{p_3(p_1, 0)}{\sqrt{p_3(p_1, z)}} \right) \exp \{-i\omega \sigma \}. \]  

(14)

Furthermore,

\[ \frac{d\sigma}{dz} = -p_3(z, p_1), \]  

(15)

which indicates that $z$ decreases with increasing time. This is indeed an up-going wave with only its final values to be determined. Further, those final values will define the “time-shift” $\sigma_0 = \sigma_0(p_1)$, as we will see below.

Equations (12-15) provide the exact solution to the problem for the differential equation (8) modulo an analytical expression for the final data.

### 3.1 Determining the final data for $W$

Here we provide the final data for the differential equation (8). Figure 1 gives us a strong indication of how to create the final data in the $(x, z)$ plane, so let us start there. The rays are directed towards decreasing $z$ and that will be assured by having $p_z$ negative. Note that in the spatial transform domain, we have assured that $p_3$ is negative and, as previously indicated, we expect that the stationary value of this variable will be $p_z$. From Figure 1, we can also see that $p_x$ is negative when $x$ is negative and $p_x$ is positive when $x$ is positive. The simplest function to have this property is $ax$, with $a > 0$. Hence we want data for which $p_x = ax$. The amplitude is not important to this discussion, we will set it equal to one. Therefore, we define the final data to be

\[ W(x, 0, \omega) = \exp\{-i\omega ax^2/2\}, \quad a > 0. \]  

(16)

For these data, the final travel time is

\[ \tau(x, 0) = ax^2/2, \quad \text{with} \quad \frac{\partial \tau(x, 0)}{\partial x} = p_x(x, 0) = ax. \]  

(17)

To determine the final data in the $(p_1, z, \omega)$-domain, it is necessary to take the Fourier transform of these data, namely,

\[ W(p_1, 0, \omega) = \int_{-\infty}^\infty \exp\{-i\omega(ax^2/2 - p_1 x)\}W(x, 0, \omega)dx. \]  

(18)
This integration can be carried out in closed form to yield

\[ W(p_1, 0, \omega) = \sqrt{\frac{2\pi}{|\omega|a}} \exp\{i\omega p_1^2/2a - i\pi/4 \cdot \text{sign}(\omega)\}. \] (19)

Given these data, we can complete the solution representation in the solution Equation (12) of the ordinary differential equation, (10). The result is

\[ W(p_1, z, \omega) = \sqrt{\frac{2\pi p_3(p_1, 0)}{|\omega|a p_3(p_1, z)}} \exp\left\{i\omega \left(\int_0^z p_3(p_1, z')dz' + p_1^2/2a\right) - i\pi/4 \cdot \text{sign}(\omega)\right\}. \] (20)

With this solution completed, we are prepared to find a representation in the \((x, z, \omega)\) domain by applying the inverse Fourier transform defined in (7) to this last representation of \(W\):

\[ W(x, z, \omega) = \sqrt{\frac{|\omega|}{2\pi a}} \exp\left\{-i\pi/4 \cdot \text{sign}(\omega)\right\} \int_{-\infty}^{\infty} dp_1 \sqrt{\frac{p_3(p_1, 0)}{p_3(p_1, z)}} \exp\{i\omega \Phi(x, z, p_1)\}. \] (21)

Here,

\[ \Phi(x, z, p_1) = \int_0^z p_3(p_1, z')dz' + p_1^2/2a - p_1 x. \] (22)

In the next section, the method of stationary phase will be applied to this solution representation.

4 Asymptotic Analysis of \(W(x, z, \omega)\)

The next step in our discussion is to apply the method of stationary phase to the solution represented by Equations (21) and (22). To that end, we need the \(p_1\)-derivatives of the phase function \(\Phi\), defined in Equation (22). Those derivatives are

\[
\begin{align*}
\frac{\partial \Phi}{\partial p_1} &= -\int_0^z \frac{p_1}{p_3(p_1, z')}dz' + \frac{p_1}{a} - x, \\
\frac{\partial^2 \Phi}{\partial p_1^2} &= -\int_0^z \frac{p_3^2(z')}{p_3^3(p_1, z')}dz' + \frac{1}{a} \\
\frac{\partial^3 \Phi}{\partial p_1^3} &= -3 \int_0^z \frac{p_3^2(z')p_1}{p_3^5(p_1, z')}dz' \\
\frac{\partial^4 \Phi}{\partial p_1^4} &= -3 \int_0^z p_3^2(z') \left\{ \frac{1}{p_3^3(p_1, z')} + \frac{5p_1^2}{p_3^5(p_1, z')} \right\} dz'.
\end{align*}
\] (23)

The stationary value of \(p_1\) is determined by setting the first derivative here equal to zero; that is,

\[ x = \frac{p_1}{a} - \int_0^z \frac{p_1}{p_3(p_1, z')}dz'. \] (24)

We cannot solve this equation analytically for \(p_1(x, z)\) even for this simple example; the equation is transcendental in \(p_1\). Instead, we must content ourselves with a parametric representation of the solution. Each choice of \(p_1\) defines a curve in \((x, z)\)-space, namely, the ray of ray theory. The final value of \(x\) on the ray (at \(z = 0\)) is just \(p_1/a\). For increasing \(z\) (moving backwards in time along the ray), we see that \(x\) increases when \(p_1\) is negative and
it decreases when $p_1$ is positive; $p_1 = 0$ is a somewhat pathological case, although the ray is well-defined–$x = 0$.

The reader should compare this description to the rays depicted in Figure 1. Indeed, the figure was generated in units of km and s with values

$$p^2(z) = p_0^2 - \alpha^2 z^2, \quad p_0 = .5, \quad \alpha = .1, \quad a = .25,$$

\[ \Rightarrow v^2(z) = \frac{1}{p_0^2 - \alpha^2 z^2}, \quad v(0) = 2. \tag{25} \]

The range of final $x$-values was -1.8 to 1.8 in steps of .1. For this range and the given value of $a$, the range of $p_1 = ax$ is - .45 to .45. In this case $p_3(0, p_1)$ will always be real with minimum value .218 for $x = 1.8$ and $z = 0$. The steepest rays turn near $z = 5$ with the turning point trending upward in $z$ and outward from the center in $x$ with larger magnitude of $p_1$. The cusp of the caustic is just above $z = 2$ and is bounded well-away from the turning points.

Before proceeding to the result of the stationary phase approximation of the integral representation of $W$ as defined by Equations (21) and (22), we remark that

$$\tau(x, z) = -\Phi(x, z, p_s(x, z)) \tag{26}$$

is the travel time along the ray defined by Equation (24). Taking the $x$-derivative here, we find that

$$p_x = \frac{\partial \tau}{\partial x} = -\frac{\partial \Phi(x, z, p_s(x, z))}{\partial x} + \left. \frac{\partial \Phi(x, z, p_1)}{\partial p_1} \right|_{p_1 = p_s(x, z)} \left. \frac{\partial p_s(x, z)}{\partial x} \right|_{p_1 = p_s(x, z)} \equiv p_x = p_s(x, z), \tag{27}$$

The first term in the rightmost expression here is readily calculated from the definition of $\Phi$ in Equation (22) to be just $p_s(x, z)$, and hence that term can be disregarded. Thus, we find that

$$\frac{\partial \tau}{\partial x} \equiv p_x = p_s(x, z), \quad (28)$$

which is the stationary value of $p_1$. This is the result claimed earlier. Of course, it then easily follows from the definition of $p_3$ in Equation (11) that

$$\frac{\partial \tau}{\partial z} \equiv p_z = -p_3(p_s(x, z), z). \tag{29}$$

For the case of a simple stationary point ($\Phi'' \neq 0$), we are now prepared to apply the stationary phase to the integral representation of the wave field defined by Equations (21) and (22). The result is a sum of three terms of the form

$$W_{ssp}(x, z, \omega) = \frac{\exp\{i\omega \Phi(x, z, p_s) + i\pi/4 \cdot \text{sign}(\omega)[\text{sign}(\Phi''(x, z, p_s(x, z))) - 1]\}}{p_3(p_1, z)} \sqrt{a|\Phi''(x, z, p_s(x, z))|},$$

subject to $p_s$ being coupled to the spatial coordinates through the stationary phase condition—equivalently, the ray equation—(24).

We remark that after this long journey the final result is of order one in $|\omega|$ as was the final data for the problem given by Equation (16). Further, the phase shift here, $i\pi/4 \cdots$, is
zero when \(\text{sign} [\Phi''(x, z, p_s(x, z))] = +1\). From the equation for \(\Phi''\) in Equation (23) we can see that this is the case for \(z\) “small enough,” namely, near zero. The phase shift that occurs when \(\Phi''\) passes through zero is related to the analysis of the caustic, where \(\Phi'' = 0\), coming next.

Note that when the observation point passes to the other side of the caustic, the phase adjustment in this last equation changes from zero to \(-\pi/2\). Thus, the phase shift is \(-\pi/2\) when the observation point moves in the direction of decreasing time. More to the point, the phase shift is \(+\pi/2\) when moving along the ray in the direction of increasing time. This is consistent with the general rule about phase shifts when a ray passes through a caustic.

Let us summarize what has been derived here. When the three stationary points of the Fourier representation of the solution given by Equations (21) and (22) are separated, the asymptotic solution is given by a sum of terms given by Equation (30), each subject to a stationarity condition which is equivalent to the ray equation, namely, Equation (24).

### 4.1 Turning Points and Caustics

Here, we will discuss the asymptotics in the neighborhood of the caustic curve, defined by \(\Phi' = 0\). However, we cannot study this curve without also addressing the turning point(s) where \(p_3(p_1, z)\) as defined in Equation (11) is zero. We begin with the latter.

For each ray, that is, for each choice of \(p_1\), we introduce the notation \(z_T(p_1)\) for the choice of \(z\) that makes \(p_3\), defined in Equation (11), equal to zero:

\[
p_3(p_1, z_T(p_1)) = \sqrt{p^2(z_T(p_1)) - p_1^2} = 0. \tag{31}
\]

From the ray equation (24), we can see that \(dx/dz\) is infinite at the turning point. More familiarly, we can reinterpret this as \(dz/dx = 0\) at the turning point; that is, the ray is horizontal there. Some turning points can be seen in Figure 1. Further, it can be seen that the depth of the turning point (and also the \(x\)-coordinate at the turning point) is different for each choice of \(p_1\). To move further back along the ray—that is, to move upward in \(z\) to times earlier than the time of turning on the ray—it is necessary to change the sign of the square root that defines \(p_3\): this continuation is not relevant to the current discussion, except that it be understood conceptually to facilitate the discussion below.

Note that the integral defining the ray in Equation (24) becomes “improper” in a classical calculus sense, but remains convergent as long the slowness has a nonzero first derivative there. (The singularity is a simple form, \(\sqrt{\text{const} \cdot (z - z_T)}\), near the turning point and known to be integrable.) On the other hand, \(\Phi''\) as defined in Equation (23) is an integral that becomes infinite when \(z \to z_T\). This calls into question the approximate wave field contribution \(W_{ssp}\) given in Equation (30). As a last subsection of this discussion, we will show that the product \(|\Phi''(x, z, p_s(x, z))|p_3(p_s, z)\) appearing in the denominator has a finite limit when \(z \to z_T\). Further, this limit provides the “right” asymptotic approximation, but that will not be shown in this paper. This pathology is a consequence of using \(z\) as the running variable along each ray. For each \(z\)-value, there are two \(x\) values, one before the ray turns and the other after the ray turns. Thus, \(x = x(z, p_s)\) is a double valued function of \(z\) for each \(p_s\). This is a general problem when rays turn, not peculiar to the \(v(z)\) case.

Next, let us consider the point where the ray is tangent to the caustic—the point on the ray where \(\Phi'' = 0\). We will show that for this example that point will lie above the turning
point on the ray, for which we will denote the $z$-coordinate by $z_T$. For this, we need only examine the expression for the second derivative of $\Phi$ in Equation (23). Note that it is positive, equal to $1/a$ when $z = 0$. Further, it approaches $-\infty$ as $z$ approaches $z_T$ along the ray. Thus, somewhere in between, that is, for some $(x, z) = (x_C, z_C)$, it must be true that the second derivative, $\Phi'' = \Phi''(x_C, z_C, p_s)$, is equal to zero:

$$\Phi''(x_C, z_C, p_s) = 0, \quad 0 < z_C < z_T.$$  (32)

The argument presented here is based on the *intermediate value theorem* of advanced calculus.

Now, in the neighborhood of this point, we use Taylor’s theorem to expand the phase function $\Phi$.

$$\Phi(x_C, z_C, p_s, p_1) \approx \Phi(x_C, z_C, p_s, p_s) + \left[\frac{(p_1 - p_s)^3}{3!} \frac{\partial \Phi(x_C, z_C, p_s, p_s(x, z_C))}{\partial p_1}\right]_{p_1 = p_s}.$$  (33)

The terms in the Taylor series involving $\Phi'$ and $\Phi''$ do not appear because both of those derivatives are zero at $p_1 = p_s$. Thus, the phase is approximately cubic as long as $\Phi'''(x_C, z_C, p_s, p_s)$ is not zero there. From its definition in (23) we see that this third derivative can only be zero when $p_1 = 0$. This was previously noted as a special case; $p_1 = 0$ defines the ray along the $z$-axis passing through the cusp of the two branches of the smooth caustic.

Excluding observation points near the cusp, we conclude that $\Phi$ is approximately cubic in $p_1$ near the caustic and, more importantly, $\Phi'$ is *approximately quadratic* near the caustic. That means that the equation $\Phi' = 0$ will have two solutions, two nearby choices of $p_1$. This can be seen in the configuration of the rays in Figure 2. Choose a point on the “lit” side of the caustic and observe two rays passing through it with nearby values of $p_1$, that is, nearly equal slopes (or dips). In the direction of increasing time, it can be seen that one ray has passed through the caustic and is traversing towards the upper surface. For this contribution to the asymptotic expansion at that point, $\Phi'' > 0$; see the discussion below Equation (30). The other ray has not yet passed through the caustic, so that for this ray, $\Phi'' < 0$. How do we know that $\Phi''$ transitions in this manner? Because $\Phi'''$ is of one sign and nonzero, except on the cusp ray, as can be confirmed from its definition in Equation (23). This assures us that $\Phi''$ is a monotonic function on all but the cusp ray. In the limit, as the observation point moves to the caustic surface, the two rays merge into one ray tangent to the caustic at the observation point.

By a similar analysis, near the cusp, the Taylor expansion is approximately quartic; the fourth derivative given in Equation (23) can be seen to be nonzero for all choices of $p_1$. Hence, there are three nearby stationary points and three nearby rays as can be seen in Figure 1. Two of the rays are approaching different branches of the smooth caustic from below, while the third ray has passed through one or the other branch of the smooth caustic, depending on the sign of $x$ at the observation point.

### 4.2 $W$ on the caustic

Here, we will make some remarks about the smooth part of the caustic curve of Figure 1. For a second order stationary point, the first and second derivative of the phase vanish and
the third derivative is nonzero. We have seen that this arises for observation points on the smooth part of the caustic. Where the order of magnitude of the contribution from simple stationary point has a factor $1/|\omega|^{1/2}$, for the second order stationary point the contribution has a multiplier $2^{-1/3}$. 

We can see here that the second multiplier is larger than the first one by the power $|\omega|^{1/6}$. This is usually not a large quantity in seismic problems; in dimensionless form, “high frequency” is characterized by dimensionless high frequency parameter values between 3 and 20. Even for a value of 64, this increase in amplitude is only a factor of two! Hence, this is not as significant of a change in the magnitude of the amplitude in seismic modeling and inversion as it is in the optics examples or the ship’s wake example of the introduction.

Nonetheless, it must be addressed because it does cause the ray theory formalism to break down; ray theory cannot accommodate a change in asymptotic order in $\omega$.

### 4.3 Uniform expansions near caustics

As noted in the introduction, there are extensions of asymptotic expansions that provide the transition in powers of $\omega$ arising as stationary points (or stationary points and branch points!) coalesce. The idea behind the theory relates to integral representations of special functions. In particular, we look for special functions whose integrals involve imaginary polynomial exponents. These exponents should have the same local behavior—when the observation point is in the caustic region—as the more general phase in the integral representation of the wave field. For example, near the smooth part of the caustic, we would want a special function whose integral representation involves a cubic polynomial exponent. In this case, the first derivative will be locally quadratic and have two solutions, hence two stationary points defining two rays. Similarly, near the cusp, we would like to have a quartic polynomial in the exponent of the integral representation.

Below, we discuss only the smooth caustic case. The special function in question is related to the Airy function, usually denoted by $\text{Ai}$ or $\text{Bi}$. Here, we use a slight modification of the Airy function. It is a solution of the ordinary differential equation

$$V''(s) + sV(s) = 0, \quad \text{with} \quad V(s) = \text{Ai}(-s) \quad \text{or} \quad \text{Bi}(-s).$$

(34)

Note that if the multiplier $s$ were instead a positive constant, say $a^2 > 0$, then the fundamental linearly independent solutions would be any two among, $\exp\{\pm ias\}$, $\sin as$, $\cos as$. These simple functions nearly characterize the behavior of $V(s)$ for large positive $s$ in the sense that the $V$’s are oscillatory, just as the simpler functions are. However, their phases are not linear in $s$ and the asymptotic expansions for large positive $s$ are of the form $s^{-1/4} \exp\{\pm 2i s^{3/2}/3\}$, $s^{-1/4} \cos\{2s^{3/2}/3\}$ or $s^{-1/4} \sin\{\pm s^{3/2}/3\}$

Now suppose that $s$ were a negative constant, say, $-a^2 < 0$. Then two linearly independent solutions to this equation would the exponentials, $\exp\{\pm as\}$, one growing exponentially at infinity, the other decaying exponentially. (Of course, we could use $\sinh$ and $\cosh$, but both of these are exponentially growing and hence less useful as a pair in physical problems

\[\text{In both of these cases, and below, the frequency must be scaled by an appropriate length scale and wave speed to make these powers of dimensionless quantities. We forego that detail here.}\]
where we are interested in solutions that are bounded at infinity.) Qualitatively, the solutions to the differential equation (34) are also growing and decaying exponentials for large values negative values of $s$, except that they grow and decay as $|s|^{-1/4} \exp\{\pm 2/3|s|^{3/2}\}$.

More surprisingly, solutions that behave like the growing or decaying exponential for large negative $s$ each behave like sines (or cosines) with phase shifted arguments for large positive $s$. On physical grounds, we would expect that the Airy function which arises in wave-like applications such as ours, would decay exponentially on the “dark side” of the caustic and would involve two wave-like functions on the lit side; this is exactly the way it turns out. More specifically, the Airy function of interest to us is $V(s) = \mathrm{Ai}(-s)$ and has the integral representation

$$V(s) = \mathrm{Ai}(-s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{i[\sigma^3/3 - s\sigma]\} d\sigma. \quad (35)$$

The integral on the right in this equation is conditionally convergent by virtue of the oscillation of $\sigma^3$, which becomes progressively more rapid (as compared to linear oscillation) as $\sigma \to \pm \infty$.

Figure 3 is a graph of the Airy function $V(s)$ for the range $-10 < s < 20$. We see here exactly the type of desired behavior described above. This function is sinusoidal–wave-like–for $s$ positive and decays exponentially for $s$ negative. The caustic will be located at $s = 0$ in the application. In the frequency domain, the solution consists of two imaginary exponentials of opposite signs, leading to one wave approaching the caustic, the other leaving the caustic as time progresses.

The integral representation of the Airy function in Equation (35) does not have the general form of the phase in the solution representation of Equation (21). We can arrive at that form for the integral above in two steps. We consider only the case $s > 0$ and introduce a change of variable of integration from $\sigma$ to $u$ defined by $\sigma = u\sqrt{s}$. In this case, the integral representation of $V(s)$ in Equation (35) is

$$V(s) = \frac{\sqrt{s}}{2\pi} \int_{-\infty}^{\infty} \exp\{is^{3/2}[u^3/3 - u]\} du. \quad (36)$$
The multiplier that we really want in the phase in order to match the form of the phase in Equation (21) is \( \omega \). We achieve that by rescaling the variable of integration \( \sigma \) and redefining \( s \) as follows:

\[
s^{3/2} = \omega \nu^{3/2} \quad \iff \quad s = \omega^{2/3} \nu, \quad \sigma \Rightarrow \sigma \omega^{1/3},
\]

in which case the previous equation for \( V \) is recast as

\[
V(\omega^{2/3} \nu) = \frac{\omega^{1/3}}{2\pi} \int_{-\infty}^{\infty} \exp \left\{ i \omega [\sigma^{3/2} - \nu \sigma] \right\} d\sigma.
\]

In this discussion, we have assumed that \( \omega > 0 \). In fact, it can be easily verified that \( V \) is a real valued function for real \( s \) and replacing positive \( \omega \) by its negative will not change the value of the function; it only changes the integrand to its complex conjugate, leaving the result of integration unchanged. So we will proceed under the assumption that \( \omega \) is positive.

Now let us examine this last representation of \( V \) in Equation (39). It has the form of the integral with oscillatory exponential as in the solution representation, Equation (21), to which we can apply the method of stationary phase. For \( \nu \) positive, there are two stationary points at \( \pm \sqrt{\nu} \), with second derivative nonzero. That is, these are two simple stationary points. As \( \nu \to 0 \), the two stationary points coalesce at zero into a single second order stationary point, whose asymptotic expansion is one term, namely \( V(0) \). That will not be true in general, but this is the simplest function whose behavior mirrors the behavior of the integral in Equation (20); that is, it mirrors the behavior of the solution \( W \) to our final value problem on the lit side and near to the smooth caustic, but away from the cusp.

Application of the method of stationary phase to \( V \) in Equation (38) leads to the following asymptotic expansion:

\[
V(\omega^{2/3} \nu) \sim \frac{\exp \left\{ 2i \omega \nu^{3/2} / 3 + i \pi / 4 \right\} - \exp \left\{ -2i \omega \nu^{3/2} / 3 - i \pi / 4 \right\}}{2i [\omega^{2/3} \nu]^{1/4} \sqrt{\pi}}, \quad \omega \nu^{2/3} >> 1.
\]

To complete the story about the asymptotics of \( V(s) \), we provide the exponentially decaying asymptotic expansion valid for large negative values of its argument, namely,

\[
V(\omega^{2/3} \nu) \sim \frac{1}{2[\omega^{2/3} \nu]^{1/4} \sqrt{\pi}} \exp \left\{ -2\omega |\nu|^{3/2} / 3 \right\}, \quad \omega \nu^{2/3} << -1.
\]

With the integral representation of \( V \) in Equation (39) in place, we can say a little more about why only one exponential arises in this last result. For \( \nu \) negative, the stationary points of the integrand in Equation (39) become imaginary, \( \pm i \sqrt{|\nu|} \). The method of stationary phase now must be replaced by the method of steepest descents [Bleistein and Handelsman, 1986] and the stationary points are now called saddle points; the surface for the real part of the exponent now looks like a saddle in the neighborhood of the stationary point, nee saddle point. This is a method that requires use of complex function theory. The first step in the analysis is to deform the path of integration onto a path through one or both of the saddle points. Cauchy’s theorem for integrals of analytic functions is used to justify this deformation. For this particular case, one can only deform the path of integration, the real \( \sigma \)-axis, on to a path through the saddle point on the positive imaginary \( \sigma \)-axis. The result in the last equation arises as the contribution from that one saddle point.
4.4 \( V' \).

The first derivative \( V' \) also plays a crucial role in the uniform asymptotic expansion. From the definition of \( V(s) \) in Equation (36), the definition of \( V' \) is

\[
V'(s) = \text{Ai}(-s) = -i \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma \exp\{i[\sigma^3/3 - so]\}d\sigma.
\]  

(41)

Again, this is a conditionally convergent integral, so that the formal process does, indeed, provide an integral representation for the derivative. The asymptotic expansion of this function—analogy with the expansion of \( V \), itself, in (39)—is

\[
V'\left(\frac{\omega^2}{3} \nu\right) \sim \left[\frac{\omega^{2/3} \nu}{2\sqrt{\pi}}\right]^{1/4} \left[\exp\left\{2i\omega \nu^{3/2} / 3 + i\pi/4\right\} + \exp\left\{-2i\omega \nu^{3/2} / 3 - i\pi/4\right\}\right], \quad \omega^{2/3} \nu >> 1.
\]  

(42)

Thus, by comparing this result with Equation (39), one can see that in this asymptotic limit, \( \omega^{-1/3} V'(\omega^{2/3} \nu) \) is of the same asymptotic order as \( V(\omega^{2/3} \nu) \). On the other hand, from the exact representation of both of these functions in Equations (36) and (42), \( \omega^{-1/3} V'(0) \) is asymptotically of lower order by \( \omega^{-1/3} \) than \( V(0) \). Consequently, both terms are needed to complete the asymptotic expansion of the wave field in WKBJ-form away from the caustic, but only \( V(0) \) arises in the leading order approximation at the caustic. Thus, an expansion containing both of these functions accomplishes the transition in asymptotic order as the observation point in the \((x, z)\)-domain ranges around the smooth part of the caustic. This is what we mean by a uniformly valid asymptotic expansion. It is an expansion in one parameter—dimensionless high frequency—while another parameter (other parameters) are allowed to vary over a range that causes the asymptotic order of the function to change.

We have now provided a brief introduction into the concepts and special functions needed to describe the wave field in the neighborhood of the smooth part of the caustic.

4.5 Describing \( W \) near the caustic in terms of \( V \).

With the machinery of the previous section in place, we are prepared to describe the solution \( W \) defined by Equations (21) and (22) when the observation point is near the smooth caustic; that is, when the stationary points are near enough to one another that a transitional—uniform—asymptotic expansion is needed to describe the wavefield. Let us return to the integral representation of \( W(x, z, \omega) \) in Equations (21) and (22), with the points on the smooth caustic defined as \( p_s(x, z_C) \) in Equation (32).

In the neighborhood of the caustic, there is one contribution from the stationary point with \( p_s \) corresponding to the ray traveling to the opposite branch of the caustic. Then, there is an additional contribution from the two nearby stationary points that coalesce on the given caustic. For this contribution, the uniform asymptotic expansion, denoted by \( W_C \) has the structure

\[
W_C(x, z, \omega) \sim \omega^{1/6} \sqrt{\frac{2\pi}{a}} \exp\left\{i\omega \Psi - i\pi/4 \cdot \text{sign}(\omega)\right\} \left[A_0 V(\omega^{2/3} \nu) + \frac{B_0}{i\omega^{1/3}} V'(\omega^{2/3} \nu)\right].
\]  

(43)

In order to define the variables in this result, we need a notation that acknowledges that there are two nearby stationary points. We denote those two points by \( p_{\pm}(x, z) \) and
distinguish between them through the condition
\[ \Phi(x, z, p_-(x, z)) > \Phi(x, z, p_+(x, z)). \] (44)

Then, we can define
\[ \Psi = \Phi(x, z, p_-) + \Phi(x, z, p_+), \quad \nu = \left[ \frac{3(\Phi(x, z, p_-) - \Phi(x, z, p_+))}{4} \right]^{2/3}, \] (45)
\[ \Rightarrow \quad 2\nu^{3/2} = \frac{\Phi(x, z, p_+) - \Phi(x, z, p_-)}{2}. \]

Here, the first form of \( \nu \) provides a direct definition of this variable, while the second form provides the expression needed in the asymptotic expansion of \( V' \) in Equation (39) and \( V'' \) in Equation (42). One can check that when \( V \) and \( V'' \) are replaced by their asymptotic expansions in the representation of Equation (43) of \( W_C \), the resulting two wave forms have the travel times of the waves at \((x, z)\) approaching and leaving the caustic.

The amplitudes \( A_0 \) and \( B_0 \) are merely the first terms of an asymptotic series in inverse powers of \( \omega \). These first two coefficients can be derived via the theory presented in Chester et al. [1957] and Bleistein and Handelsman [1986]. The result of the calculations following that theory leads to the following values for the coefficients.

\[ A_0 = \frac{1}{2} \left[ \frac{p_3(p_-, 0)}{p_3(p_-, z)} \sqrt{2v^{1/4}} \frac{\Phi''(x, z, p_-) - \Phi''(x, z, p_+)}{\sqrt{|\Phi''(x, z, p_-)|}} \right], \]
\[ B_0 = \frac{1}{2\sqrt{v}} \left[ \frac{p_3(p_-, 0)}{p_3(p_-, z)} \sqrt{2v^{1/4}} \frac{\Phi''(x, z, p_-)}{\sqrt{|\Phi''(x, z, p_-)|}} - \frac{p_3(p_+, 0)}{p_3(p_+, z)} \sqrt{2v^{1/4}} \frac{\Phi''(x, z, p_+)}{\sqrt{|\Phi''(x, z, p_+)|}} \right]. \] (46)

In these expressions, there is apparently singular behavior in the limit as the stationary points coalesce; that is, as the observation point moves to the caustic. We know that \( \nu \to 0 \) and \( \Phi''(x, z, p_\pm) \to \) in this limit. However, one can show that the quotient remains finite and nonzero by writing all of the coefficients as Taylor series in the distance from the caustic:
\[ \frac{\sqrt{2v^{1/4}}}{|\Phi''(x, z, p_\pm)|} \to \left| \frac{2}{\Phi''(x, z, p_C)} \right|^{1/3}. \] (47)

Further, the expression for \( B_0 \) has a difference quotient—the square bracketed term divided by \( \sqrt{\nu} \)—with both numerator and denominator approaching zero as the observation point approaches the caustic. Again the quotient is finite, approaching a derivative of the average in \( A_0 \) with respect to the argument \( \sqrt{\nu} \).

To explain these limits in greater detail requires a more in-depth discussion of the derivation of the asymptotic expansion than is worthwhile here. However, we remark that on the caustic, where \( \nu = 0 \), the limiting value of the uniform asymptotic expansion of \( W_C \) given
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by Equation (43) is

\[ W_C(x, z, \omega) \sim \omega^{1/6} \left[ \frac{p_3(p_C, 0)}{2\pi p_3(p_C, z)} \right]^{1/3} \frac{2}{\Phi'''(x, z, p_C)} \exp \left\{ i\omega\Psi - i\pi/4 \cdot \text{sign}(\omega) \right\} . \]  

(48)

As predicted earlier, this contribution on the caustic is \( O(\omega^{1/6}) \), while the isolated simple stationary point contributions given in Equation (30) are \( O(1) \). Thus the amplitude of the solution on the caustic is larger than the amplitude away from the caustic by this same \( O(\omega^{1/6}) \). As noted above, this is not particularly large in seismic applications. However, ray theory or the method of stationary phase for fixed order of stationary point cannot effect the transition in order in \( \omega \) necessary to describe the asymptotic solution in a seamless manner. It is the representation in terms of the Airy function and its derivative, given by Equation (42) that can effect this transition. This is a “high frequency” asymptotic expansion that remains uniformly valid as the distance from the caustic varies in some neighborhood of the smooth part of the caustic, bounded away from the cusp.

In summary, we have provided an asymptotic expansion for \( W_C \) in Equation (43) that remains finite as the observation point approaches the caustic. This improves on the failure of the WKBJ expansion near the caustic. On the other hand, away from the caustic, this representation reverts to the WKBJ expansion when the Airy function and its derivative are replaced by their asymptotic expansions.

This result also justifies seeking an extension of the ray theory for the true-amplitude one-way wave equations to the Airy function form, just as was done by Kravtsov [1964a, b] and by Ludwig [1966]. That derivation is still in process, with the eikonal equation having been derived, but not the modified transport equation.

Figure 4:

400 traces of input data for which the rays are as depicted in Figure 1. Arrival time of the pulse at \( z = 0 \) is \( t = 0.1 + 0.06255 \times 10^{-6} x^2 \).
4.6 Relationship to the solution of the problem for the full wave equation.

The exact solution of the ordinary differential equation, (10), is the leading order asymptotic solution in the same Fourier domain for the full wave equation that would be derived by the WKBJ method. Hence, the amplitudes and phases of the solution at the stationary points for this one-way wave equation agree with the amplitudes and phases for the leading WKBJ solution to the full wave equation. In computing the elements of the uniform asymptotic expansion, $W_C$, given by Equation (40), we carried out the same computations on the same travel times and phases as we would for the uniform asymptotic expansion of the solution to the full wave equation. Consequently, the result that we derive is exactly the same as the one for the full wave equation.

![Figure 5](image)

Figure 5:

a: Real part of the back-projected wave at 20hz. Note the match of the caustic with the rays of Figure 1. b: Imaginary part of the back-projected wave.

5 Numerical confirmation for this example.

In addition to this analytical solution we solved the same problem numerically to confirm the asymptotic analysis. Figure 4 shows the observed data for the numerical test. These data represent a band limited delta function whose arrival at the upper surface is delayed quadratically with offset from the center of the acquisition range. The output is shown as wave fronts at 20hz in Figures 5a and Figures 5b. They clearly show the same caustic as the rays of Figure 1 show, now through the high intensity amplitude on the caustic. The wave fronts that have passed through the caustic are convex-upward; the wave field from outside the caustic region is convex-downward. The transition occurs at the caustic. The continuation of the convex-downward propagating wave in the V-shaped region formed by the caustic is best seen in the interference pattern near the cusp, displayed in an enlargement in Figure 6. Note that the wave field remains finite, although certainly stronger near the
smooth part of the caustic, where we generated an analytical uniform expansion, and near the cusp, where we did not generate a uniform analytical result.

6 Summary and conclusions.

We have presented an example of a smooth caustic for the true-amplitude one-way wave equation in 2D that governs the propagation of up-going waves. The underlying trick was to introduce final data for which the back-propagation to the interior of the domain starts the rays off in such a manner that the dip of the rays becomes progressively more shallow as the starting point moves away from the origin.

For this example, the solution representation by Fourier inversion of a problem for the true-amplitude one-way wave equation produces the “right” uniform asymptotic expansion in the neighborhood of the smooth caustic. That expansion starts out with a linear combination of an Airy function and its derivative. In contrast to ray theory, the coefficients in the expansion–analogous to the amplitude in ray theory–remain bounded when the observation point moves to the caustic. The caustic itself is characterized by the argument of the Airy function being zero there.

This result strongly suggests that an extension of the ray theory formalism to deal properly with smooth caustics ought to work. Experience with generating extensions of ray theory in the neighborhood of other anomalies of rays suggests that formalisms for edge diffraction, head waves and shadow boundaries arising from wedges (pinch-outs) and slip faults ought
to be possible, as well.

References


