Amplitude calculations for 3-D Gaussian beam migration using complex-valued traveltimes

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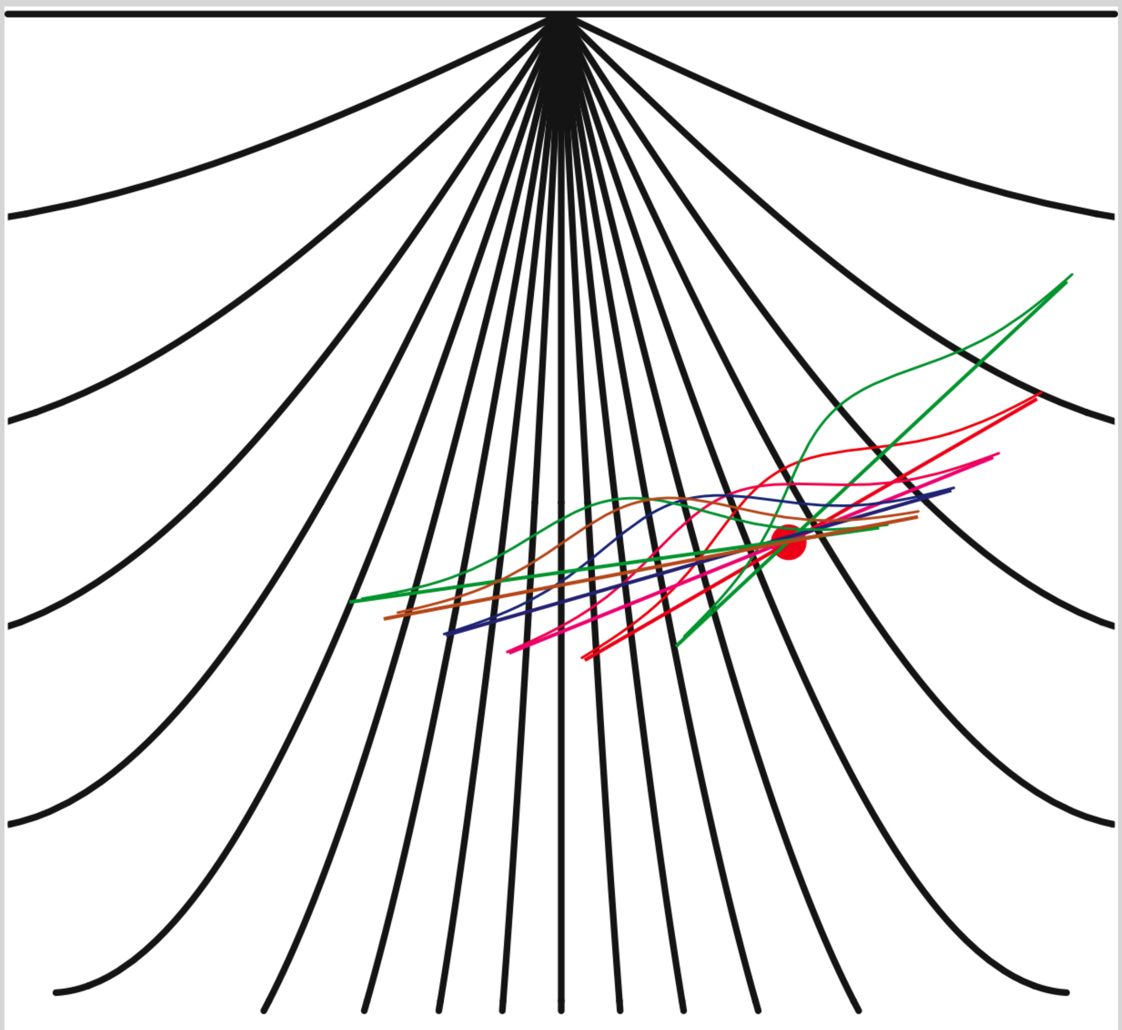
***Presenter** 

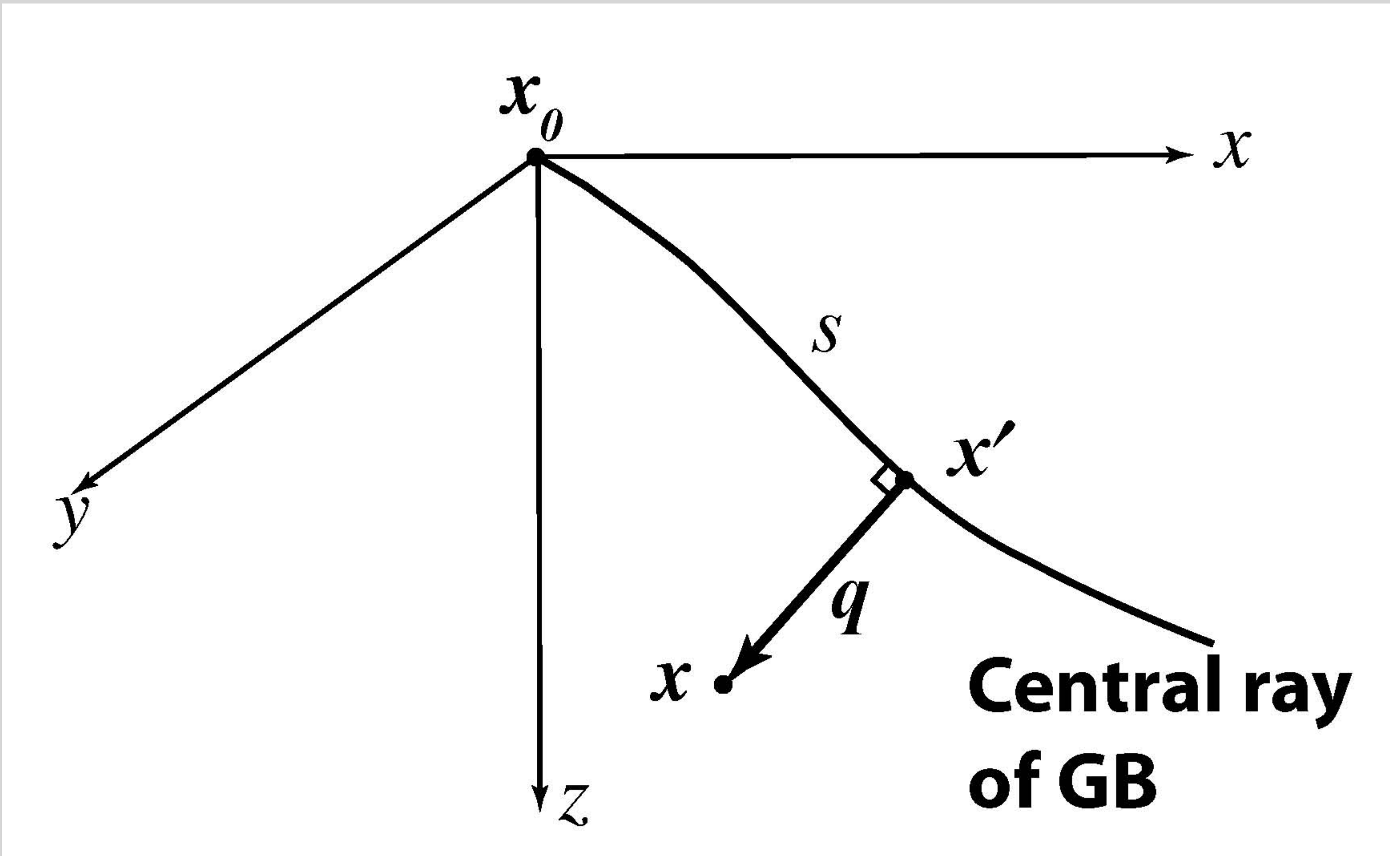
Gaussian beams

- Extend ray theoretic modeling to complex exponents
- Waves: sum over beams

Sum over beams for Green's function



One beam



Kirchhoff migration with GB’s

- Requires two Green’s functions

  Each requires integral/sum over beams

  – 2D: two additional integrals

  – 3D: four additional integrals

Ross Hill's method

- **Methods of steepest descent for integrals with complex exponents**
  - **2D: two integrals ⟶ one**
  - **3D: four integrals ⟶ two**

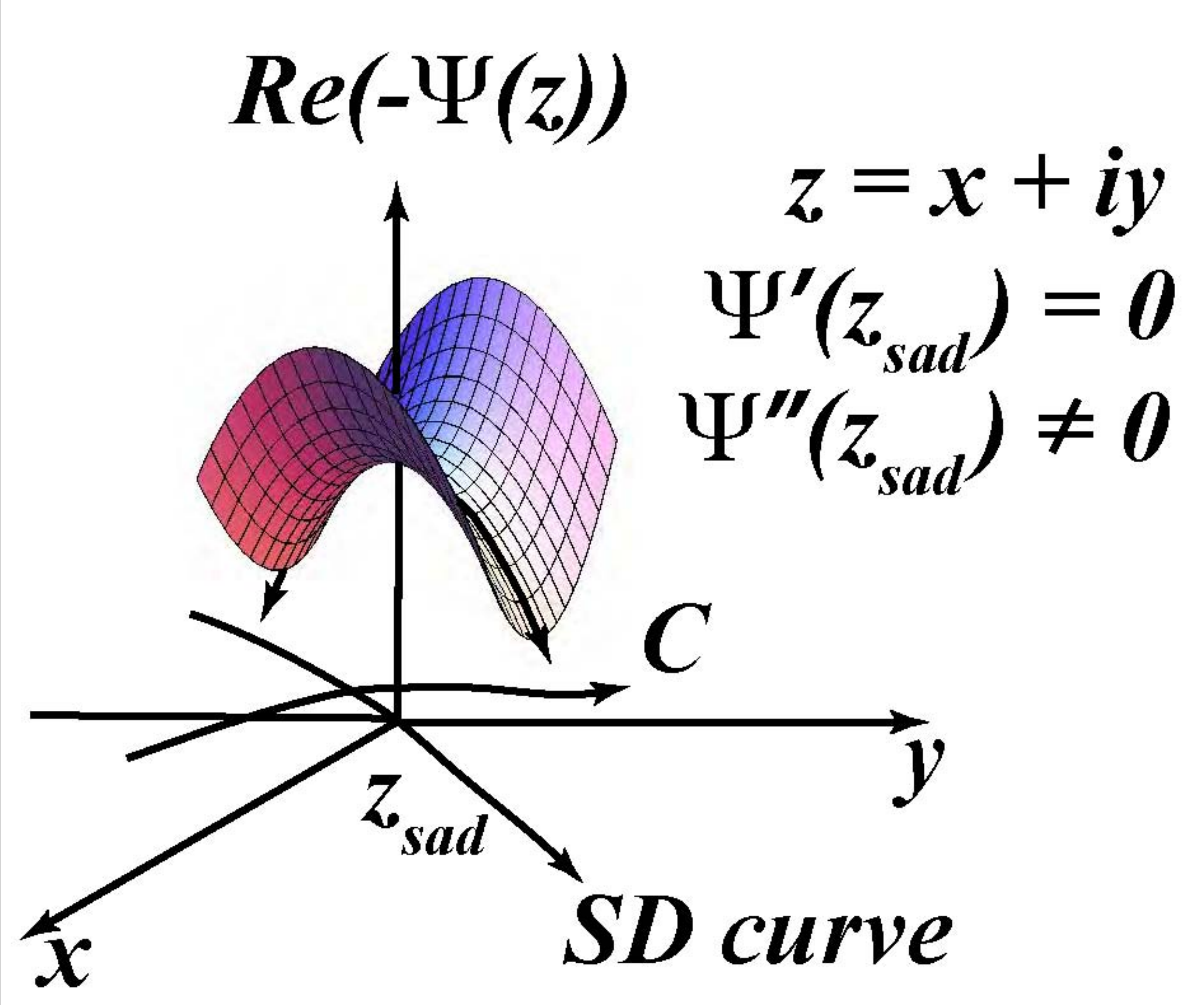
Prototype integral in 1D

$$I(\omega) = \int f(z) \exp\{-\omega \Psi(z)\} dz$$

$$\frac{d\Psi}{dz} = 0, \quad z = z_{sad}$$

$$\Psi_{zz} = \left. \frac{d^2\Psi}{dz^2} \right|_{z=z_{sad}} \neq 0$$

Method of steepest descent



Objective

- Deform contour C on to steepest descent path
- $Im(\Psi)$ constant on steepest descent path $\longrightarrow$
- Only $Re(\Psi)$ varies on SD path

Formula

$$I(\omega) = \int f(z) \exp\{-\omega \Psi(z)\} dz$$

$$I(\omega) \sim \sqrt{\frac{2\pi}{\omega \Psi_{zz}}} f(z_{sad}) \exp\{-\omega \Psi(z_{sad})\}$$

Formula

$$I(\omega) \sim \sqrt{\frac{2\pi}{\omega\Psi_{zz}}} f(z_{sad}) \exp\{-\omega\Psi(z_{sad})\}$$

Phase comes from

$$-\arg(\Psi_{zz})/2$$

Iterated Method of Steepest Descent

$$\int f(\boldsymbol{z}) \exp\{-\omega\Psi(\boldsymbol{z})\} dz_1 dz_2, \quad \boldsymbol{z} = (z_1, z_2)$$

$$\nabla_z \Psi(\boldsymbol{z}) = \boldsymbol{0}, \quad \boldsymbol{z} = \boldsymbol{z}_{sad} = (z_{1sad}, z_{2sad})$$

$$\det[\boldsymbol{\Psi}] \neq 0. \quad \boldsymbol{\Psi} = \left[\frac{\partial^2 \Psi}{\partial z_i \partial z_j}\right]$$

Iterated Method of Steepest Descent

$$\int f(\boldsymbol{z}) \exp\{-\omega\Psi(\boldsymbol{z})\} dz_1 dz_2, \quad \boldsymbol{z} = (z_1, z_2)$$

$$I(\omega) \sim \frac{2\pi}{\omega} \frac{f(\boldsymbol{z}_{sad})}{\sqrt{\det[\boldsymbol{\Psi}(\boldsymbol{z}_{sad})]}} \exp\{-\omega\Psi(\boldsymbol{z}_{sad})\}$$

Prototype integral over beams

$$I(\boldsymbol{x}, \boldsymbol{x}_s, \boldsymbol{x}_r, \omega) = \int_{D_{xs}} \frac{dp'_{s1} dp'_{s2}}{p'_{s3}} \int_{D_{xr}} \frac{dp'_{r1} dp'_{r2}}{p'_{r3}}$$

$$\cdot A^*_{GB}(\boldsymbol{x}'_s, \boldsymbol{x}_s) A^*_{GB}(\boldsymbol{x}'_r, \boldsymbol{x}_r) \exp\{-\omega \Psi(\boldsymbol{x}'_s, \boldsymbol{x}_s, \boldsymbol{x}'_r, \boldsymbol{x}_r)$$

$$\Psi(\boldsymbol{x}'_s, \boldsymbol{x}_s, \boldsymbol{x}'_r, \boldsymbol{x}_r) = i[T(\boldsymbol{x}'_s, \boldsymbol{x}_s) + T(\boldsymbol{x}'_r, \boldsymbol{x}_r)]^*$$

p's: initial transverse slownesses in Cartesian coordinates

Hill's transformation of coordinates

Offset slownesses

$$p'_{h1} = p'_{r1} - p'_{s1}, \quad p'_{h2} = p'_{r2} - p'_{s2}$$

Midpoint slownesses

$$p'_{m1} = p'_{r1} + p'_{s1}, \quad p'_{m2} = p'_{r2} + p'_{s2}$$

- **Apply iterated steepest descent to integrals over offset slownesses**
- **Integrate numerically in midpoint slownesses**

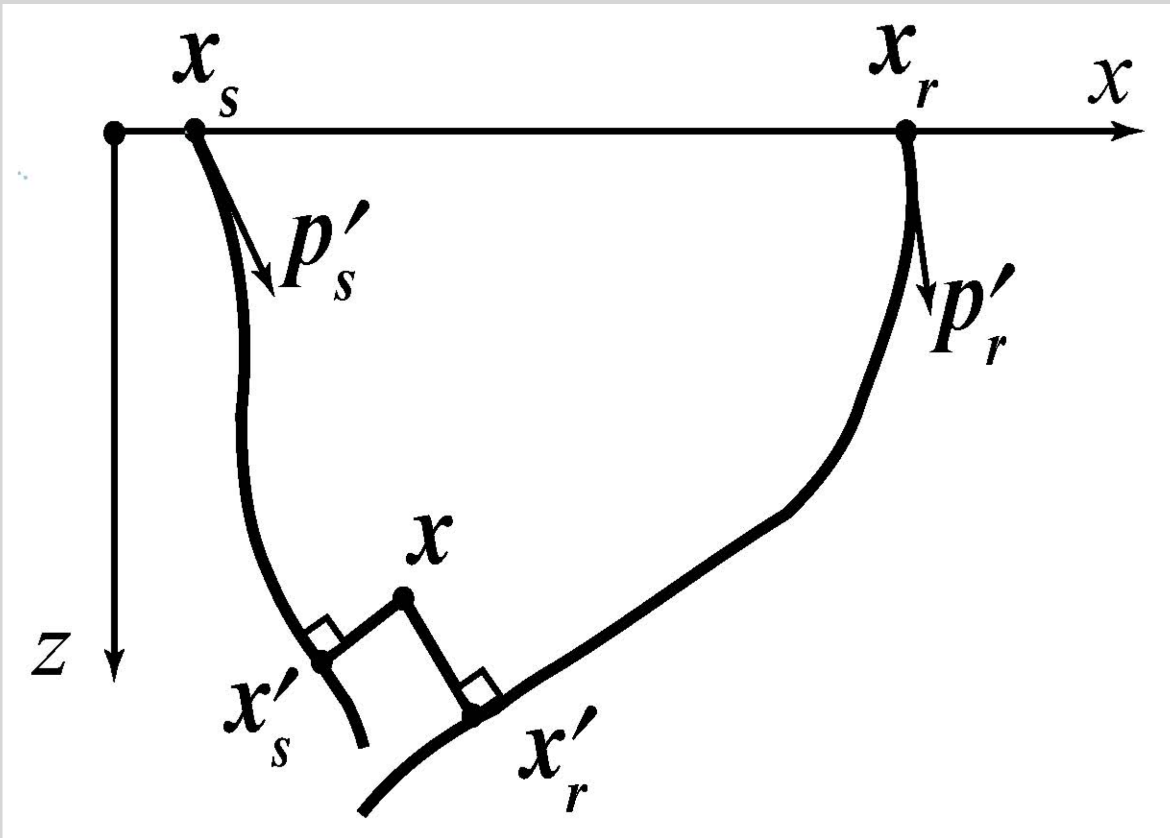
After iterated steepest descent calculation

$$I \sim \frac{\pi}{2\omega}\int_{D_m}\frac{dp'_{m1}dp'_{m2}}{\sqrt{\det[\boldsymbol{\Psi}]}}A^*_{GB}(\boldsymbol{x}'_s,\boldsymbol{x}_s)A^*_{GB}(\boldsymbol{x}'_r,\boldsymbol{x}_r)$$

$$\cdot\exp\{-i\omega[T(\boldsymbol{x}'_s,\boldsymbol{x}_s)+T(\boldsymbol{x}'_r,\boldsymbol{x}_r)]^*\},\ \boldsymbol{p}_h=\boldsymbol{p}'^{\mathrm{sad}}_h(\boldsymbol{p}'_m)$$

$$\boldsymbol{\Psi}=\left[\frac{\partial^2\Psi}{\partial p_{hI}\partial p_{hJ}}\right]\qquad \det[\boldsymbol{\Psi}]\neq 0$$

Typical p_m



$$\frac{\partial^2 \Psi}{\partial p'_{hj} \partial p'_{hk}} = i \left[\frac{\partial^2 T(\boldsymbol{x}'_r, \boldsymbol{x}_r)}{\partial p'_{rj} \partial p'_{rk}} + \frac{\partial^2 T(\boldsymbol{x}'_s, \boldsymbol{x}_s)}{\partial p'_{sj} \partial p'_{sk}} \right]^*, \quad j,k = 1,2$$

?????

We do know about

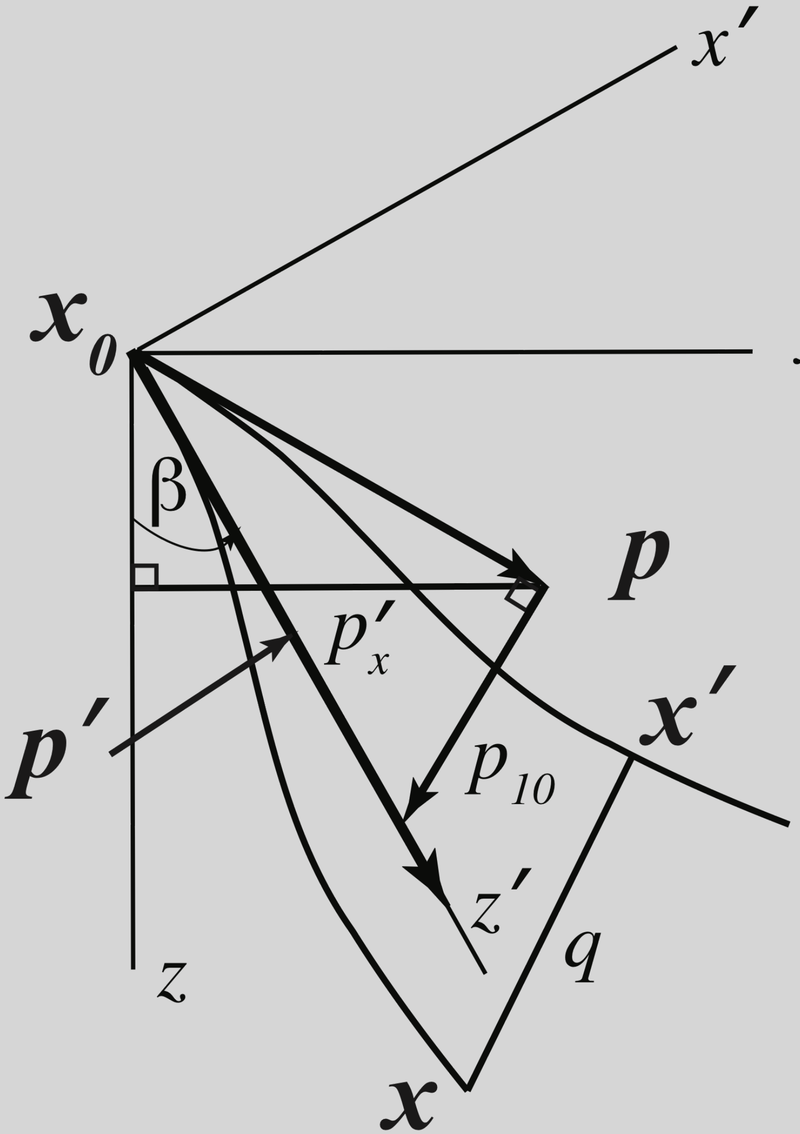
$$\mathrm{M} = \mathbf{T}_q \;=\; \left[\frac{\partial^2 T}{\partial q_\mu \partial q_\nu} \right]_{\boldsymbol{q}=\mathbf{0}}$$

Objective

Transform derivatives in p_r' and p_s' to derivatives in q_r and q_s

Two steps

2D case



- $x_0 = x_s$ or x_r
- Rotate $\longrightarrow \beta$
- $p'_x \longrightarrow p_{10}$
- $p_{10} \longrightarrow q$

3D Case

- Two angles, β_1, β_2
- Rotation matrix, Γ

3D Case

- Two angles, β_1, β_2

- Rotation matrix, Γ

- Defines ray-centered p_0 in terms of Cartesian p'
- Ray-centered ray theory:

$$p_0 \longrightarrow q$$

Matrix Ψ

$$\Psi = -\, i\omega_r w_0^2 \left\{ \frac{1}{\mathrm{v}_{0r}(0)} \Gamma_r^{\mathrm{T}} \mathrm{Q}_{GBr}^{-1} \mathrm{Q}_{2r} \Gamma_r + \frac{1}{\mathrm{v}_{0s}(0)} \Gamma_s^{\mathrm{T}} \mathrm{Q}_{GBs}^{-1} \mathrm{Q}_{2s} \Gamma_s \right\}^*$$

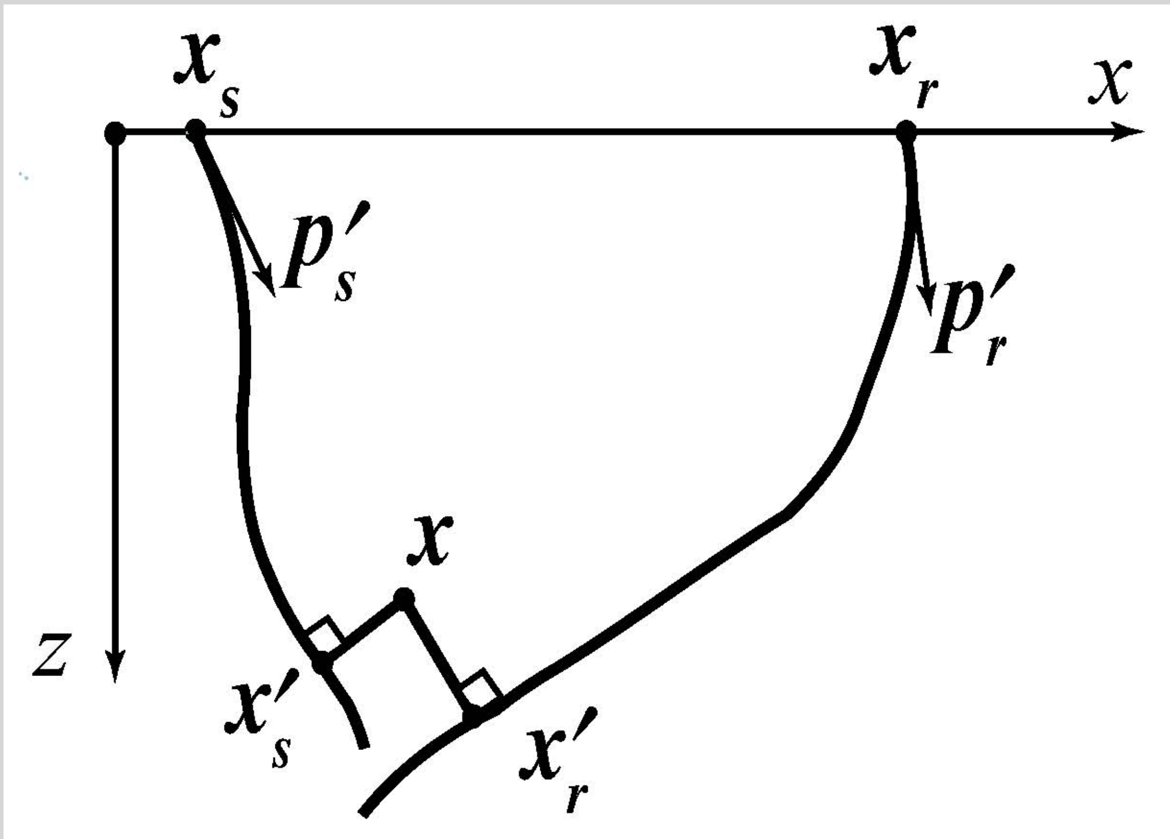
- **Γ's: Rotation matrices**
- **Q_2's: Červený's fundamental solutions for point sources**

Matrix Ψ

$$\Psi = -i\omega_r w_0^2 \left\{ \frac{1}{\mathrm{v}_{0r}(0)} \Gamma_r^{\mathrm{T}} \mathrm{Q}_{GBr}^{-1} \mathrm{Q}_{2r} \Gamma_r + \frac{1}{\mathrm{v}_{0s}(0)} \Gamma_s^{\mathrm{T}} \mathrm{Q}_{GBs}^{-1} \mathrm{Q}_{2s} \Gamma_s \right\}^*$$

Compute Γ's on central rays, *not* on ray through image point!

Typical p_m



Summary and Conclusions

- Start from double integrals in p_s' and in p_r'
- Transform to double
- integrals in p_h' and p_m'
- Apply steepest descent to double integral in p_h'

Summary and Conclusions

- **Need determinant of matrix Ψ**
- **Transform derivatives in p_h' to derivatives in q for rays from source and receiver**
- **Obtain the Hessian matrix $[\Psi]$**