Amplitude Calculations for 3-D Gaussian Beam Migration using Complex-valued Traveltimes

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Abstract

Gaussian beams are often used to represent Green’s functions in three-dimensional Kirchhoff-type true-amplitude migrations because such migrations made using Gaussian beams yield superior images to similar migrations using classical ray-theoretic Green’s functions. Typically, the integrand of a migration formula consists of two Green’s functions, each describing propagation to the image point—one from the source position and the other from the receiver position.

The use of Gaussian beams to represent each of these Green’s functions in 3D introduces two additional double integrals when compared to a Kirchhoff migration using ray-theoretic Green’s functions, thereby adding a significant computational burden. Hill [2001] proposed a method for reducing those four integrals to two, compromising slightly on the full potential quality of the Gaussian beam representations for the sake of more efficient computation. That approach requires a two-dimensional steepest descent analysis for the asymptotic evaluation of a double integral. The method requires evaluation of the complex traveltimes of the Gaussian beams as well as the amplitudes of the integrands at the determined saddle points. In addition, it is necessary to evaluate the determinant of a certain (Hessian) matrix of second derivatives. Hill [2001] did not report on this last part; thus, his proposed migration formula is kinematically correct but lacks correct amplitude behavior. In this paper, we derive a formula for that Hessian matrix in terms of dynamic ray tracing quantities. We also show in a simple example how the integral that we analyze here arises in a true amplitude migration formula.

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Second derivatives, complex traveltime.

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1 Introduction

In oil and gas exploration, a seismic reflection experiment consists of an exploding or vibrating source of energy near the Earth’s surface sending elastic waves into the Earth. The seismic waves propagate inside the Earth, and waves reflected from interfaces where material properties change return to the Earth’s surface to be recorded by an array of receivers. A seismic survey consists of hundreds or thousands of such experiments in close proximity to one another, each producing a seismic record that gives a distorted picture of the Earth’s subsurface. Although reflectors are visible on the records, they tend to be misplaced laterally and obscured by diffracted energy. Further, the records are recorded in time, so they do not provide an image of the Earth’s subsurface in depth.

The goal of seismic migration is to undistort the data recorded by the seismic survey, producing accurate maps of reflector locations. The basis for migration is wave propagation theory: given a reasonably accurate profile of seismic wave velocities inside the Earth, the physical waves are simulated by numerical wavefields propagating from the source and receiver locations, and the occurrence of reflection at locations inside the Earth is simulated by an imaging condition involving the wavefields at those locations. There is a vast geophysical literature on migration and associated problems, including problems of analyzing migrated amplitudes and estimating the wave velocities. Some of these problems are treated by Claerbout [1985]. The problem of interest in this paper is true-amplitude migration, which aims to preserve reflection amplitudes to the point where amplitudes of migration data provide accurate estimates of reflection coefficients as a function of source-receiver offset or incidence angle at reflector locations.

1.1 Kirchhoff migration

As with all migration methods, true-amplitude Kirchhoff migration requires downward propagation of the source wavefield and downward propagation of the observed data to image points in the subsurface. The downward propagations are accomplished using Green’s identity, operating on data evaluated along one surface to obtain data at a deeper surface via a convolution-type integral of the data with a Green’s function. We would also downward propagate sources other than point sources as a convolution-type integral. This is Kirchhoff migration [Schneider, 1978]. Each of these propagation processes requires evaluation of Green’s functions at the image point, one from a source, one from a receiver. When Gaussian beam representations of these Green’s functions are used, it is necessary to generate the Gaussian beams themselves in neighborhoods of the image points. Then, to obtain the Gaussian beam representation of each Greens function in 3D, it is necessary to carry out a 2D integration of Gaussian beams over all takeoff angles where rays are in the vicinity of the image point. By contrast, the Greens functions for standard Kirchhoff migration, derived from classical asymptotic ray theory, require only a complex function evaluation with no additional integrations.
1.2 Gaussian beam migration

Multiplying the Green’s functions together, as required by migration theory, results in the need to evaluate four nested integrals for Gaussian beam migration, as opposed to the multiplication of two complex numbers for classical ray-theoretic Green’s functions. Hill [2001] suggested a method for reducing those four additional integrals to two. Hill’s method first replaces integrals over source and receiver ray parameters with integrals over midpoint and offset ray parameters. He then applies the method of steepest descent for integrals with complex exponents to the (innermost) integrals over offset parameters, leaving the (outermost) integrals over midpoint parameters to be computed numerically. He provides a technique for determining the critical (saddle) points and evaluating the complex traveltime and amplitude in the Kirchhoff integral formula. However, for true amplitude integrity, the steepest descent approximation of the integral also requires including the determinant of the Hessian matrix of second derivatives of the complex traveltime with respect to the two offset ray parameters as an additional adjustment factor in the amplitude of the asymptotic approximation. Hill did not evaluate that determinant; thus his method suffers a slight amplitude and phase fidelity. That is, its peak amplitude on reflectors cannot be guaranteed to be proportional to a specular reflection coefficient.

1.3 The Hessian and the method of steepest descent for integrals

The Hessian we seek is a sum of two other Hessians, one with respect to initial transverse ray parameters for the rays from the source to the image point, the other, the same for the rays from the receiver. Those matrices are needed to compute the sum of matrices before evaluating the needed determinant. Červený and Pšenčík [1983] and Červený [2001], Section 5.8, provide the tools for determining those matrices as well as other details of Gaussian beams which come from the dynamic equations using ray tracing, complex-valued traveltime and complex-valued ray amplitude.

The steepest descent method needs the evaluation of the Hessians in offset ray parameters for each choice of the midpoint ray parameters over which the numerical integration is yet to be done. It is important to note that at those saddle points for which one and/or the other of the central rays from source and receiver misses the image point, the imaginary part of the complex traveltime is positive, and the integrand has exponential decay. Only when both rays from source and receiver pass through the saddle point is the imaginary part of the traveltime equal to zero. Thus, the remaining outermost integrations are dominated by the region of midpoint ray parameters for which central rays pass nearby the image point.

1.4 Two Hessians

We can exploit this observation to simplify the evaluation of the two Hessians at the saddle points in offset ray parameters. Those Hessians contain terms linear in the offset variables—the $q$’s of ray-centered coordinates. When central rays from both source and receiver pass through the image point, those $q$’s are all zero, producing the zero of the imaginary traveltime. Directly evaluating the coefficients of the terms linear in the $q$’s would require finite
difference approximations to derivative operators. This evaluation would be compromised by numerical artifacts arising from differencing quantities that are only moderately well behaved. As an alternative, we propose an approximate evaluation of the two Hessian matrices. In our approximation, we neglect those contributions that are linear in the $q$'s. Near the regions of dominant contribution to the integral, this causes insignificant error because the $q$'s are small; elsewhere the error is limited by the exponential decay of the integrand. The resulting approximation of the Hessians can then be evaluated in terms of dynamic ray quantities. Our approximation has been empirically shown to be adequate for estimating reflection coefficients in 2D [Gray and Bleistein, 2009].

### 1.5 The 2D problem

The problem in 2D requires the reduction of two single integrals of Gaussian beams over takeoff angles representing the two Green’s functions, one from the source, one from the receiver. In 2D there is only one orthogonal variable, say $q$, and the dynamic variables are also scalars, $Q$ and $P$. The second derivative we need is a scalar given by the sum of two other known scalars. That case is discussed by Gray and Bleistein [2009] who present a true amplitude Gaussian beam migration in 2D. They derived the second derivative indirectly by a method that did not expose the approximation we introduce here, although the same approximation is implicit in the final formula. Numerical examples confirm the claim of true amplitude migration. We show one of those examples in Section 6. The second derivatives we need are the derivatives of the complex traveltimes with respect to the initial transverse ray parameters—for rays from source and receiver—of rays in a Cartesian coordinate system. We do not have direct knowledge about how those derivatives propagate. On the other hand, we know about the second derivatives of complex traveltime with respect to the orthogonal coordinate $q$ for rays from source and receiver; these derivatives are expressible in terms of $Q$ and $P$. Thus, we need to connect the two second derivatives—one for rays from the source, the other for rays from the receiver—through changes of variables and the derivatives of those transformations. This is the key to determining the desired second derivatives of complex traveltime in terms of computed quantities of dynamic ray tracing.

As a first step, we need to transform the underlying Cartesian coordinate system to a Cartesian system defined by the initial direction of the ray from a source or receiver to the image point; eventually, we will not need to know that ray which might not be one of the discrete central rays of Gaussian beams used to compute the Green’s function at $x$. In Figure 1, that initial point for all rays is denoted by $x_0$, which can be the source point $x_s$ or the receiver point $x_r$. The angle of the initial direction of this ray is denoted by $\beta$ and the initial direction along the ray from $x_0$ through $x$ is denoted by the initial slowness vector $p_x$ in the figure. This transformation is completely geometric, defined by a matrix of rotation of coordinates.

The vector $p'_x$ depicts the initial direction of the ray from $x_0$ through $x'$ along a central ray of a Gaussian beam. The integration variable in Hill’s [2001] method in 2D is the initial ray parameter $p'_x$, the $x$-component of $p'$ in the Cartesian coordinate system $(x, z)$. The method of steepest descent of a single integral, such as the integral over directions of the rays of the Gaussian beams, requires the second derivative with respect to this integration variable $p'_x$. The 3D depiction is more complicated because there are two transverse slowness
Second derivatives, complex traveltime.

Figure 1: Rotation of Cartesian coordinates in 2D to the direction of the ray from the initial point $x_0$ to the image point $x$. The initial slowness vector of this ray is $p_x$. The rotation angle with respect to the vertical is $\beta$. The vector $p'$ is the initial direction of the central ray of a Gaussian beam through a point $x'$ which is connected to the image point $x$ by the vector $q$. $p'_x$ is the transverse coordinate of $p'$ in the unrotated coordinate system; this is the variable of integration over an ensemble of Gaussian beams used to compute the Green’s function at the point $x$. The direction of $q$ is orthogonal to the tangent to the central ray at $x'$. The ray equations in ray-centered coordinates describe the propagation of the orthogonal displacement $q$ between the point $x'$ and $x$. The traveltime at the image point $x$ is approximated to quadratic order in $q$, as differential equations in the arc length $s$ along the central ray through $x'$. The differential equation for $q$ is coupled with a differential equation for the propagation of $p_1 = \partial \tau / \partial q$. The initial value of $p_1$ is the orthogonal projection of $p'$ on to the initial direction defined by $p_x$.

The dynamic quantities in 2D, scalars $Q$ and $P$, are described by differential equations in $s$ along the ray through $x'$. The second derivatives of traveltimes with respect to $q$ are given by the quotient $PQ^{-1}$ for appropriate initial conditions on these dynamic variables. For classical asymptotic ray theory with real traveltimes, the initial conditions for $Q$ and $P$ are real; for Gaussian beams with complex traveltimes, the initial conditions for $Q$ and $P$ are complex. Thus, once the second derivative of traveltime with respect to $p_x$ is transformed to a second derivative with respect to $q$, we can write that second derivative with respect to $p_x$ in terms of the dynamic quantities $Q$ and $P$. This latter second derivative is exactly the one coordinates to deal with.
that we need to complete the amplitude formula in the method of steepest descent; this was done in Gray and Bleistein [2009], although we used an indirect method to compute that scalar second derivative—simpler to determine than the determinant of second derivatives that we need in 3D.

1.6 Returning to the 3D problem

The next step after the rotation of coordinates is to transform from initial slownesses in the ray-centered coordinate system to the $q$’s mentioned above. This is a standard matrix of propagation of a local ray tube in ray-centered coordinates. It is given by a matrix solution $Q$ of the dynamic ray tracing equations with appropriate initial data for the pair, $Q$, $P$. After these steps, the original Hessian of the complex traveltime in Cartesian slownesses can be written in terms of the Hessian of the same complex traveltime with respect to $q = (q_1, q_2)$.

One other subtlety occurs in the derivation of the Hessian in $q$. In ray-centered coordinates, the arc length along a ray is an independent third parameter with the given location determined by marching out to a specific value of the arc length $s$ and then moving orthogonally to the ray along a vector $q$. However, the image point $x$ is the same for the entire family of Gaussian beams in the integral representing a Green’s function. As a result of that, the arc length can be different for each ray that “carries” a Gaussian beam, so that $s = s(q)$, as we show below by examining the case of constant velocity in the discussion beginning on page 21; on the other hand, when there are no caustics in the ray family, we can also write down $q = q(s, p_{10}, p_{20})$; the latter pair on the right here are the two initial slownesses that generalize $p_{10}$ shown in Figure 1 to 3D.

With an explanation of the approximation we make along the way, we arrive at an approximate expression for the desired Hessian of the complex traveltime in Cartesian slownesses that we want in terms of the Hessian of the same complex traveltime with respect to $q$. The latter Hessian is surrounded by the Jacobians of the two transformations of coordinates and their transposes. The formula for the Hessian of one set of Cartesian slownesses—sources or receivers—is stated in equation (50). The final formula for the sum of Hessians needed in this analysis is stated in equation (51).

All quantities in (51) are determined on the discrete set of central rays that are computed. Thus, after a difficult theory, once we have the answer, the original rotation to the ray through the image point $x$ is not needed. This is explained in context.

1.7 Content of the sections

Section 2 contains background information necessary for this analysis and establishes a baseline of notation. The asymptotic formula we need to evaluate is stated in equation (19), with $I$ in that equation related to the product of Green’s functions as defined by equation (15). The Hessian matrix that we need to determine at a saddle point is defined by equation (18). It is written in terms of the Hessians of transverse slownesses for source and receiver in equation (21).

Section 3 is devoted to the analysis of either of these last two slownesses; they both require the same analysis, one for source coordinates, one for receiver coordinates. Hence, a
single analysis suffices for both. Section 4 provides the final formula for the Hessian matrix that we seek.

In Section 5, we present a true amplitude common-shot Kirchhoff migration formula and show that exactly the sort of integral we analyze in this paper appears in that formula. The methodology we present is driven by the fact that the exponent that appears in the migration operator is the complex conjugate of the sum of complex traveltimes of the separate Green’s functions. That structure is generic and not specific to our example; hence, our derivation applies to the kernel of the integral migration operator for any integration operator using Gaussian beams.

Finally, in Section 6, we provide the 2D example that appeared in Gray and Bleistein [2009]. This example supports our claimed accuracy of the method presented here.

The two major results derived in this paper are as follows.

1. The asymptotically-approximate determinant of a sum of two Hessian matrices that are required to complete the 2D steepest descent formula that provides “true amplitude” of the Gaussian beam migration formula in 3D.

2. Strictly speaking, the steepest descent evaluation of integrals is not available in more than one complex variable, but iteration under reasonable assumptions provides us with a formula that is almost to be expected in a double integral with a complex exponent. This analysis is presented in Appendix A.

2 Preliminaries

We need various pieces of background information and notation in order to explain our analysis. That will be done in this preliminary section. Here is a list of what will be discussed.

1. The kinematic and dynamic equations governing propagation of paraxial rays in ray-centered coordinates.

2. The dynamic quantities of the propagator matrix—Červený [2001], Section 4.3.1—and the relationship to the dynamic quantities of Gaussian beams.

3. The complex traveltimes from source and receiver to an image point; their dependencies on the original slownesses as well as their dependencies on the transformed slownesses proposed by Hill [2001].

4. The Hessian we need to evaluate and its relation to the Hessians in the original slowness variables from source and receiver to image point.

2.1 The kinematic and dynamic equations governing propagation on paraxial rays in ray-centered coordinates

This is item 1 in the list above. We begin by considering a ray defined by Cartesian ray theory with initial point $x_0$ and passing through $x'$ as in Figure 2, the 3D version of Figure
Second derivatives, complex traveltime.

Figure 2: Central ray of a Gaussian beam from $x_0$ through $x'$, connected along an orthogonal vector $q$ to the point $x$.

1, omitting the slowness vectors. The point $x_0$ might be the initial point for a ray from a source location, $x_0 = x_s$, or a receiver location, $x_0 = x_r$.

The initial direction of the ray from $x_0$ through $x$ defines the rotation direction in 3D of the new coordinate system in 3D. Associated with that ray is an orthogonal ray-centered coordinate system, $(s, q_1, q_2)$. Figure 3 on page 9 shows the initial orientation of the unit vectors $(\hat{t}, \hat{e}_1, \hat{e}_2)$ in an $(s, q_1, q_2)$ coordinate system. In these coordinates $s$ is arc length along the central ray, as depicted in Figure 2, and $q_1$ and $q_2$ are the orthogonal coordinates to the central ray, also orthogonal to one another.

Figure 3: The polar angles define the initial slowness in the Cartesian system. The ray through $x$ has initial direction $\hat{p}'$. $\hat{e}_1$, $\hat{e}_2$ are the other initial orthogonal coordinates in this system. Furthermore, the rotation angles $\beta_1$ and $\beta_2$ are depicted. Each of the central rays of Gaussian beams, as in Figure 2, has its own such triple of initial directions.

In Figure 2, we depict the coordinates of the point $x$ in terms of the ray-centered coordinates $(s, q_1, q_2)$ on the central ray through $x'$.

In these coordinates, $q_1 = q_2 = 0$ on the central ray through $x'$. Nearby central rays of Gaussian beams from the same point $x_0$ are defined by initial directions that are different...
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from the initial direction of this central ray.

We need a notation for the velocity and its derivatives along the central ray through \( x' \):

\[
v_a(s) = v(s, 0, 0), \quad v_{a,s} = \left. \frac{\partial v(s, 0, 0)}{\partial s} \right|_{q_1 = q_2 = 0},
\]

\[
v_{0,i} = \left. \frac{\partial v}{\partial q_i} \right|_{q_1 = q_2 = 0}, \quad v_{0,i,j} = \left. \frac{\partial^2 v}{\partial q_i \partial q_j} \right|_{q_1 = q_2 = 0}, \quad I, J = 1, 2, \quad \mathbf{V} = \begin{bmatrix} v_{0,ij} \end{bmatrix}.
\]

Kinematic behavior of the central rays is determined by describing the propagation of the coordinates \((q_1, q_2)\) for each arc length \( s \) along the central ray and the slownesses of the real traveltime \( \tau \) of asymptotic ray theory, defined by

\[
p_1 = \frac{\partial \tau}{\partial q_1}, \quad p_2 = \frac{\partial \tau}{\partial q_2}.
\]

In our application, we need to move from initial slownesses \((p'_x, p'_y)\) of this central ray in the Cartesian coordinates system of the defined integral of interest to the initial ray-centered slownesses \((p_{10}, p_{20})\).

The kinematic equations are then given by

\[
\frac{dq_i}{ds} = v_0 p_i, \quad q_i(0) = 0,
\]

\[
\frac{dp_i}{ds} = -\frac{v_{0,i,j}}{v_0^2} q_j, \quad p_i(0) = p_{10}, \quad I, J = 1, 2.
\]

Summation on repeated capital indices from 1 to 2 is understood. The propagation of the transverse slownesses \((p_1, p_2)\) is determined. We determine the \( q \)'s of interest through the connection of the central ray to the image point.

These equations can also be written in vector form as

\[
\frac{dq}{ds} = v_0 \mathbf{p}, \quad \mathbf{q}(0) = \mathbf{0},
\]

\[
\frac{dp}{ds} = -\frac{1}{v_0^2} \mathbf{V} \mathbf{q}, \quad \mathbf{p}(0) = \mathbf{p}_0,
\]

with

\[
\frac{d\tau}{ds} = \frac{1}{v}.
\]

In the last equation, \( \mathbf{V} \) is the matrix defined in equation (1) and the vectors are vertical arrays of the \( q \)'s and \( p \)'s, respectively.

We turn now to the dynamic equations of ray centered coordinates. These are equations for two \( 2 \times 2 \) matrices, \( \mathbf{Q} \) and \( \mathbf{P} \). This is item 2 of our list in the introduction to this section. These matrices are, in turn, related to the Hessian matrix of second derivatives of the traveltime with respect to \( \mathbf{q} \) as follows:

\[
\mathbf{M} = \begin{bmatrix} \frac{\partial^2 \tau}{\partial q_i \partial q_j} \end{bmatrix}_{q=0}, \quad I, J = 1, 2, \quad \mathbf{M} = \mathbf{PQ}^{-1},
\]
with $Q$ and $P$ satisfying the equations

$$
\frac{dQ}{ds} = v_0 P,
$$

$$
\frac{dP}{ds} = -\frac{1}{v_0^2} VQ.
$$

These are the same equations as the kinematic equations (4), except now for $2 \times 2$ matrices. We refrain from stating initial conditions since we will define different solutions to these equations below by imposing different initial conditions. Furthermore, when we introduce Gaussian beams through complex-valued initial conditions for $Q$ and $P$, we will use $T$ instead of $\tau$ for the resulting complex traveltime. Then, $\tau$ remains the real traveltime of classical asymptotic ray theory. However, the relation between the Hessian of $T$ with respect to $q$ and the matrices $M, Q$ and $P$ stated in equation (5) will remain the same.

2.2 The dynamic quantities of the propagator matrix.

Červený [2001], Section 4.3.1, introduces the propagator matrix $\Pi(x', x_0)$. This is a matrix of four columns and four rows made up of two different solutions of the dynamic ray equations (6) with different initial conditions as follows.

$$
\Pi(x_0, x_r) = \begin{bmatrix}
Q_1(x', x_0) & Q_2(x', x_0) \\
P_1(x', x_0) & P_2(x', x_0)
\end{bmatrix}.
$$

The initial conditions for these two pairs of solutions are

$$
Q_1(x_0, x_0) = I, \quad Q_2(x_0, x_0) = 0,
$$

$$
P_1(x_0, x_0) = 0, \quad P_2(x_0, x_0) = I.
$$

The solution pair $Q_1(x', x_0)$ and $P_1(x', x_0)$ arise naturally when describing propagation of plane waves in (real) asymptotic ray theory. The solution pair $Q_2(x', x_0)$ and $P_2(x', x_0)$ arise naturally when describing the propagation from a point, as for the Green’s function. The initial data for $\Pi$ is a $4 \times 4$ identity matrix. Furthermore, when the columns are viewed as solutions of the system of differential equations (6), it can be shown that they are linearly independent in the sense that the Wronskian—the determinant of the matrix $\Pi$—is nonzero for all values of $s$.

Remark

It is useful to understand the dimensions of the two solutions comprising the propagator matrix. Note that the elements of $Q_1$ are dimensionless quantities, given initially by a matrix of ones and zeroes. Similarly, the elements of $P_2$ are dimensionless. With $Q_1$ dimensionless, the second differential equation in (6) reveals that the dimensions of the elements of $P_1$ are $T/L^2$—TIME/LENGTH-SQUARED. Similarly, with $P_2$ dimensionless, the dimensions of $Q_2$ are $L^2/T$—LENGTH-SQUARED/TIME.
For our physical applications, we will choose the dimension of the elements of $Q$ to be LENGTH and the dimensions of the elements of $P$ to be slowness, inverse velocity—TIME/LENGTH. Thus, when adding together linear combinations of the two sets of fundamental solutions, the dimensionality of the coefficients of these solutions will have corresponding dimensionality to yield the desired dimensionality of the sum.

Now let us consider a complex-valued traveltime, $T$, instead of $\tau$ used above in equation (5). Thus, instead of equation (5), we write

$$M_{GB} = \left[ \frac{\partial^2 T}{\partial q_i \partial q_j} \right]_{q=0}$$

(9)

The pair of functions $Q(x', x_0) = Q_{GB}(x', x_0)$ and $P(x', x_0) = P_{GB}(x', x_0)$ is a solution of the dynamic equations (6) subject to the initial conditions

$$Q_{GB}(x_0, x_0) = \frac{\omega_r w_0^2}{v_0(0)} I,$$

$$P_{GB}(x_0, x_0) = \frac{i}{v_0(0)} I,$$

(10)

In the first equation, $\omega_r$ is a reference frequency and $w_0$ is a length scale. At the reference frequency $\omega_r$, $w_0$ is the initial "standard deviation" of the Gaussian exponential that arises in the description of the Gaussian beam. For example, in a homogeneous medium with initial arc length zero and initial traveltime zero, the exponential part of the Gaussian beam is

$$\exp \left\{ i\omega T \right\} = \exp \left\{ i\omega \frac{|x' - x_0|}{V_0} - \frac{\omega |q|^2}{2 [\omega_r w_0^2 + iV_0|x' - x_0|]} \right\}.$$

(11)

It is easy to check our claim about $w_0$ by setting $x = x_0$ in this equation and $\omega = \omega_r$.

As a result of the linearity in the differential equations, we can write the two matrices $Q_{GB}$ and $P_{GB}$ in terms of the elemental matrices of equation (8) as follows:

$$Q_{GB}(x', x_0) = \frac{\omega_r w_0^2}{v_0(0)} Q_1(x', x_0) + \frac{i}{v_0(0)} Q_2(x', x_0),$$

$$P_{GB}(x', x_0) = \frac{\omega_r w_0^2}{v_0(0)} P_1(x', x_0) + \frac{i}{v_0(0)} P_2(x', x_0).$$

(12)

2.3 Gaussian beams and the complex traveltimes $T$

Here we proceed to item 3 in our list at the beginning of this section. The Gaussian beam representation of a Green’s function $G(x, x_0, \omega)$ is given by Hill [2001] as

\footnote{Hill [1990, 2001] uses subscripts $x$, $y$ and $z$. We use numerical subscripts because we need them for index notation in the matrices to follow.}
In these equations, the superscript T denotes “transpose.” The vector $p'$ is the initial Cartesian slowness vector of the ray that propagates from $x_0$ to $x'$. The ensemble of central rays in this integral covers a region around the image point $x$. (We do not need to know which ray goes through $x$. This claim will be explained near the end of Section 4 starting on page 24.) For each ray, a vector $q$ connects point $x'$ to the image point $x$ and is perpendicular to the ray at $x'$; $s$ is the arc length along that ray to the point $x'$. Thus, the complex traveltime $T$ is implicitly a function of the transverse slownesses, $p'_1$, $p'_2$, and $\tau(s)$ is the traveltime of the classical asymptotic ray theory as noted above. In the application where $x$ is fixed and the initial direction of the ray changes, the arc length $s$ at the upper limit of the integral defining $T$ is also a function of the initial slownesses on the ray from $x_0$ through $x'$ and the image point $x$.

Any correlation-type migration imaging condition involves a product of such Green’s functions from source and receiver to image point:

$$G(x, x_s, x_r, \omega) = G^*(x, x_s, \omega)G^*(x, x_r, \omega),$$  \hfill (14)

with $^*$ denoting complex conjugate here and below. When we use equation (13) for each Green’s function, we obtain

$$G(x, x_s, x_r, \omega) = -\frac{\omega^2 \omega_s^2 w_0^4}{4 \pi^2 V^3(\frac{V^3}{2}(x_r))} I(x, x_s, x_r, \omega),$$

$$I(x, x_s, x_r, \omega) = \int_{x_s}^{x_r} \int_{x_r}^{x_s} \frac{dp'_1 dp'_2}{p'_3} \frac{dp'_3}{p'_3},$$  \hfill (15)

$$\Psi(x'_s, x'_s, x'_r, x'_r) = i[T(x'_s, x'_s) + T(x'_r, x'_r)]^*.$$

Hill [2001] proposes the change of variables

$$p'_1 = p'_r - p'_s, \quad p'_2 = p'_r + p'_s, \quad p'_r = \frac{p'_m + p'_h}{2}, \quad p'_s = \frac{p'_m - p'_h}{2},$$

$$p'_m = p'_r + p'_s, \quad p'_h = p'_r - p'_s, \quad p'_r = \frac{p'_m + p'_h}{2}, \quad p'_s = \frac{p'_m - p'_h}{2}.$$  \hfill (16)
The variables in $p_h$ may be viewed as an offset slowness vector and the variables in $p_m$ may be viewed as a midpoint slowness vector.

This change of variables produces an expressions for $I$ with the $(p'_{m1}, p'_{m2})$ integrals outermost. He then proposes that the method of steepest descent be applied to the two integrals in the variables $p_{h1}'$ and $p_{h2}'$. While acknowledging the approximate nature of doing this, he carries it out as a kinematic process, neglecting a complete amplitude and phase analysis. Therefore, he needs only to determine the saddle points in these two variables for any given choice of the other pair, $p'_{m1}$ and $p'_{m2}$. He then proposes to compute the integrals over $p'_{m1}$ and $p'_{m2}$ numerically.

In Appendix A we show how to calculate the leading order asymptotic expansion of an integral such as $I$ in equation (15) by the method of steepest descent. This method calculates that expansion by applying the method of steepest descent in one dimension iterively. The method properly accounts for amplitude as well as complex exponent in that expansion. We assume that for each value of $p'_m = (p'_{m1}, p'_{m2})$ there exists a “simple saddle point”

$$p_{h}^{\text{sad}}(p'_m) = (p_{h1}^{\text{sad}}(p'_m), p_{h2}^{\text{sad}}(p'_m)),$$

for which

$$\frac{\partial \Psi}{\partial p'_{h1}} = \frac{\partial \Psi}{\partial p'_{h2}} = 0, \quad p_h = p_{h}^{\text{sad}}(p'_m),$$

$$\det[\Psi] \neq 0, \quad \Psi = \left[ \frac{\partial^2 \Psi}{\partial p_{hI} \partial p_{hJ}} \right], \quad I, J = 1, 2, \quad p_h = p_{h}^{\text{sad}}(p'_m).$$

The last constraint here, $\det[\Psi] \neq 0$, makes this saddle point “simple.” This is a standard assumption on the Hessian matrix $\Psi$ in two-dimensional integrals.

**Remark** The condition, $\det[\Psi] \neq 0$

In one dimension, a stationary point is “simple” if the second derivative of the exponent is nonzero at that point. In this case, the difference,

$$\Psi(p) - \Psi(p^{\text{sad}}) \approx \Psi''(p^{\text{sad}})(p - p^{\text{sad}})^2/2,$$

is quadratic. The stationary phase formula relies on this quadratic approximation. The stationary point is called ”higher order” if the second derivative at the stationary or saddle point is zero as well; the local approximation is at least cubic and the leading order asymptotic approximation is changed accordingly. See, for example, Bleistein and Handelsman [1986]. For higher dimensional integrals with a real traveltime, (imaginary exponent), this idea is extended by using properties of Hessian matrices. In particular, these matrices are symmetric, their eigenvalues are real and their eigenvectors are orthogonal. Bleistein and Handelsman [1975, 1986] show how these properties allow a rotation of coordinates to the directions of the eigenvectors, called the “principal directions” of the Hessian. In this case, the function $\Psi$ is locally a sum of signed squares, with the signs of the separate terms depending on the signs of the individual eigenvalues. The same is possible for purely real exponents, such as occur in Laplace-type integrals (op.
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cit.), except that now both of the eigenvalues of the Hessian matrix of second derivatives have to be of one sign. This leads to the asymptotic expansion derived by the one-dimensional stationary phase or Laplace method formula, respectively, applied to the integrations in each of the principal directions.

For a complex valued traveltime $\Psi$, we do not have the same theory for its complex valued Hessian matrix. However, we show in Appendix A that the same condition naturally arises in order to solve for the second derivative of the second variable of integration after having determined the first integral by the formula for the leading order asymptotic approximation by the method of steepest descent. See equation (A.20).

Under these assumptions, the iterated method of steepest descent leads to the following asymptotic formula for $I$ in equation (14).

$$I(x, x_s, x_r, \omega) \sim \frac{\pi}{2\omega} \int_{D_m} \frac{dp'_m dp'_n}{\sqrt{\det[\Psi]}} A^*_C (x'_s, x_s) A^*_C (x'_r, x_r)$$

$$\cdot \exp\{ -i\omega[ T(x'_s, x_s) + T(x'_r, x_r)]^* \}, \quad p_h = p^\text{sad}_h (p^'_m). \tag{19}$$

The objective of this paper is evaluate $\det[\Psi]$ appearing in this formula. For that purpose, let us first rewrite the matrix $\Psi$ in terms of the separate Hessian matrices for $T(x'_s, x_s)$ and $T(x'_r, x_r)$ appearing in the definition of $\Psi$ in equation (14).

To begin this process, we observe first that

$$\frac{\partial \Psi}{\partial p'_{hj}} = \frac{\partial p'_{rj}}{\partial p'_{hj}} \frac{\partial \Psi}{\partial p'_{rj}} + \frac{\partial p'_{sj}}{\partial p'_{hj}} \frac{\partial \Psi}{\partial p'_{sj}},$$

$$= \frac{\partial \Psi}{\partial p'_{rj}} - \frac{\partial \Psi}{\partial p'_{sj}}$$

$$= i \left[ \frac{\partial T(x'_r, x_r)}{\partial p'_{rj}} - \frac{\partial T(x'_s, x_s)}{\partial p'_{sj}} \right]^*, \quad j = 1, 2. \tag{20}$$

In the second line in this equation, we used equation (16) to evaluate the derivatives of the elements of $p_h$ with respect to the elements of $p_r$ and $p_s$. In the third line, we replaced $\Psi$ by the separate complex traveltimes, the $T$'s, using the definition of $\Psi$ in equation (15).

Differentiating one more time leads to the following representation of the elements of the Hessian matrix of second derivatives of $\Psi$:

$$\frac{\partial^2 \Psi}{\partial p'_{hj} \partial p'_{hk}} = i \left[ \frac{\partial^2 T(x'_r, x_r)}{\partial p'_{rj} \partial p'_{rk}} + \frac{\partial^2 T(x'_s, x_s)}{\partial p'_{sj} \partial p'_{sk}} \right]^*, \quad j, k = 1, 2. \tag{21}$$

This is item 4 of the list at the beginning of this section. In the next section we analyze the two matrices on the right hand side of this equation.

We remark that each of the matrices on the right hand side of equation (21) will be singular at a caustic of the central rays $\{a\}$ from the receiver (first matrix), or $\{b\}$ from the
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source (second matrix), respectively. This would undo the original purpose of using Gaussian beams to represent the Green's functions. However, the occurrence of both matrices being singular simultaneously in such a manner as to create a singular sum of matrices is expected to be relatively rarer than singular behavior of one or the other. The sum of the singular matrices being zero requires an alignment of the eigenvectors attached to the pair of zero eigenvalues. In fact, one situation where we know this will occur is when the image point is on a caustic of the incident rays. In that case, the incident rays and the reflected rays lie in a plane, and the eigenvectors of the zero eigenvalues are colinear. In Figure 4, we show a 2D example of a family of rays (solid black) incident on a line. The reflected rays (dashed black) continue the caustic of the incident rays. In fact, those rays emanate from the “image points” of the sources points at 0 km depth. This example is a cartoon of the more general 3D case. Incident and reflected rays start out in the same plane, so that the slice might be thought of as arising from the normal to the local tangent to the reflector surface at the caustic point. Further, if the reflector were curved, that curvature would only slightly distort the directions of the reflected (black) rays. However, those new rays would still exhibit the same caustic as the incident rays.

The true amplitude Kirchhoff migration theory is not valid at caustics in any case. We would not use a caustic reflection point as a place to estimate a reflection coefficient. We only need to regularize the sum of matrices in equation (21) so that the zero determinant of the sum of those two matrices does not cause the running sum producing the true amplitude migration to blow up. So the question of inversion as we define it is moot. Thus, we would content ourselves with some appropriate regularization of the matrix sum.

3 The Hessian matrix $T$

The right hand side of equation (21) contains two Hessian matrices of identical structure. In this section, we show how they can be written in terms of the matrix elements of the propagator matrix $\Pi$ of equation (6). There is no need to distinguish between source and receiver here through the subscripts $s$ and $r$ and so we speak of a generic Hessian with respect
to initial transverse slownesses. Thus, we write
\[ T = [T_{jk}], \quad T_{jk} = \frac{\partial^2 T(x', x)}{\partial p'_j \partial p'_k}, \quad j, k = 1, 2. \] (22)

As discussed in the Introduction, we proposed to determine this Hessian by transforming from Cartesian coordinates to ray-centered coordinates and relating this Hessian to the Hessian of \( T \) with respect to the offset coordinates \((q_1, q_2)\) from a central ray.

We propose to do this in two steps.

1. Transform the Hessian from the variables \( p' \) of the integral in Cartesian coordinates to the initial ray-centered slownesses orthogonal to the central ray \((p_{10}, p_{20})\). Implicit in this transformation is a transformation also of the Cartesian coordinates, \( x' \) to the ray-centered coordinates \((s, q_1, q_2)\). This transformation is described below.

2. Transform the Hessian in \((p_{10}, p_{20})\) into a Hessian in \((q_1, q_2)\). The Jacobian of this transformation arises from the mapping by rays from a point source.

A subtlety in Step 1 of this analysis needs discussion. Each \((p'_1, p'_2)\) of the Cartesian integration variables defines a new central ray. Each central ray has its own ray-centered coordinate system. In each individual coordinate system, the initial slownesses \((p_{10}, p_{20})\) orthogonal to the central ray are zero on the central ray. By changing coordinate systems with each central ray, we cannot transform derivatives with respect to the variables \((p'_1, p'_2)\) into derivatives with respect to \((p_{10}, p_{20})\) because the coordinate system of the latter slownesses keeps changing. That is, there is only one Cartesian coordinate system, but each ray has its own ray-centered coordinate system.

Instead, we introduce the central ray from \( x_0 \) to \( x \) as a reference. We then evaluate the initial values \( p_0 \) of the vector \( p' \) on each paraxial ray in a fixed ray-centered coordinate system measured from the initial direction of this reference ray. In this rotated coordinate system \( p_0 \) is just the new representation of the vector \( p' \). In Figure 5, we depict the ray from \( x_0 \) to \( x \) as well the ray \( x_0 \) to \( x' \). We also show the polar angles \((\beta_1, \beta_2)\) of the initial direction of the ray to \( x \).

Once we have this first transformation from \( p' \) to the initial values \( p_0 \) in place, the second transformation from the initial values of \((p_{10}, p_{20})\) to \((q_1, q_2)\) is simply a mapping by rays that can be expressed in terms of the appropriate dynamic matrix \( Q_2 \).

Gaussian beams were used for a similar purpose as we are doing here in the asymptotic analysis of the integral over beams in Hill [2001], Appendix B. He used such transformations in homogeneous media to compare his beam representation to the known leading order ray-theoretic asymptotic approximation of Green’s functions. In that appendix, the integral over beams is calculated as an integral in polar angles measured from the distinguished central ray through \( x \) as opposed to an integration of slownesses \( p'_s \) or \( p'_r \) through the points \( x'_s \) or \( x'_r \), respectively, as in the representations of the Green’s functions imbedded in the right hand side in equation (15). The use of polar coordinates as in Hill’s appendix was not really necessary; in fact, Hill’s [2001] Gaussian beam migration integrates initially over the slownesses \( p'_s \) or \( p'_r \) as we are doing here in equation (15). However, in his appendix Hill carried out the asymptotic analysis in homogeneous media; that is all he needed for the
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Figure 5: The reference ray from $x_0$ to $x$ and the ray from $x_0$ to $x'$. The initial ray direction of the first ray is defined by the polar angles $(\beta_1, \beta_2)$. The initial direction of this ray is defined by the values of $p'$. The initial vector $p_0$ is the vector $p'$ in the coordinate system of the central ray through $vx'$. The purpose of comparison of leading order asymptotic representations of Green’s functions; the calculation in homogeneous media is particularly simple in polar coordinates. Here, however, for the Hessian matrices $T$ of equation (22) for sources and receivers and eventually for the matrix $\Psi$ of equation (18), we need to do a similar analysis in heterogeneous media. Implicit in our analysis is the methodology for calculating the asymptotic expansion that Hill [2001] did, but in heterogeneous media. This is not new: Červený [2001] describes asymptotic expansions for more arbitrary changes of coordinates in heterogeneous media. His method relies on a theorem in Korn & Korn [1968] where the authors show how to simultaneously diagonalize a symmetric complex-valued matrix such as our Hessian matrix $T$ in equation (22). It is not straightforward to use Červený’s method to go from the general discussion of the asymptotic expansion of the double integral to the specific example we have here. Hence, we proceed with the transformations and then they will be applied to facilitate the evaluation of the $\det[\Psi]$. Thereafter we apply the method of Appendix A of this paper to determine the asymptotic expansion in $p'_h$ in equation (19).

We now proceed to the outlined analysis.

3.1 Transformation to a Hessian with respect to $p_0$.

Figure 3 on page 9 shows the initial directions of a ray in Cartesian and ray-centered coordinates. The vector $\hat{p}'$ is the initial direction of the ray through $x$. In the original Cartesian coordinate system, $\hat{p}' = (\hat{p}'_1, \hat{p}'_2, \hat{p}'_3)$. As a first step, we rotate the Cartesian coordinate system through the angles $\beta_1$ and $\beta_2$ of the figure, with $\hat{p}'$ being the initial direction of the new $z$-axis. In the new coordinate system, $\hat{p}' = v p' = (0, 0, 1)$.

The rotation to the new Cartesian coordinates from the old is accomplished by first carrying out a rotation around the $z$-axis through the angle $\beta_2$ and then by a rotation through the angle $\beta_1$ about the new $y$-axis with unit vector $\hat{e}_2$. We therefore write the transformation of coordinates as a product of two matrix multiplications on the elements of
an arbitrary slowness vector $p'$ as follows:

$$
p_0 = \begin{bmatrix} p_{10} \\
p_{20} \\
p_{30} \end{bmatrix} = \Gamma_1(\beta_1)\Gamma_2(\beta_2) \begin{bmatrix} p'_1 \\
p'_2 \\
p'_3 \end{bmatrix} = \Gamma_1(\beta_1)\Gamma_2(\beta_2)p'.
$$

(23)

Here

$$
\Gamma_1(\beta_1) = \begin{bmatrix} \cos \beta_1 & 0 & \sin \beta_1 \\
0 & 1 & 0 \\
-\sin \beta_1 & 0 & \cos \beta_1 \end{bmatrix}, \quad \Gamma_2(\beta_2) = \begin{bmatrix} \cos \beta_2 & \sin \beta_2 & 0 \\
-\sin \beta_2 & \cos \beta_2 & 0 \\
0 & 0 & 1 \end{bmatrix},
$$

(24)

$$
\Gamma_1(\beta_1)\Gamma_2(\beta_2) = \begin{bmatrix} \cos \beta_1 \cos \beta_2 & \cos \beta_1 \sin \beta_2 & \sin \beta_1 \\
-\sin \beta_1 \cos \beta_2 & \cos \beta_1 \cos \beta_2 & 0 \\
-\sin \beta_1 - \cos \beta_1 \sin \beta_2 \sin \beta_1 & \cos \beta_1 \cos \beta_2 & \cos \beta_1 \cos \beta_2 \sin \beta_1 \end{bmatrix}.
$$

Now in equation (22), we can apply the chain rule to rewrite the Hessian with respect to $p'_1, p'_2$ as

$$
T = [T_{jk}], \quad T_{jk} = \frac{\partial^2 T(x', x)}{\partial p'_j \partial p'_k} = \frac{\partial^2 T(x', x)}{\partial p_{\lambda 0} \partial p_{\kappa 0}} \frac{\partial p_{\lambda 0}}{\partial p_{\lambda 0}} \frac{\partial p_{\kappa 0}}{\partial p_{\kappa 0}} \frac{\partial p'_j}{\partial p'_j} \frac{\partial p'_k}{\partial p'_k}, \quad j, k = 1, 2, \quad \lambda, \kappa = 1, 2.
$$

(25)

Summation from 1 to 2 over the repeated indices $\lambda$ and $\kappa$ is understood in this equation.

This rotation of the representation of the initial slownesses is also the rotation of the initial directions of the coordinate system. It makes the direction of the post rotation $z$-axis be along the initial tangent of the ray from $x_0$ through $x$.

We can calculate the matrix of derivatives among the slownesses in equation (25) from equation (23) relating these variables. We simply differentiate both sides of equation (25) with respect to $p'_1$ and $p'_2$. In doing so, we must take care to note that $p'_3$ is a function of $p'_1$ and $p'_2$ through the eikonal equation in Cartesian coordinates, namely,

$$
p'_1^2 + p'_2^2 + p'_3^2 = 1/v^2.
$$

(26)

When we carry out these calculations, we find that

$$
\left[ \frac{\partial p_{\lambda 0}}{\partial p'_j} \right] = \begin{bmatrix} \frac{\cos \beta_2}{\cos \beta_1} & \frac{\sin \beta_2}{\cos \beta_1} \\
-\sin \beta_2 & \cos \beta_2 \end{bmatrix} = [\Gamma_{\lambda j}] = \Gamma, \quad \lambda, j = 1, 2.
$$

(27)

In analogy with equation (25), let us now introduce the notation

$$
T_0 = [T_{\lambda 0}], \quad T_{\lambda 0} = \frac{\partial^2 T(x', x)}{\partial p_{\lambda 0} \partial p_{\kappa 0}}, \quad \lambda, \kappa = 1, 2.
$$

(28)

Then we can rewrite $T$ in equation (25) as

$$
T_{jk} = T_{\lambda 0} \Gamma_{\lambda j} \Gamma_{\lambda k}, \quad j, k = 1, 2; \quad T = \Gamma^T T_0 \Gamma.
$$

(29)

We sum over the repeated indices $\lambda, \kappa$ in the elements form of equation (28) for $T_{jk}$. Again, the superscript $T$ denotes transpose in the matrix form of this same equation for $T$.

Equation (29) completes the transformation of the Hessian of the complex traveltime in the initial slownesses $p'$ to a Hessian in the initial slownesses $p_0$. 

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3.2 Transformation to a Hessian with respect to $q$

Varying $p_0$ leads to different rays, $q = q(p_0, s)$. Furthermore, the arc length $s$ along the central ray also changes with $p_0$ because the image point $x$ is fixed: $s = s(q)$. As an example, consider the configuration of Figure 5 on page 18 for homogeneous media where the rays from $x_0$ to $x'$ are straight lines. Similar to the complex exponent in equation (11) we find that the complex traveltime in equation (13) is

$$T = \frac{s}{V_0} + \frac{i|q|^2}{2[\omega_0^2 + iV_0s]}, \quad s = \sqrt{|x' - x_0|^2 - q^2}. \quad (30)$$

As claimed, $s = s(q)$. We could equally have confirmed that $s$ was a function of $p'$ or $p_0$, but the confirmation at this stage is much easier.

![Figure 6: Central rays from the source and receiver propagating to points $x'_s$ and $x'_r$. Those points connect to the image point $x$ by straight lines that are orthogonal to the respective rays at $x'_s$ and $x'_r$, respectively. The exponential decay of the respective complex traveltime from source and receiver arises from quadratic forms in the respective $q$'s defining the vectors $x - x'_s$ and $x - x'_r$. This configuration might be typical for a choice of $p'_m$ different from $p'_m$, for which both rays pass through $x$ and the $q$'s are all equal to zero. The difference of the transverse parts of the vector difference $p'_r - p'_s$ is the 2D vector $p_h$ in the acquisition plane, while the transverse part of the sum $p'_r + p'_s$ is the 2D vector $p'_m$ in the acquisition plane.]

Therefore, we consider a change of variables from $p_0$ to $q$. In that case, we write

$$\frac{\partial T}{\partial p_{\lambda_0}} = \frac{\partial T}{\partial q_{\mu}} \frac{\partial q_{\mu}}{\partial p_{\lambda_0}}, \quad \lambda, \kappa = 1, 2, \quad (31)$$

and

$$\frac{\partial^2 T}{\partial p_{\lambda_0} \partial p_{\kappa_0}} = \frac{\partial^2 T}{\partial q_{\mu} \partial q_{\nu}} \frac{\partial q_{\mu}}{\partial p_{\lambda_0}} \frac{\partial q_{\nu}}{\partial p_{\kappa_0}} + \frac{\partial T}{\partial q_{\mu}} \frac{\partial^2 q_{\mu}}{\partial p_{\lambda_0} \partial p_{\kappa_0}}, \quad \lambda, \kappa = 1, 2. \quad (32)$$
As previously, summation over the repeated indices $\mu$ and $\nu$ from 1 to 2 is to be understood.

Viewing this Hessian in $q$ helps us to understand a difficulty in the evaluation process: in general, the saddle points $p_h^{sad}(p_m')$ do not place the central rays through $x$, but rather through some $x'$ or $x''$ as in Figure 6. In this case, $Im[T] \neq 0$ for one and/or the other central ray from source and receiver. Hence the total traveltime will have some exponential decay. For a certain $p_m'$, say, $p_m'^0$, both rays for the saddle point in $p_h$ will pass through $x$; that is, $q_s = q_r = 0$ and $Im[T] = 0$, as well. This is the choice of $p_m'$ that would be the saddle point in $p_m'$ if we were to estimate the integral in $p_m'$ by the method of steepest descent. For this choice of $p_m' = p_m'^0$, all of the first derivatives with respect to slowness are zero, since the first derivatives with respect to the elements of $p_h = p_h^{sad}(p_m')$ are already equal to zero. With all of these derivatives equal to zero, the first derivatives with respect to the elements of $q$ must also be zero here, as well, when $p_h = p_h^{sad}(p_m')$ and $p_m' = p_m'^0$. Furthermore, from the equation for equation (13), we can see that the first derivatives of the traveltime $T$ are linear in the $q$’s.

From the point of view of leading order asymptotic analysis, the smooth variations of the amplitude of the integrand away from $p_m'^0$ matter little as long as that amplitude is correct at $p_m'^0$. The Hessian that we are trying to evaluate is a piece of the amplitude in the integral over $p_m'$, so this observation applies to the evaluation of the Hessian. This is true whether we evaluate the integral numerically or with some asymptotic formula. We avoid using an asymptotic formula because we cannot a priori guarantee the nature of the saddle point—whether it is a simple or higher order saddle point of the complex traveltime as a function of $p_m'$. The numerical evaluation will be correct in either case, but it only depends on the amplitude near $p_m'^0$; $Im[T] \neq 0$ away from this value of $p_m'$, and the error in amplitude of the integrand in equation (19) for $I$ is exponentially damped by the value of $Im[T]$. As noted in the introduction, the approximation we propose here has been empirically shown to be adequate for estimating the reflection coefficient in 2D as demonstrated in the numerical example in the last section of this paper.

Thus, we can evaluate the formula for the Hessian in $p_0$ on the right hand side of equation (25)—admittedly incorrect for $p_m' \neq p_m'^0$—by setting all first derivatives of $T$ equal to zero and also setting $q = 0$. Now, in equation (32), we simplify the right hand side to write

$$\frac{\partial^2 T}{\partial p_{\lambda 0} \partial p_{\kappa 0}} = \frac{\partial^2 T}{\partial q_{\mu} \partial q_{\nu}} \frac{\partial q_{\mu}}{\partial p_{\lambda 0}} \frac{\partial q_{\nu}}{\partial p_{\kappa 0}}, \quad \lambda, \kappa = 1, 2. \tag{33}$$

Furthermore, we will evaluate all derivatives at $q = 0$.

As an aside, let us examine the exponent $T$ for constant velocity, equation (30).

The first and second derivatives of that traveltime with respect to $q$ are as follows

$$\frac{\partial T}{\partial q_{\mu}} = -\frac{1}{V_0} q_{\mu} + \frac{i q_{\mu}}{[\omega_r w_0^2 + i V_0 s]} - \frac{i |q|^2}{[\omega_r w_0^2 + i V_0 s]^2} \frac{q_{\mu}}{s}, \quad \mu = 1, 2, \tag{34}$$

$$\frac{\partial^2 T}{\partial q_{\mu} \partial q_{\nu}} = -\frac{1}{V_0} \left[ \frac{\delta_{\mu \nu}}{s} + \frac{q_{\mu} q_{\nu}}{s^3} \right] + \frac{i \delta_{\mu \nu}}{[\omega_r w_0^2 + i V_0 s]} + O(|q|^2), \quad \mu, \nu = 1, 2,$$

$$s = \sqrt{|x' - x_0|^2 - q^2}.$$

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In the equation for the Hessian $\frac{\partial^2 T}{\partial q_\mu \partial q_\nu}$ above, $\delta_{\mu \nu}$ is the Kronecker delta function, equal to one for $\mu = \nu$ and equal to zero otherwise.

Recall the discussion below equation (32) where we argued that quadratic corrections amplitude functions would affect the integration over slownesses negligibly. The most dominant part of the integrand was the neighborhood of the image point. For rays nearby that point, the quadratic correction is small. For rays further away the quadratic decay arising from $T$ again makes the error negligible. Thus, there is no reason to write down those last terms in the second line in equation (34) to make the point that this Hessian is a complicated function of $q$, even for constant velocity. Neglecting the quadratic terms in the second derivative of $T$ in equation (34) we obtain the approximation

$$\frac{\partial^2 T}{\partial q_\mu \partial q_\nu} = -\delta_{\mu \nu} \frac{V_0}{|\mathbf{x}' - \mathbf{x}_0|} + \frac{i \delta_{\mu \nu}}{|\omega_r w_0^2 + i V_0 |\mathbf{x}' - \mathbf{x}_0|}$$

$$= -\frac{\delta_{\mu \nu} \omega_r w_0^2}{|\omega_r w_0^2 + i V_0 |\mathbf{x}' - \mathbf{x}_0|}, \quad q = 0. \tag{35}$$

In the first form here we can identify the two quotients with the matrix solutions of the dynamic ray equations (6). In matrix form

$$\mathbf{T}_q = \left[ \frac{\partial^2 T}{\partial q_\mu \partial q_\nu} \right]_{q=0} = -\mathbf{P}_2 \mathbf{Q}_2^{-1} + \mathbf{P}_{GB} \mathbf{Q}_{GB}^{-1} = \mathbf{P}_{GB} \mathbf{Q}_{GB}^{-1} - \mathbf{P}_2 \mathbf{Q}_2^{-1} \tag{36}$$

$$= \mathbf{Q}_{GB}^{-1} \mathbf{P}_{GB}^T - \mathbf{P}_2 \mathbf{Q}_2^{-1} = \mathbf{Q}_{GB}^{-1} \left[ \mathbf{P}_{GB}^T \mathbf{Q}_2 - \mathbf{Q}_{GB}^T \mathbf{P}_2 \right] \mathbf{Q}_2^{-1}.$$
that $\Delta$ is constant on rays, given by its initial value. The initial values for $Q_{GB}$ and $P_{GB}$ are stated in equation (10) and the initial values of $Q_2$ and $P_2$ are stated in equation (8). Using these values we find that
\[
\Delta = P^T_{GB}Q_2 - Q^T_{GB}P_2 = -\frac{\omega_r w_0^2}{V_0} I = -\frac{\omega_r w_0^2}{v_0(0)} I. \tag{39}
\]

This is an identity that arises from the simplectic properties of the propagator matrix as discussed by Červený [2001] in section 4.3.2, starting on page 281 and introduced earlier in Červený and Pšencík [1983] without resorting to the simplectic method. Note that the derivation of differential equation (38) did not rely on the fact that the velocity is constant in the specific example. Hence, even for heterogeneous media, $\Delta$ is given by its initial value on the ray.

Furthermore, the mechanics of the derivation were not peculiar to the particular choices of $Q$’s and $P$’s. It relied solely on the differential equations for $Q$’s and $P$. Thus, if we redefined $\Delta$ for any pairs of $Q$’s and $P$’s satisfying the dynamic differential equations (10), it would still be true that $\Delta$ is constant on the rays, given by its initial value or by its value at any point on the ray, for that matter.

To complete this discussion, we now use the identity for $\Delta$ in the last equation (39) to rewrite $T_q$ in equation (36) as
\[
T_q = -\frac{\omega_r w_0^2}{v_0(0)} Q_2^{-1}Q_{GB}^{-1} - \frac{\omega_r w_0^2}{v_0(0)} Q_{GB}^{-1}Q_2^{-1}. \tag{40}
\]

In the second form here, we have exploited the fact that each of the matrices is symmetric. We took a transpose of the product and then removed the transpose of the matrix $Q_{GB}^{-1}$.

We return now to the heterogeneous case, continuing the analysis of the Hessian on the right hand side of equation (33). Let us begin by considering the first derivatives in that equation. We now show that the matrix of derivatives of $(q_1, q_2)$ with respect to $(p_{10}, p_{20})$ is just $Q_2$ by following Červený [2001], Section 4.1.7. To do so, let us set
\[
\tilde{Q} = \left[ \frac{\partial q_\mu}{\partial p_{\lambda 0}} \right], \quad \tilde{P} = \left[ \frac{\partial p_\mu}{\partial p_{\lambda 0}} \right]. \tag{41}
\]

This pair of vectors $q_\mu, p_\mu$ satisfies the kinematic ray equations and initial conditions stated in equation (4). When we differentiate those equations with respect to the components of $p_0$ we obtain exactly the dynamic ray equations (6) for the matrices $\tilde{Q}, \tilde{P}$. Then, differentiation of the initial conditions of equation (4) yields exactly the initial conditions of the functions $Q_2, P_2$ given in equation (8). That is,
\[
\tilde{Q} = \left[ \frac{\partial q_\mu}{\partial p_{\lambda 0}} \right] = Q_2. \tag{42}
\]

We use this identity and equation (33) to obtain
\[
T_0 = Q_2 T_q Q_2, \quad T_q = \left[ \frac{\partial^2 T}{\partial q_\mu \partial q_\nu} \right], \quad \mu, \nu = 1, 2. \tag{43}
\]
3.2.1 Analysis of $T_q$

Here, we follow the discussion of Červený [2001], Section 5.8.3. Figure 5 on page 18 shows the central ray of a Gaussian beam. The Gaussian beam is to be evaluated at the point $x$, thus defining the vector $q$ connecting a point on the ray $x'$ to the point $x$. In equation (13) we see that $T$ is a sum of two traveltimes, the first being the central ray real traveltime $\tau(s)$ of asymptotic ray theory connecting $x_0$ to $x'$. The second term is the complex part of the traveltime given by the standard quadratic form in $q$.

It is straightforward to evaluate the Hessian of the second term here with respect to $q$ at $q = 0$. The first term requires a little more effort.

With a slight abuse of notation, let us consider the real ray-theoretic traveltime $\tau(x, x_0)$ from $x_0$ to $x$. We can write a quadratic approximation of that traveltime in terms of the traveltime on the central ray through $x'$ as

$$\tau(x, x_0) = \tau(x', x_0) + \frac{1}{2} q P_2 Q_2^{-1} q. \quad (44)$$

This equality allows us to write the traveltime $\tau(x', x_0)$ in terms of the fixed traveltime $\tau(x, x_0)$ minus a quadratic form in the vector $q$; that is,

$$\tau(x', x_0) = \tau(x, x_0) - \frac{1}{2} q P_2 Q_2^{-1} q. \quad (45)$$

We use this equation in equation (13) to rewrite the complex traveltime $T(x', x_0)$ as

$$T(x', x_0) = T(x, x_0) + \frac{1}{2} q^T \left[ P_{GB} Q_{GB}^{-1} - P_2 Q_2^{-1} \right] q. \quad (46)$$

Now, when we calculate the second derivative here at $q = 0$, we find that $T_q$ defined in equation (43) is given by

$$T_q = P_{GB} Q_{GB}^{-1} - P_2 Q_2^{-1}, \quad q = 0, \quad (47)$$

in heterogeneous media. This is exactly the same as equation (36) for $T_q$ in homogeneous media. As noted in the discussion below that equation, the simplification of $T_q$ did not rely on the fact that the velocity was constant. Thus, the final form of $T_q$ in equation (40) is valid here as well and we write for the heterogeneous case.

$$T_q = -\frac{\omega r w_0^2}{v_0(0)} Q_2^{-1} Q_{GB}^{-1}, \quad q = 0. \quad (48)$$

We now have all of the pieces needed to express the Hessian matrix $\Psi$ in equation (21).

4 The approximation of $\Psi$

We now move back through the sequence of formulas for the various Hessians that we introduced in the previous section. Then we will take the formula for the Hessian with respect to $p'$ and apply it to both the source and receiver traveltimes to obtain a formula for $\Psi$ in equation (21).
To begin, we use \( T_q \) as defined in equation (48) to rewrite \( T_0 \) in equation (43) as
\[
T_0 = -\frac{\omega_r w_0^2}{v_0(0)} Q_2 Q_2^{-1} Q_{GB}^{-1} Q_2 = -\frac{\omega_r w_0^2}{v_0(0)} Q_{GB}^{-1} Q_2
\]  
(49)

Now that we have \( T_0 \), we can go back a further step by substituting this representation into equation (29) to express the Hessian with respect to \( p' \) in terms of \( T_0 \) as follows.
\[
T = -\frac{\omega_r w_0^2}{v_0(0)} \Gamma^T Q_{GB}^{-1} Q_2 \Gamma.
\]  
(50)

The matrix \( \Gamma \) appearing on the right side of this equation is defined in terms of the angles \( \beta_1 \) and \( \beta_2 \) in equation (27). It is the matrix of transformation from Cartesian slownesses to ray-centered slownesses; the slowness vector defines the initial direction of the tangent of the ray from the initial point \( x_0 \) to the image point \( x \). Further, recall that the various \( Q \)'s and \( P \)'s here are solutions of the dynamic ray equations (6) with initial conditions for \( Q_{GB} \) and \( P_{GB} \) given in equation (10) and the initial conditions for \( Q_2 \) and \( P_2 \) given in equation (8).

Equation (21) tells us that we must add two Hessians of the form defined by the last equation, (50), in order to obtain the Hessian \( \Psi \) for the complex traveltime \( \Psi \) which is related to the two complex traveltimes from source and receiver by equation (15). The only difference in the two components is that one is for sources and the other is for receivers. We can accomplish that distinction by introducing subscripts \( s \) and \( r \) in the right hand side of equation (50). Thus we find that
\[
\Psi = -i\omega_r w_0^2 \left\{ \frac{1}{v_{or}(0)} \Gamma_r^T Q_{GBr}^{-1} Q_{2r} \Gamma_r + \frac{1}{v_{os}(0)} \Gamma_s^T Q_{GBs}^{-1} Q_{2s} \Gamma_s \right\}^*.
\]  
(51)

Note here that the subscript \( r \) in \( \omega_r \) denotes “reference frequency” and is the same for Hessians associated with the traveltimes from source and receiver.

The matrices \( Q_{2r} \) and \( Q_{2s} \) are each singular if \( x'_r \) or \( x'_s \), respectively, is located at a caustic of the ensemble of central rays from their respective initial points. In order for this sum of matrices to be singular, the singularities of the two matrix products would have to “line up;” that is, their eigenvectors associated with the zero eigenvalue would have to be colinear. We would expect such an occurrence to be relatively rarer than for one and/or the other of the matrix products to be singular. Nonetheless, some regularization of this matrix may be required in numerical computations.

As noted earlier, we can expect the Hessian matrix \( \Psi \) to be singular when the image point is located at a caustic of incident rays. However, the underlying inversion theory is not applicable at such a point for estimation of a reflection coefficient, so this anomaly is moot as regards that theory.

In the Introduction and in Section 2.3 we claimed that we did not need to know the actual initial slownesses of the ray from the initial point \( x_0 = x_s \) or \( x_r \) to the image point \( x \).

We explain now why we do not need to know those rotation angles on the ray from the source or receiver through the image point explicitly. It is necessary to assume that the actual rays that we do compute are close sufficiently dense that the discrete sum over rays is
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a sufficiently accurate approximation of the continuous integral over rays to satisfy whatever numerical accuracy criterion we impose. That means we can think of the discrete sum as having the properties of the integral.

We will use the matrices $\Gamma_r$ and $\Gamma_s$ on the rays that we actually compute. This again is justified by the discussion below equation (32). This is in error except for the distinguished ray pair from source and receiver through the image point. Those errors for other rays are linearly small for small values of the $q$’s nearby and exponentially small for $q$’s further away. This suffices for our leading order approximation of the the integral $I$ in equation (19). Then, all quantities in equation (15) are determined on the central rays of the computed Gaussian beams. As we will see in the common shot example of Section 5—in particular in equation (60)—the reflectivity in that example needs to be computed on the actual central rays of Gaussian beams that we compute. Hence, we can use Hill’s original variables $p_m$ and $p_h$ or even in the variables $p'_r$ and $p'_s$ of equation (15).

4.1 Special case: $V = 0$

In many applications, the matrix of second derivatives $V$ defined by equation (1) is set equal to zero. This is equivalent to assuming a piecewise constant or piecewise linear velocity model. As a consequence, in equation (6) we see that the dynamic quantity $P$ is a constant given by its initial value. Furthermore, the dynamic equations now separate into scalar equations for the pairs $Q_{ij}$ and $P_{ij}$ for fixed $ij$ and there is no interaction with other elements of the matrix; that is, the matrix equations reduce to scalar equations for each of these pairs. Further, for the case of piecewise constant velocity, the matrices $Q$ and $P$ become diagonal with the same functions propagating along each diagonal because the initial data are diagonal.\(^2\) Since the initial data are the same for each diagonal element of $Q$ and $P$ in this case, each matrix is a scalar times the identity matrix; We call such a matrix a scalar matrix. Thus, the propagation is reduced to determining the scalar $Q$’s and $P$’s of 2D propagation.

With all of this simplifications, we can replace the matrix of $\Psi$ by

$$\Psi = -i\omega w_0^2 \left\{ \frac{1}{v_{0r}(0)}Q_{2r}Q_{GBr}^{\Gamma_r^T \Gamma_r} + \frac{1}{v_{0s}(0)}Q_{2s}Q_{GBs}^{\Gamma_s^T \Gamma_s} \right\}^*.$$\hspace{1cm}(52)

From equation (27), it is straightforward to calculate the products of $\Gamma$’s appearing here. They are

$$\Gamma_r^T \Gamma_r = \begin{bmatrix} \cos^2 \beta_{1r} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{[v_{0r}(0)p_{3r}]^2} \\ 0 & 1 \end{bmatrix},$$\hspace{1cm}(53)

\(^2\)If the velocity is linear in some direction, then that direction and the initial direction of each ray form a plane. The initial values of $Q$ and $P$ will be different in the in-plane and out-of-plane directions.
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\[ \Gamma_s^T \Gamma_s = \begin{bmatrix} \frac{1}{\cos^2 \beta_{1s}} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\|v_{0s}(0)p'_{3s}\|^2} & 0 \\ 0 & 1 \end{bmatrix}. \]

Because these rotation matrices are not scalar matrices, \( \Psi \) is not a scalar matrix; that is, they are not scalar multiples of the identity matrix.

5 Common-shot inversion with Gaussian beams; an example

We will show a particular example of a common-shot inversion formula for which the product of Green’s functions leads to an integral of the form of \( I \) in equation (15).

We start from the classic imaging condition for common-shot data [Claerbout, 1971],

\[ \mathcal{R}(\mathbf{x}, \mathbf{x}_s, \theta) = \frac{1}{2\pi} \int \frac{U(\mathbf{x}, \mathbf{x}_s, \omega)}{D(\mathbf{x}, \mathbf{x}_s, \omega)} d\omega. \]  

(54)

This formula for the reflectivity \( \mathcal{R} \) is asymptotically equivalent to the Kirchhoff inversion formula when \( U \) and \( D \) are replaced by their ray-theoretic asymptotic expansions; see Keho and Beydoun [1988], Hanitzsch [1997], Zhang et al [2003] and Bleistein et al [2005]. The function \( U(\mathbf{x}, \mathbf{x}_s, \omega) \) is generated by downward propagating \( U(\mathbf{x}_r, \mathbf{x}_s, \omega) \), the observed response to a point source at \( \mathbf{x}_s \) at the receivers denoted by \( \mathbf{x}_r \). The function \( D(\mathbf{x}, \mathbf{x}_s, \omega) \) is the full-bandwidth downward propagating wave from a point source, that is, it is the downward part of the Green’s function denoted by \( G(\mathbf{x}, \mathbf{x}_s, \omega) \). Both wave fields are propagated into the Earth in some background model of the wavespeed.

Let us replace \( D \) by \( G \) in the integrand of equation (54) for \( \mathcal{R} \) and then multiply numerator and denominator by \( G^* \), the complex conjugate of the downward propagating wave from the source. This leads to

\[ \mathcal{R}(\mathbf{x}, \mathbf{x}_s) = \frac{1}{2\pi} \int \frac{U(\mathbf{x}, \mathbf{x}_s, \omega)G^*(\mathbf{x}, \mathbf{x}_s, \omega)}{G(\mathbf{x}, \mathbf{x}_s, \omega)G^*(\mathbf{x}, \mathbf{x}_s, \omega)} d\omega. \]  

(55)

Next, we propose to replace the Green’s functions in the denominator here by their leading order ray-theoretic asymptotic expansions and observe that only a product of amplitudes \( A \), independent of \( \omega \) survive, the multiplication:

\[ \mathcal{R}(\mathbf{x}, \mathbf{x}_s) = \frac{1}{2\pi A^2(\mathbf{x}, \mathbf{x}_s)} \int U(\mathbf{x}, \mathbf{x}_s, \omega)G^*(\mathbf{x}, \mathbf{x}_s, \omega)d\omega. \]  

(56)

In ray-centered coordinates

\[ \frac{1}{A^2(\mathbf{x}, \mathbf{x}_s)} = (4\pi)^2 \frac{v_0(\mathbf{x}_s)}{v_0(\mathbf{x})} \frac{\text{det}[Q_2(\mathbf{x}, \mathbf{x}_s)]}{v_0(\mathbf{x})}. \]  

(57)

The determinant \( \text{det}[Q_2(\mathbf{x}, \mathbf{x}_s)] \) will be equal to zero at caustics of the rays from the source to the output point; some regularization is required to avoid the zeroes there. That is not important to the discussion here.
When we use this formulas for $A$ in equation (56) for the reflectivity, we find that

$$R(x, x_s) = \frac{8\pi v_0(x_s)}{v_0(x)} \det[Q(x, x_s)] \int U(x, x_s, \omega)G^*(x, x_s, \omega)d\omega. \quad (58)$$

The downward propagated field $U(x, x_s, \omega)$ can be written in terms of the surface data as

$$U(x, x_s, \omega) = 2i\omega \int_{z_r=0} p_{3r}' G^*(x, x_r, \omega)U(x_r, x_s, \omega)dx_rdy_r \quad (59)$$

in a standard manner by using Green’s theorem. Here, $p_{3r}'$ is the third component of $p'_r$.

Substitution of this representation of $U(x, x_s, \omega)$ into equation (58) for the reflectivity leads to

$$R(x, x_s) = \frac{8\pi v_0(x_s)}{v_0(x)} \det[Q(x, x_s)] \int 2i\omega \int p_{3r}' U(x_r, x_s, \omega)dx_rdy_r \mathcal{G}(x, x_s, x_r, \omega) \quad (60)$$

In this equation, $\mathcal{G}(x, x_s, x_r, \omega)$ is defined in equation (14) and was the starting point for the analysis of this paper. It is not our intention to continue with the discussion of this true amplitude migration formula or any other. We only wanted to show that the integral $I$ that we analyzed asymptotically here does arise in true-amplitude migration formulas as claimed.

6 A numerical example in 2D

In Gray and Bleistein [2009], we presented numerical examples in 2D for a common-shot data set. The asymptotic technique derived and implemented there is the 2D analog of what we have presented here. Both a deconvolution imaging condition with a structure similar to the reflectivity formula of equation (54) and a correlation imaging formula similar to the reflectivity formula of equation (60) were tested. See Gray and Bleistein [2009] for details.

The best results were obtained for the deconvolution imaging condition and we present those here in Figure 7. This example models reflections from density contrasts in a medium of constant velocity 2000 m/s. Four horizontal reflectors with identical reflection coefficients are placed at depths of 1000, 2000, 3000, and 4000 m. A single shot record, with a recording aperture of 7000 m on either side of the shot point, is migrated. Half-opening angles were limited to 60° in the migration.

Within the reflection aperture for each reflector the output confirms the “true-amplitude” claim but for numerical noise from the computation and from the peak search.

7 Summary and conclusions

We have obtain a formula for the Hessian matrix of complex traveltimes with respect to transverse Cartesian slowness variables [equation (50)]. We believe that this formula is new. That allows us to add two Hessian matrices of this form to obtain yet another Hessian matrix for the sum of complex traveltimes with respect to Cartesian “offset slownesses,” equation (51). The determinant of this Hessian is needed for the asymptotic reduction of four integrals in source and receiver transverse slownesses to two integrals. The formula we evaluate in the
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Figure 7: Left: Migrated image (deconvolution imaging condition) from a single shot record in a constant-velocity medium with horizontal reflectors caused by identical density contrasts. Migrated amplitudes are similar for all reflectors. Amplitude artifacts near the maximum offsets are migration aperture truncation effects. Right: Peak amplitude as a function of offset for all reflectors. Amplitude fall-off corresponds to the peak amplitudes along the migration aperture artifacts of the left figure.

The first step [equation (50)] requires approximating an earlier representation of that Hessian at the saddle point of the travelttime with respect to the “midpoint slowness.”

In two dimensions, the steepest descent analysis is much simpler. There, two integrals in transverse slownesses were reduced to a single integral by the method of steepest descent. The example of Gray and Bleistein [2009], provides a confirmation of “true-amplitude” claim for this method.

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A Iterated Method of Steepest Descent in Two Variables

In Section 2.3, we claimed a result for the asymptotic expansion of a pair of iterated integrals obtained by the method of steepest descent—specifically for the asymptotic expansion of the integral $I(x, x_s, x_r, \omega)$ of equation (15). To the best of the authors' knowledge, the derivation of the explicit asymptotic formula leading to the asymptotic expansion of this integral presented in equation (19) has not appeared in the open literature. In that equation, $\Psi$ is given by equation (51). We derive the claimed leading order asymptotic expansion of equation (19) here. The derivation of the matrix form of $\Psi$ in equation (51) is the major discussion of Section 2.3.

The classical method of steepest descent for a one-dimensional integral does not generalize to higher dimensions. It would require the analog of the Cauchy integral theorem in two complex variables—essentially four variables. The difficulty is that in higher dimensions a closed curve does not contain an interior domain as it does in one complex variable; that is, in two dimensions. On the other hand, we show here that the iterated application of the one-complex variable method of steepest descent really behaves much like the corresponding application of the multidimensional method of stationary phase, with the complex determinant of the Hessian matrix of the complex traveltime appearing in the amplitude of the asymptotic expansion. This is an alternative to the method used by Červený [1982] and repeated in Červený [2001]. His method rests on a theorem in linear algebra that gives conditions under which the sum of two complex-valued symmetric matrices can be simultaneously diagonalized. His reference for that result is a paper in the Russian literature, not easily accessible in the western literature. To date, we have not found the counterpart of this theorem in the western literature.

A.1 Steepest descent in one dimension

Before beginning the analysis of the double integral, we remind the reader of the corresponding asymptotic expansion in one variable. We introduce the integral

$$I(\omega) = \int f(z) \exp\{-\omega \Psi(z)\} \, dz.$$  \hfill (A.1)

Here, the interval of integration is real\(^3\) and positively oriented, but both functions in the integrand are complex-valued. We assume that the exponent has a saddle point that we can characterize by the equation

$$\frac{d\Psi}{dz} = 0, \quad z = z_{sad}.$$  \hfill (A.2)

Furthermore, we assume that the second derivative is nonzero at the saddle point,

$$\Psi_{zz} = \frac{d^2\Psi}{dz^2} \bigg|_{z=z_{sad}} \neq 0;$$  \hfill (A.3)

\(^3\)The constraint of the path of integration to the real axis is not necessary. This can be generalized, but the generalization is not needed for our application.
Ψ_{zz} may be complex. We seek the leading order asymptotic expansion of this integral for "large" values of the parameter ω.

The Taylor expansion of Ψ near the saddle point has the form

\[ Ψ(z) − Ψ(z_{sad}) = \frac{1}{2} Ψ_{zz}(z − z_{sad})^2 + \ldots. \]  

(A.4)

The direction of steepest descent in \( z − z_{sad} \) at the saddle point is the direction in which this second order approximation is real and positive, which provides the direction of maximal exponential decay of the integrand. In terms of the phases (denoted by arg in the complex variable literature) of the factors in the complex product appearing on the right side of the last equation, this condition becomes,

\[ \arg(Ψ_{zz}) + 2 \arg(z − z_{sad}) = 0, \ 2\pi, \ \ldots. \]  

(A.5)

We expect that the direction of choice will be a rotation of the contour of integration—the real line, positively oriented—through an acute angle. Thus from the two unique choices of direction here we choose

\[ \arg(z − z_{sad}) = −\arg(Ψ_{zz})/2, \]  

(A.6)

such that the oriented direction with this angle makes an acute angle with the direction of the (real) path of integration. Application of the formula for evaluation of the integral of equation (A.1) by the method of steepest descent\(^4\) [Bleistein, 1984, equation (7.3.11)] with positive ω leads to the following.

\[ I(ω) \sim \sqrt{\frac{2\pi}{ω|Ψ_{zz}|}} f(z_{sad}) \exp \{−ωΨ(z_{sad}) − i \arg(Ψ_{zz})/2\} \]

(A.7)

\[ = \sqrt{\frac{2\pi}{ωΨ_{zz}}} f(z_{sad}) \exp \{−ωΨ(z_{sad})\}. \]

Notice that by defining the integrand on the right hand side of equation for \( I(ω) \) with a minus sign in the exponent, the phase adjustment in the first line here provides exactly the right factor so that the denominator can be expressed as the (principal value) square root of the second derivative. If we had defined the exponent without that minus sign, then we would have need an additional phase shift of \( ±\pi/2 \) in the right hand exponents of this last equation.

The method of steepest descent relies on deforming the path of integration onto a path where the exponential difference Ψ(z) − Ψ(z_{sad}) remains real and increasing and then applying the more basic Laplace method [Bleistein, 1984] to the integral on that steepest descent contour.

\(^4\)There we use \( \lambda = −ω \) as the large parameter. The minus sign modifies the argument of the directions of steepest descent in equation (A.6).
A.2 Iterated steepest descent in two dimensions

This is an extension not found in texts on the method of steepest descent. We consider the iterated integral of equation (15). For this purpose, we introduce indexed variables and consider the integral

\[ I(\omega) = \int f(z) \exp\{-\omega \Psi(z)\} dz_1 dz_2, \quad z = (z_1, z_2). \quad (A.8) \]

Here, the domain of integration is real, although when considered as iterated integrals, we will allow for deformations into the complex \( z_1 \)-plane to obtain the asymptotic expansion of the integral with respect to \( z_1 \) and similarly for \( z_2 \). Both functions in the integrand are complex-valued.

We assume that the exponent has a saddle point in both variables that we can characterize by the equation

\[ \nabla_z \Psi(z) = 0, \quad z = z_{sad} = (z_{1sad}, z_{2sad}). \quad (A.9) \]

Furthermore, we assume that the Hessian matrix—the matrix of second derivatives—of \( \Psi \) is non-singular at this saddle point; that is

\[ \det[\Psi] \neq 0. \quad \Psi = \left[ \frac{\partial^2 \Psi}{\partial z_i \partial z_j} \right], i, j = 1, 2. \quad (A.10) \]

We will also assume that

\[ \left. \frac{\partial^2 \Psi}{\partial z_1^2} \right|_{z=z_{sad}} \neq 0. \quad (A.11) \]

If this were not the case, we could rotate the coordinate system to make this so; such rotations use matrices with determinant equal to one, so that they do not affect the final formula for the asymptotic expansion of \( I(\omega) \) in equation (19).

Let us consider the integration in \( z_1 \) alone in equation (A.8). From equation (A.10), that integral has a saddle point when

\[ \left. \frac{\partial \Psi(z_1, z_2)}{\partial z_1} \right|_{z=z_{sad}} = 0. \quad (A.12) \]

By the assumption of the existence of the saddle point in equation (A.10), we know that this equation has a solution for \( z_2 = z_{2sad} \), at which point, \( z_1 = z_{1sad} \). By the implicit function theorem, equation (A.12) has a unique solution in the neighborhood of \( z = z_{sad} \) by virtue of the assumption of equation (A.11) that the second derivative with respect to \( z_1 \) is nonzero at the saddle point. Therefore we can write

\[ z_1 = Z(z_2), \quad \frac{\partial \Psi(Z(z_2), z_2)}{\partial z_1} \equiv 0 \quad (A.13) \]

in some neighborhood of \( z = z_{sad} \).

We can now write down the asymptotic expansion with respect to \( z_1 \) of the iterated integral \( I(\omega) \) in equation (A.8) by using the formula of equation (A.7) for that asymptotic expansion in one variable:

\[ I(\omega) = \sqrt{\frac{2\pi}{\omega}} \int \frac{f(Z(z_2), z_2)}{\sqrt{\Psi_{z_1 z_1}(Z(z_2), z_2)}} \exp\{-\omega \Psi(Z(z_2), z_2)\} dz_2. \quad (A.14) \]
We write down the first derivative with respect to $z_2$ of this redefined exponent $\Psi$ by applying the chain rule to deal with the dependence of the first variable of $\Psi$ on $z_2$.

\[
\frac{d\Psi(Z(z_2), z_2)}{dz_2} = \frac{\partial \Psi}{\partial z_1} dZ + \frac{\partial \Psi}{\partial z_2}.
\]  
(A.15)

The first term here is identically equal to zero as a result of the stationarity condition in equation (A.13). Therefore, let us rewrite the first derivative here accordingly and then write down the second derivative, as well.

\[
\frac{d\Psi(Z(z_2), z_2)}{dz_2} = \frac{\partial \Psi}{\partial z_2},
\]

\[
\frac{d^2\Psi(Z(z_2), z_2)}{dz_2^2} = \frac{\partial^2 \Psi}{\partial z_2^2} + \frac{\partial^2 \Psi}{\partial z_2 \partial z_1} dZ + \frac{\partial^2 \Psi}{\partial z_2 \partial z_1} dZ.
\]  
(A.16)

From the first line here, we see that the condition that the total derivative with respect to $z_2$ of this new exponent be equal to zero is exactly the condition that the partial derivative with respect to $z_2$ be equal to zero. This is just the requirement that the second component of the gradient of $\Psi$ be equal to zero in equation (A.9). We then conclude that the saddle point occurs at $z_2 = z_{2\text{sad}}$ for which point we also have $z_1 = z_{1\text{sad}}$. In summary, the dual saddle point obtained by setting the gradient of the original exponent equal to zero is the same as the simultaneous saddle point in two separate variables determined by iterated application of the method of steepest descent.

Now we must evaluate the second derivative of $\Psi$ in equation (A.15) at the saddle point. To this end, we must first express the derivative of $Z(z_2)$ in terms of derivatives of $\Psi$. The function $Z(z_2)$ is defined implicitly in equation (A.13). We differentiate that equation with respect to $z_2$:

\[
\frac{\partial^2 \Psi(Z(z_2), z_2)}{\partial z_1^2} \frac{dZ(z_2)}{dz_2} + \frac{\partial^2 \Psi(Z(z_2), z_2)}{\partial z_1 \partial z_2} \equiv 0.
\]  
(A.17)

The coefficient of the derivative of $Z$ is nonzero at the saddle point—equation (A.11)—so we can divide by it and conclude that

\[
\frac{dZ(z_2)}{dz_2} = -\frac{\partial^2 \Psi(Z(z_2), z_2)}{\partial z_1 \partial z_2} \left[ \frac{\partial^2 \Psi(Z(z_2), z_2)}{\partial z_1^2} \right]^{-1}.
\]  
(A.18)

We substitute this value of the first derivative into the second line of equation (A.15) to obtain the representation we seek for the second derivative of $\Psi$ with respect to $z_2$.

\[
\frac{d^2\Psi(Z(z_2), z_2)}{dz_2^2} = \left[ \frac{\partial^2 \Psi}{\partial z_1^2} \frac{\partial^2 \Psi}{\partial z_2^2} - \left( \frac{\partial^2 \Psi}{\partial z_1 \partial z_2} \right)^2 \right] \left[ \frac{\partial^2 \Psi(Z(z_2), z_2)}{\partial z_1^2} \right]^{-1}
\]

\[
= \det[\Psi] \left[ \frac{\partial \Psi(Z(z_2), z_2)}{\partial z_1^2} \right]^{-1}.
\]  
(A.19)
We again apply the asymptotic expansion formula of equation (A.7) to the integral of equation (A.14) and find that

\[ I(\omega) \sim \frac{2\pi}{\omega} \frac{f(z_{sad})}{\sqrt{\det[\Psi(z_{sad})]}} \exp\{-\omega\Psi(z_{sad})\}. \tag{A.20} \]

This is the formula that we applied to obtain the asymptotic expansion of the Gaussian beam representation of \( I \) in equation (15) to obtain the asymptotic expansion of \( I \) in equation (19).

We have presented this general formula here because it has application to other multidimensional integrals in the analysis of Gaussian beams. In particular, it is applicable to the asymptotic expansion of the basic Green’s function itself in heterogeneous media. Hill’s [2001] derivation is in homogeneous media. This derivation is an alternative to one presented by Červený [1982] and Červený [2001].