Seismic Inversion with One-way Wave Equations

Yu Zhang,[1] Guanquan Zhang,[2] and Norman Bleistein[3]

Abstract

One way wave operators are powerful tools for forward modeling and inversion. However, their implementation involves introduction of the square root of an operator as a pseudo-differential operator. Exact representations of such square roots are illusive, except in the simplest of cases. Here, singling out depth as the preferred direction of propagation, we introduce a representation of the square root operator as an integral in which a rational function of the transverse Laplacian appears in the integrand. This allows us to solve the resulting one-way wave equations with the simple device of introducing an auxiliary function that satisfies a lower dimensional wave equation in transverse variables only. We verify that ray theory for these one-way wave equations leads to one-way eikonal equations and the correct leading order transport equation for the full wave equation. We then

[1]Veritas DGC Inc., 10300 Town Park Drive, Houston, TX 77072
[2]Chinese Academy of Sciences, Beijing, 100080 P. R. China
[3]Center for Wave Phenomena, Department of Geophysics, Colorado School of Mines, Golden, CO 80401-1887 USA

introduce appropriate boundary conditions at $z = 0$ to generate waves at depth whose quotient leads to a reflector map and estimate of the ray-theoretical reflection coefficient on the reflector. This method is known as "true amplitude wave equation migration" in the geophysics literature. Computer output confirms the accuracy of the method for this inversion.

Motivation: Consider the wave equation,

$$\frac{1}{v^2}\frac{\partial^2 W}{\partial t^2} - \nabla^2 W = 0. \tag{1}$$

For constant wavespeed, rewrite in frequency domain in the form

$$\mathcal{L}W = \frac{\partial^2 W}{\partial z^2} + k_z^2 W = \left[\frac{\partial}{\partial z} \mp ik_z\right]\left[\frac{\partial}{\partial z} \pm ik_z\right] W = 0. \tag{2}$$

Here,

$$\begin{aligned} k_z &= \operatorname{sign}(\omega)\sqrt{\frac{\omega^2}{v^2} - \bar{k}^2} = \frac{\omega}{v}\sqrt{1 - \frac{(v\bar{k})^2}{\omega^2}}, \\ \bar{k} &= \sqrt{k_x^2 + k_y^2}. \end{aligned} \tag{3}$$

Further, the solutions of the full second order wave equation are actually solutions of the two one-way wave equations

$$\left\{\frac{\partial}{\partial z} \mp ik_z\right\} A_{\pm} \exp\{\pm ik_z z\} = 0. \tag{4}$$

Here, for z-positive being downward, the upper signs yield a down-going solution and the lower signs yield an upgoing solution.

We would like one-way wave equations for variable velocity, as well. However,

$$\begin{aligned}\left[\frac{\partial}{\partial z} \mp ik_z\right]\left[\frac{\partial}{\partial z} \pm ik_z\right] W &= \frac{\partial^2 W}{\partial z^2} + k_z^2 W \pm i\,\frac{\partial k_z}{\partial z}\, W \\ &= \frac{\partial^2 W}{\partial z^2} + k_z^2 W \mp i\,\frac{1}{v}\,\frac{dv}{dz}\,\frac{\omega^2}{v^2 k_z}\, W.\end{aligned} \tag{5}$$

In this last equation, the first two terms affect amplitude and phase, while the last term will modify the leading order amplitude. To see this better, set

$$\bar{p} = \frac{\bar{k}}{\omega}, \quad p_z = \frac{k_z}{\omega} = \frac{1}{v}\sqrt{1 - (v\bar{p})^2}. \tag{6}$$

Then, the eikonal and transport equations for (2) are

$$\left[\frac{d\tau}{dz}\right]^2 = p_z^2 \quad \Longrightarrow \quad \frac{d\tau}{dz} = \pm p_z, \quad \text{and} \quad 2\frac{d\tau}{dz}\frac{dA}{dz} + \frac{d^2\tau}{dz^2}A = 0, \tag{7}$$

whereas for (4), the corresponding equations are

$$\frac{d\tau_\pm}{dz} = \pm p_z \quad \text{and} \quad 2\frac{d\tau_\pm}{dz}\frac{dA_\pm}{dz} + \frac{d^2\tau_\pm}{dz^2}A_\pm \mp \frac{1}{v^3 p_z}\frac{dv}{dz}A_\pm = 0. \tag{8}$$

True amplitude one-way wave propagation: Correcting these amplitude equations leads to the following one-way wave equations as pseudo-differential operator equations

$$\mathcal{L}_{\pm} W = \left[\frac{\partial}{\partial z} \pm \Lambda \right] W + \Gamma W = 0 \tag{9}$$

Here, Λ and Γ are pseudo-differential operators with symbols λ and γ, respectively:

$$\lambda = i k_z = \frac{i\omega}{v}\sqrt{1 - \frac{(v\bar{k})^2}{\omega^2}}, \qquad \gamma = -\frac{1}{2k_z} = \frac{v_z}{2v}\left(1 + \frac{(v\bar{k})^2}{\omega^2 - (v\bar{k})^2}\right). \tag{10}$$

We can show that the solutions of these equations satisfy the one-way eikonal equations and transport equation in (7) for the full wave equation. A crucial identity from G. Q. Zhang [1993] for proving this result is

$$\lambda = ik_z = \frac{i\omega}{v}\left\{1 - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\,\frac{(v\bar{k})^2}{\omega^2 - s^2(v\bar{k})^2}\,ds\right\}. \tag{11}$$

This identity expresses the operator symbol for k_z without any square roots of operators. The denominator here represents the inverse of a transverse wave operator, essentially a convolution with an appropriate Green's function. The same inverse operator, with $s = 1$ appears in γ. With this identity, the proof of the above-mentioned claim becomes straightforward, but tedious.

First, we write

$$\mathrm{L}_T(s) = -\left\{\frac{\partial^2}{\partial t^2} - s^2 (v(x,y,z)\nabla_{Tx})^2\right\} \tag{12}$$

In this equation

$$(v(x,y,z)\nabla_{Tx})^2 = \left(v(x,y,z)\frac{\partial}{\partial x}\right)^2 + \left(v(x,y,z)\frac{\partial}{\partial y}\right)^2$$

Then we can write the symbolic equations

$$\Lambda = -\frac{1}{v}\frac{\partial}{\partial t}\left\{I - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\mathrm{L}_T^{-1}(s)(v(x,y,z)\nabla_{Tx})^2 ds\right\} \tag{13}$$

and

$$\Gamma = \frac{v_z}{2v}\left(I - \mathrm{L}_T^{-1}(1)(v(x,y,z)\nabla_{Tx})^2\right). \tag{14}$$

True amplitude wave equation migration: The process of producing a reflector map in the Earth from observed surface data is called *migration* in the geophysical literature. See, for example, Claerbout [1971,1985]. (Recorded data is moved from its position on a space-time plot to its position on a space-space plot via the migration process.) When one-way wave propagators are used for obtaining a reflector map, the method is known as *wave equation migration*. The standard method uses the one-way propagators of (2), even for heterogeneous media.

More specifically, supposed that the reflected wave field from a single source experiment is observed at $z = 0$ for all time. Then the source and observed wavefields are assumed to be solutions of the equations

$$\begin{cases} \left(\frac{\partial}{\partial z} + \Lambda\right) D = 0, \\ \\ D(x, y, z = 0; \omega) = -\delta(\vec{x} - \vec{x_s}), \end{cases} \tag{15}$$

and

$$\begin{cases} \left(\frac{\partial}{\partial z} - \Lambda\right) U = 0, \\ \\ U(x, y, z = 0; \omega) = Q(x, y; \omega) \end{cases} \tag{16}$$

where D is the downgoing (source) wavefield and U is the upgoing (observed) wavefield. The image is then produced as an *impedance* or *reflectivity* function at every image point defined by

$$R(x, y, z) = \int \frac{U(x, y, z; \omega)}{D(x, y, z; \omega)} d\omega. \tag{17}$$

The key to this imaging method is that the construct/destructive interference between the phases of the two waves produces a large amplitude where the reflectors reside and a small amplitude where they do not.

While this result produces a reflector map, it does not provide accurate amplitude information. To achieve that, we need to use, instead, the solutions of our modified true amplitude one-way wave equations (9). That is, we introduce p_D and p_U as solutions of the following problems.

$$\begin{cases} \left(\frac{\partial}{\partial z} + \Lambda - \Gamma\right) p_D(x, y, z; \omega) = 0, \\ p_D(x, y, z = 0; \omega) = -\frac{1}{2}\Lambda^{-1}\delta(\vec{x} - \vec{x}_s), \end{cases} \tag{18}$$

and

$$\begin{cases} \left(\frac{\partial}{\partial z} - \Lambda - \Gamma\right) p_U(x, y, z; \omega) = 0, \\ p_U(x, y, z = 0; \omega) = Q(x, y; \omega). \end{cases} \tag{19}$$

Also, we modify the imaging condition (17) to be the quotient of the wave fields p_D and p_U:

$$R(x, y, z) = \int \frac{p_U(x, y, z; \omega)}{p_D(x, y, z; \omega)} d\omega. \tag{20}$$

See Zhang et al. [2001, 2002].

By asymptotic analysis for analytic model data, we can show that this output is the same as the *reflectivity function* of Kirchhoff inversion (true amplitude Kirchhoff migration) as in the theory of Bleistein [1987] and Bleistein et al. [2001]. That result predicts an output for each reflector in the form of a bandlimited delta function of normal distance to the reflector with a peak amplitude in known proportion to the geometrical optics reflection coefficient.

Numerical Results. To show how true amplitude common-shot migration works, we apply it to a 2-D horizontal reflector model in a medium with velocity $v = 2000 + 0.3z$. The input data (Figure 1) is a single shot record over four horizontal reflectors from density contrast. Figure 2-left shows the migrated shot record using the conventional common-shot migration algorithm (17). The peak amplitudes along the four migrated reflectors are shown in the right. This method has a phase error: note the multiplication by i in Λ on the right side in (18) as opposed to the lack of such a phase shifting factor on the right side of (15). The consequent phase error has been corrected during the migration. However, the migrated amplitudes are poor, especially on the reflector at depth $z = 1000m$ along which the reflection angles vary over a wide range. (This method has incorrect angular dependence when compared to true amplitude reflectivity or the geometrical optics reflection coefficient at each point.) Figure 3 shows results of true amplitude common-shot migration (20). From the right plot, we clearly see that the true amplitude algorithm recovers the reflectivity accurately, aside from the edge effects and small jitters caused by interference with wraparound artifacts.

Conclusions: Common-shot migrations offer good potential of imaging complex structures, but the conventional formulations of such migrations produce incorrect migrated amplitudes. The migration method we proposed calibrates common-shot migrations by correcting both their amplitude and phase behavior. The new method actually builds a bridge between true amplitude Kirchhoff migration and the migrations based on one-way wavefield extrapolation.

References

Bleistein, N., 1987, On the imaging of reflectors in the earth. Geophysics **52**, 931-942.

Bleistein, N., J. K. Cohen and J. W. Stockwell, Jr., 2001, *Mathematics of Multidimensional Seismic Imaging, Migration and Inversion*: Springer-Verlag, New York.

Claerbout, J. F., 1971, Toward a unified theory of reflector imaging: Geophysics, **36**, 3, 467-481.

Claerbout, J., 1985, Imaging the Earth's Interior: Blackwell Scientific Publications, Inc.

Zhang, G., 1993, System of coupled equations for upgoing and downgoing waves: Acta Math. Appl. Sinica, **16**, 2, 251-263.

Zhang, Y., Sun, J., Gray, S. H., Notfors, C., and Bleistein, N., 2001, Towards accurate amplitudes for one-way wavefield extrapolation of 3D common-shot records: 71st Ann. Mtg., Soc. Expl. Geophys., Expanded Abstracts.

Zhang, Y., Sun, J., Gray, S. H., Notfors, C., Bleistein, N., and Zhang, G., 2002, True amplitude migration using common-shot one-way wavefield extrapolation: 64th International Meeting of the EAGE, Expanded Abstracts.