

# Scale Factor for Ray Theoretic Green's Function Amplitudes

by  
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## 1 Introduction

This is a short note on the scaling factors for the amplitude of the ray-theoretic Green's function for the scalar wave equation or Helmholtz equation. We use Appendix E of Bleistein, et al. [2001] (referenced as MMSIMI below) as a point of departure. Indeed, we view this discussion as a follow-on to Sections E.4.1, E.5.2 and E.4.2, and where the initial data along the rays for all of the constituents of the ray theoretic solution are discussed in 3D, 2.5D and 2D, respectively.

The new feature here is that we discuss the initial data for the amplitude for generic ray coordinates. A feature of this general result is that the scale factor turns out to be independent of the scale of the ray parameter, thus being the same if we use time, arclength, or the special variable  $\sigma$  that characterizes 2.5D out-of-plane geometrical spreading.

## 2 The Ray Equations for the 3D Green's Function.

The starting point for this discussion is the equations that govern the propagation of the rays, phase and leading order amplitude of ray theory. The leading order term of the ray theoretic solution in 3D has the form

$$u(\mathbf{x}, \omega) = A_{3D}(\mathbf{x})e^{i\omega\tau(\mathbf{x})}. \quad (1)$$

In this equation and for the remainder of this section, we emphasize that this is the amplitude in 3D by that subscript. For this solution, the rays emanate from a single point, say  $\mathbf{x} = \mathbf{x}_0$ , where we can take the travel time,  $\tau$  to be zero and the amplitude to have the "right" singular behavior, namely that

$$A_{3D}(\mathbf{x}) \approx \frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|}, \quad \mathbf{x} \text{ "near" } \mathbf{x}_0. \quad (2)$$

In this equation, the right side describes the singular behavior of the leading order amplitude of the asymptotic Green's function in the neighborhood of the point source.

Here, we quote from MMSIMI, equations (E.2.10) and (E.3.2) for the governing differential equations of ray theory, except that we use  $\gamma_3$  for the independent variable along the rays in the present discussion. Thus, we write

$$\begin{aligned}
\frac{d\mathbf{x}}{d\gamma_3} &= 2\lambda\mathbf{p}, \quad \mathbf{x}(\gamma_1, \gamma_2, 0) = \mathbf{x}_0; \\
\frac{d\mathbf{p}}{d\gamma_3} &= 2\lambda(\mathbf{x})\nabla\left[\frac{1}{c^2(\mathbf{x})}\right] = 2\lambda p(\mathbf{x})\nabla p(\mathbf{x}) = -\frac{2\lambda}{c^3(\mathbf{x})}\nabla c(\mathbf{x}), \quad \mathbf{p}(\gamma_1, \gamma_2, 0) = \mathbf{p}_0(\gamma_1, \gamma_2). \\
\frac{d\tau}{d\gamma_3} &= \frac{2\lambda}{c^2(\mathbf{x})} = 2\lambda p^2(\mathbf{x}), \quad \tau(\gamma_1, \gamma_2, 0) = 0.
\end{aligned} \tag{3}$$

In these equations, we have used

$$\nabla\tau = \mathbf{p}, \quad \mathbf{p} \cdot \mathbf{p} = \frac{1}{c^2(\mathbf{x})} = p^2(\mathbf{x}), \tag{4}$$

and we have taken the initial value of  $\mathbf{p}$  on the ray to be a function of the two variables,  $(\gamma_1, \gamma_2)$ , denoted by  $\mathbf{p}_0(\gamma_1, \gamma_2)$ . Each choice of  $(\gamma_1, \gamma_2)$  labels a ray and defines its initial direction through the first of the ray equations where the direction of the tangent to the ray at any point is seen to be just the direction of  $\mathbf{p}$ .

The equation governing the propagation of the amplitude, (E.3.2), is

$$2A_{3D}\nabla\tau(\mathbf{x}) \cdot \nabla A_{3D}(\mathbf{x}) + A_{3D}^2\nabla^2\tau(\mathbf{x}) = \nabla \cdot (A_{3D}^2\nabla\tau(\mathbf{x})) = 0. \tag{5}$$

Without going into the details here that can be found in MMSIMland and many other places, we write down the main conclusion of this last equation, namely that

$$A_{3D}^2\mathbf{p} \cdot \hat{\mathbf{n}}dS\Big|_{\gamma_{30}}^{\gamma_3} = 0. \tag{6}$$

Here,  $dS$  is a differential surface area element mapped by the rays from a level,  $\gamma_{30}$  to another level  $\gamma_3$ . Further, we may write that

$$dS = \left| \frac{\partial\mathbf{x}}{\partial\gamma_1} \times \frac{\partial\mathbf{x}}{\partial\gamma_2} \right| d\gamma_1 d\gamma_2.$$

The advantage of this representation is that the differential surface element in  $(\gamma_1, \gamma_2)$  is just  $d\gamma_1 d\gamma_2$ . Hence, it remains unchanged as the family of rays propagate in  $\gamma_3$ . Therefore, the entire change in the surface area element in the physical space is characterized by the cross product appearing in this equation. That is,

$$A_{3D}^2\mathbf{p} \cdot \hat{\mathbf{n}} \left| \frac{\partial\mathbf{x}}{\partial\gamma_1} \times \frac{\partial\mathbf{x}}{\partial\gamma_2} \right| \Big|_{\gamma_{30}}^{\gamma_3} = 0. \tag{7}$$

Now, let us use the first line of the ray equations in (3) to set

$$\mathbf{p} = \frac{1}{2\lambda} \frac{d\mathbf{x}}{d\gamma_3}$$

and further observe that

$$\hat{\mathbf{n}} \left| \frac{\partial \mathbf{x}}{\partial \gamma_1} \times \frac{\partial \mathbf{x}}{\partial \gamma_2} \right| = \frac{\partial \mathbf{x}}{\partial \gamma_1} \times \frac{\partial \mathbf{x}}{\partial \gamma_2},$$

in order to rewrite (7) in terms of the triple scalar product which is also a determinant. That is,

$$\begin{aligned} A_{3D}^2 \mathbf{p} \cdot \hat{\mathbf{n}} \left| \frac{\partial \mathbf{x}}{\partial \gamma_1} \times \frac{\partial \mathbf{x}}{\partial \gamma_2} \right|_{\gamma_3} &= A_{3D}^2 \mathbf{p} \cdot \hat{\mathbf{n}} \left| \frac{\partial \mathbf{x}}{\partial \gamma_1} \times \frac{\partial \mathbf{x}}{\partial \gamma_2} \right|_{\gamma_{30}}, \quad \Rightarrow \\ \frac{A_{3D}^2}{2\lambda} \left| \frac{d\mathbf{x}}{d\gamma_3} \cdot \frac{\partial \mathbf{x}}{\partial \gamma_1} \times \frac{\partial \mathbf{x}}{\partial \gamma_2} \right|_{\gamma_3} &= \frac{A_{3D}^2}{2\lambda} \left| \frac{d\mathbf{x}}{d\gamma_3} \cdot \frac{\partial \mathbf{x}}{\partial \gamma_1} \times \frac{\partial \mathbf{x}}{\partial \gamma_2} \right|_{\gamma_{30}}, \quad \Rightarrow \\ \frac{A_{3D}^2}{2\lambda} \left| \frac{\partial(\mathbf{x})}{\partial(\boldsymbol{\gamma})} \right|_{\gamma_3} &= K^2(\gamma_1, \gamma_2). \end{aligned} \quad (8)$$

In this equation, we have evaluated the expression in (7) at the fixed point,  $\gamma_{30}$  as a constant in  $\gamma_3$ ,  $K^2(\gamma_1, \gamma_2)$ . The second line uses the observations above to rewrite all the factors multiplying the amplitude as a triple scalar product. The third line reinterprets that triple scalar product as the Jaobian of transformation from the variables  $\mathbf{x}$  to  $\boldsymbol{\gamma}$  induced by the ray equations.

Now, the solution for  $A_{3D}(\mathbf{x}(\boldsymbol{\gamma}))$  can be written as

$$A_{3D}(\mathbf{x}(\boldsymbol{\gamma})) = \frac{K_{3D}(\gamma_1, \gamma_2) \sqrt{2\lambda(\mathbf{x}(\boldsymbol{\gamma}))}}{\sqrt{\left| \frac{\partial(\mathbf{x})}{\partial(\boldsymbol{\gamma})} \right|}} = \frac{K_{3D}(\gamma_1, \gamma_2) \sqrt{2\lambda(\mathbf{x}(\boldsymbol{\gamma}))}}{\sqrt{J_{3D}(\boldsymbol{\gamma})}}. \quad (9)$$

In the next subsection, by examining the solution near the singular point, it will be verified that the Jacobian has the singular behavior characterized in (2). Thus, we need only pick the ‘‘constant’’  $K_{3D}(\gamma_1, \gamma_2)$  correctly in order to obtain exactly the behavior indicated by that equation.

## 2.1 Determining $K_{3D}(\gamma_1, \gamma_2)$ for the 3D Green’s Function.

In order to determine  $K_{3D}(\gamma_1, \gamma_2)$ , we need to examine the solution of the ray equations in the neighborhood of  $\mathbf{x}_0$ . The following results are fairly straightforward to verify from the

ray equations (3).

$$\mathbf{x} \approx \mathbf{x}_0 + 2\lambda(\mathbf{x}_0)\mathbf{p}_0\gamma_3, \quad \Rightarrow \quad |\mathbf{x} - \mathbf{x}_0| \approx 2\lambda(\mathbf{x}_0)|\mathbf{p}_0|\gamma_3, \quad (10)$$

$$\frac{\partial \mathbf{x}}{\partial \gamma_j} \approx 2\lambda(\mathbf{x}_0)\frac{\partial \mathbf{p}_0}{\partial \gamma_j}\gamma_3, \quad j = 1, 2.$$

Now, one can verify that

$$\begin{aligned} \frac{\partial(\mathbf{x})}{\partial(\boldsymbol{\gamma})} &\approx 8\lambda^3(\mathbf{x}_0)\gamma_3^2 \det \begin{bmatrix} \mathbf{p}_0 \\ \frac{\partial \mathbf{p}_0}{\partial \gamma_1} \\ \frac{\partial \mathbf{p}_0}{\partial \gamma_2} \end{bmatrix} \\ &= 8\lambda^3(\mathbf{x}_0)\gamma_3^2 p_0 \left| \frac{\partial \mathbf{p}_0}{\partial \gamma_1} \times \frac{\partial \mathbf{p}_0}{\partial \gamma_2} \right|. \end{aligned} \quad (11)$$

To determine  $K_{3D}(\gamma_1, \gamma_2)$ , we need only compare the approximate solution in (2) with the approximate solution in terms of the parameters  $\boldsymbol{\gamma}$  that is deduced by using the results derived just above. Thus, we will use the Jacobian in the previous equation and the distance function in (10) in the solution representation (9):

$$\begin{aligned} A_{3D} &\approx \frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|} \approx \frac{1}{8\pi\lambda(\mathbf{x}_0)|\mathbf{p}_0|\gamma_3} \\ &\approx \frac{K_{3D}(\gamma_1, \gamma_2)\sqrt{2\lambda(\mathbf{x}_0)}}{2\lambda(\mathbf{x}_0)\gamma_3\sqrt{2\lambda(\mathbf{x}_0)p_0\left|\frac{\partial \mathbf{p}_0}{\partial \gamma_1} \times \frac{\partial \mathbf{p}_0}{\partial \gamma_2}\right|}} \\ &\approx \frac{K_{3D}(\gamma_1, \gamma_2)}{2\lambda(\mathbf{x}_0)\gamma_3\sqrt{p_0\left|\frac{\partial \mathbf{p}_0}{\partial \gamma_1} \times \frac{\partial \mathbf{p}_0}{\partial \gamma_2}\right|}} \end{aligned} \quad (12)$$

In this equation, the rightmost equality in the first line simply uses the last equality in the first line of (10). The next line uses the solution representation (9) with the Jacobian

evaluated from (11). Finally, then, the last line is just a simplification of the previous result. Next we compare the second and last expressions here to determine  $K_{3D}(\gamma_1, \gamma_2)$ . The result is

$$K_{3D}(\gamma_1, \gamma_2) = \frac{1}{4\pi} \frac{\sqrt{\left| \frac{\partial \mathbf{p}_0}{\partial \gamma_1} \times \frac{\partial \mathbf{p}_0}{\partial \gamma_2} \right|}}{\sqrt{p_0}} = \frac{1}{4\pi} \sqrt{c(\mathbf{x}_0) \left| \frac{\partial \mathbf{p}_0}{\partial \gamma_1} \times \frac{\partial \mathbf{p}_0}{\partial \gamma_2} \right|}} \quad (13)$$

It can be seen, here, that  $K_{3D}(\gamma_1, \gamma_2)$  is independent of  $\lambda(\mathbf{x})$ . Thus, whether  $\tau$  (travel time),  $\sigma$  (the 2.5D out-of-plane spreading factor) or  $s$  (arclength) is used as the independent variable along the rays, the constant is the same and the solution is

$$A_{3D}(\mathbf{x}(\gamma)) = \frac{1}{4\pi} \sqrt{c(\mathbf{x}_0) \left| \frac{\partial \mathbf{p}_0}{\partial \gamma_1} \times \frac{\partial \mathbf{p}_0}{\partial \gamma_2} \right| \frac{\sqrt{2\lambda(\mathbf{x}(\gamma))}}{\sqrt{J_{3D}(\gamma)}}}. \quad (14)$$

We refrain from simplifying the powers of 2 here because the choice of ray parameter will lead to a “natural” choice of  $2\lambda$  rather than  $\lambda$ . Note that in our approximate solution  $J_{3D} = O((2\lambda)^3)$  so that the quotient appearing under the square roots is  $O((2\lambda)^{-2})$  and the amplitude is then  $O((2\lambda)^{-1})$ , which correctly matches the scale factor in  $1/|\mathbf{x} - \mathbf{x}_0|$  in the second expression in (12). This is just another way of seeing that  $K_{3D}$  is appropriately independent of  $\lambda$ .

## 2.2 Examples of $K_{3D}$

. Here, two examples of the determination of  $K_{3D}$  will be presented. The first one uses the polar coordinates of  $\mathbf{p}_0$  and the second uses the first two components of  $\mathbf{p}_0$  for  $(\gamma_1, \gamma_2)$ .

### 2.2.1 $K_{3D}$ for initial polar angles as ray-labeling parameters.

Here, we set

$$\mathbf{p}_0 = p_0(\mathbf{x}_0)(\sin \gamma_1 \sin \gamma_2, \cos \gamma_1 \sin \gamma_2, \cos \gamma_2) \quad (15)$$

and find that

$$\left| \frac{\partial \mathbf{p}_0}{\partial \gamma_1} \times \frac{\partial \mathbf{p}_0}{\partial \gamma_2} \right| = p^2(\mathbf{x}_0) \sin \gamma_2. \quad (16)$$

In this case, then, by substituting into (13) and (14), we find that

$$K_{3D}(\gamma_1, \gamma_2) = \frac{1}{4\pi} \sqrt{p(\mathbf{x}_0) \sin \gamma_2} = \frac{1}{4\pi} \sqrt{\frac{\sin \gamma_2}{c(\mathbf{x}_0)}} \quad (17)$$

and

$$A_{3D}(\mathbf{x}(\boldsymbol{\gamma})) = \frac{1}{4\pi} \sqrt{\frac{2\lambda(\mathbf{x}(\boldsymbol{\gamma})) \sin \gamma_2}{c(\mathbf{x}_0) J_{3D}(\boldsymbol{\gamma})}}. \quad (18)$$

### 2.2.2 $K_{3D}$ for initial values, $(p_{01}, p_{02})$ , as ray-labeling parameters.

In this case, we set

$$\mathbf{p}_0 = \left( p_{01}, p_{02}, p_{03} = \sqrt{p_0^2 - p_{01}^2 - p_{02}^2} \right) \quad (19)$$

and find that

$$\left| \frac{\partial \mathbf{p}_0}{\partial \gamma_1} \times \frac{\partial \mathbf{p}_0}{\partial \gamma_2} \right| = \frac{p_0}{p_{03}}. \quad (20)$$

Consequently, in this case,

$$A_{3D}(\mathbf{x}(\boldsymbol{\gamma})) = \frac{1}{4\pi} \sqrt{\frac{2\lambda(\mathbf{x}(\boldsymbol{\gamma}))}{p_{03} J_{3D}(\boldsymbol{\gamma})}}. \quad (21)$$

## 3 Two-and-one-half Dimensions (2.5D).

Two-and-one-half dimensions (2.5D) is the name we use for 3D propagation in the  $(x, z)$ -plane for a medium in which the wave speed is independent of  $y$ ; that is,  $c = c(x, z)$ . Results for 2.5D can be deduced from the 3D wave propagation results of the previous section when we specialize to this type of medium. We describe that process here and then derive the corresponding constant for the Green's function,  $K_{2.5D}$ .

The major consequence of a  $y$ -independent wave speed arises in the ray equation for  $p_2$  in (3):

$$\frac{dp_2}{d\gamma_3} = 0, \quad (22)$$

so that  $p_2$  is given by its initial value,  $p_{20}$ . It then makes sense to take this as one of the ray-labeling parameters; that is, set  $\gamma_2 = p_{20}$ . Now, from the differential equation for  $y = x_2$  in (3), we find that

$$x_2 = x_{20} + \gamma_2 \gamma_3. \quad (23)$$

We remark that the solution we seek, is still the 3D Green's function, but specialized to in-plane propagation. From this equation, we conclude that such in-plane propagation arises when we set  $p_{20} = 0$ , which we will do, below. For the moment, note that the ray equations in (3) reduce now to the 2D equations in  $(x_1, x_3, p_1, p_3, \tau)$ . Thus, the in-plane propagation

is a solution of the same system of equations as arises in 2D; however, the amplitude must still honor 3D geometrical spreading.

We can now complete the story of the initial values,  $\mathbf{p}_0$ , as

$$\mathbf{p}_0 = (p_T \sin \gamma_1, \gamma_2, p_T \cos \gamma_1), \quad p_T = \sqrt{p_0^2 - \gamma_2^2} = \sqrt{c(\mathbf{x}_0)^{-2} - \gamma_2^2}. \quad (24)$$

Next, let us consider the 3D Jacobian for this case, when we evaluate it at  $p_{20} = 0$ . First, note from (23) that for any choice of the constant  $p_{20}$

$$\frac{\partial x_2}{\partial \gamma_1} = 0.$$

Then,

$$\begin{aligned} J_{3D}(\gamma_1, 0, \gamma_3) &= \left\| \frac{\partial(\mathbf{x})}{\partial(\boldsymbol{\gamma})} \right\|_{\gamma_2=0} = \left| \det \begin{bmatrix} \cdot & 0 & \cdot \\ \cdot & 2\lambda\gamma_3 & \cdot \\ \cdot & 0 & \cdot \end{bmatrix} \right| \\ &= 2\lambda\gamma_3 \frac{\partial(x_1, x_3)}{\partial(\gamma_1, \gamma_3)} = 2\lambda\gamma_3 J_{2D}(\gamma_1, \gamma_3). \end{aligned} \quad (25)$$

We also need to calculate the cross produce in (13) by using (24) and setting  $\gamma_2 = 0$ . That result is

$$\begin{aligned} \left\| \frac{\partial \mathbf{p}_0}{\partial \gamma_1} \times \frac{\partial \mathbf{p}_0}{\partial \gamma_2} \right\|_{\gamma_2=0} &= p_0 |(\cos \gamma_1, 0, -\sin \gamma_1) \times (0, 1, 0)| \\ &= p_0 = 1/c(\mathbf{x}_0). \end{aligned} \quad (26)$$

By using these last two results in the equations for  $K$  and  $A$  in (13) and (14), we find the corresponding 2.5D results,

$$K_{2.5D}(\gamma_1) = \frac{1}{4\pi}; \quad A_{2.5D}(\mathbf{x}(\boldsymbol{\gamma})) = \frac{1}{4\pi \sqrt{\gamma_3 J_{2D}(\boldsymbol{\gamma})}}. \quad (27)$$

Here,  $\boldsymbol{\gamma} = (\gamma_1, \gamma_3)$  and the result is specific to the choice of  $\gamma_1$  being the angle that the in-plane ray makes with the vertical direction. This is a convenient choice that leads to a simplest representation.

## 4 Two Dimensions (2D).

In two dimensions, the Green's function has a slightly different form compared to (1); namely,

$$u(\mathbf{x}, \omega) = \frac{A_{2D}(\mathbf{x})}{\sqrt{|\omega|}} e^{i\omega\tau(\mathbf{x}) + i\pi/4 \text{sgn}(\omega)} = \frac{A_{2D}(\mathbf{x})}{\sqrt{-i\omega}} e^{i\omega\tau(\mathbf{x})}. \quad (28)$$

For this solution, again the rays emanate from a single point, say  $\mathbf{x} = \mathbf{x}_0$ , where we can take the travel time,  $\tau$  to be zero and the amplitude to have the "right" singular behavior,

$$A_{2D}(\mathbf{x}) \approx \frac{1}{2\sqrt{2\pi|\mathbf{x} - \mathbf{x}_0|/c(\mathbf{x}_0)}}. \quad (29)$$

We need to modify the ray equations (3) by interpreting  $\mathbf{x} = (x_1, x_3)$  and  $\mathbf{p} = (p_1, p_3)$  being two dimensional vectors with only two ray parameters,  $\boldsymbol{\gamma} = (\gamma_1, \gamma_3)$ . That is,

$$\begin{aligned} \frac{d\mathbf{x}}{d\gamma_3} &= 2\lambda\mathbf{p}, \quad \mathbf{x}(\gamma_1, 0) = \mathbf{x}_0; \\ \frac{d\mathbf{p}}{d\gamma_3} &= 2\lambda(\mathbf{x})\nabla \left[ \frac{1}{c^2(\mathbf{x})} \right] = 2\lambda p(\mathbf{x})\nabla p(\mathbf{x}) = -\frac{2\lambda}{c^3(\mathbf{x})}\nabla c(\mathbf{x}), \quad \mathbf{p}(\gamma_1, 0) = \mathbf{p}_0(\gamma_1). \\ \frac{d\tau}{d\gamma_3} &= \frac{2\lambda}{c^2(\mathbf{x})} = 2\lambda p^2(\mathbf{x}), \quad \tau(\gamma_1, 0) = 0. \end{aligned} \quad (30)$$

Similarly, (6) is replaced by

$$A_{2D}^2 \mathbf{p} \cdot \hat{\mathbf{n}} ds \Big|_{\gamma_{30}}^{\gamma_3} = 0. \quad (31)$$

In this case,  $ds$  is differential arclength mapped by rays between the level,  $\gamma_{30}$ , and the level  $\gamma_3$ . Furthermore, in place of the equation for  $dS$ , we write,

$$ds = \left| \frac{\partial \mathbf{x}}{\partial \gamma_1} \right| d\gamma_1.$$

Again, the differential  $d\gamma_1$  is a constant along the rays, so that (31) leads to the following counterpart of (7):

$$A_{2D}^2 \mathbf{p} \cdot \hat{\mathbf{n}} \left| \frac{\partial \mathbf{x}}{\partial \gamma_1} \right| \Big|_{\gamma_{30}}^{\gamma_3} = 0. \quad (32)$$

As in the 3D case, let us use the first line of the ray equations, this time in (30), to set

$$\mathbf{p} = \frac{1}{2\lambda} \frac{d\mathbf{x}}{d\gamma_3}$$

and further observe that

$$|\mathbf{p} \cdot \hat{\mathbf{n}}| \left| \frac{\partial \mathbf{x}}{\partial \gamma_1} \right| = |\mathbf{p} \times \hat{\mathbf{t}}| \left| \frac{\partial \mathbf{x}}{\partial \gamma_1} \right| = \frac{1}{2\lambda} \left| \frac{\partial \mathbf{x}}{\partial \gamma_1} \times \frac{d\mathbf{x}}{d\gamma_3} \right| = \frac{1}{2\lambda} \left| \frac{\partial(\mathbf{x})}{\partial(\gamma)} \right| = \frac{1}{2\lambda} |J_{2D}|.$$

In this equation,  $\hat{\mathbf{t}}$  is the unit tangent in the direction of  $\partial \mathbf{x} / \partial \gamma_1$ . Consequently, in place of (9), we now have the equation

$$A_{2D}(\mathbf{x}(\gamma)) = \frac{K_{2D}(\gamma_1) \sqrt{2\lambda(\mathbf{x}(\gamma))}}{\sqrt{\left| \frac{\partial(\mathbf{x})}{\partial(\gamma)} \right|}} = \frac{K_{2D}(\gamma_1) \sqrt{2\lambda(\mathbf{x}(\gamma))}}{\sqrt{|J_{2D}(\gamma)|}}. \quad (33)$$

#### 4.1 Determining $K_{2D}(\gamma_1)$ for the 2D Green's Function.

This discussion follows along the lines of Section 2.1. That is, we examine the solution of the ray equations (30) for  $\mathbf{x}$  near  $\mathbf{x}_0$ . Thus, the analog of equations (10) and (11) are

$$\mathbf{x} \approx \mathbf{x}_0 + 2\lambda(\mathbf{x}_0) \mathbf{p}_0 \gamma_3, \quad \Rightarrow \quad |\mathbf{x} - \mathbf{x}_0| \approx 2\lambda(\mathbf{x}_0) |\mathbf{p}_0| \gamma_3, \quad (34)$$

$$\frac{\partial \mathbf{x}}{\partial \gamma_1} \approx 2\lambda(\mathbf{x}_0) \frac{\partial \mathbf{p}_0}{\partial \gamma_1} \gamma_3.$$

Now, one can verify that

$$\begin{aligned} \frac{\partial(\mathbf{x})}{\partial(\gamma)} &\approx 4\lambda^2(\mathbf{x}_0) \gamma_3 \det \begin{bmatrix} \mathbf{p}_0 \\ \frac{\partial \mathbf{p}_0}{\partial \gamma_1} \end{bmatrix} \\ &= 4\lambda^2(\mathbf{x}_0) \gamma_3 p_0 \left| \frac{\partial \mathbf{p}_0}{\partial \gamma_1} \right|. \end{aligned} \quad (35)$$

We now proceed as in (12):

$$\begin{aligned}
A_{2D} &\approx \frac{1}{2\sqrt{2\pi|\mathbf{x} - \mathbf{x}_0|/c(\mathbf{x}_0)}} \approx \frac{c(\mathbf{x}_0)}{4\sqrt{\pi\lambda(\mathbf{x}_0)\gamma_3}} \\
&\approx \frac{K_{2D}(\gamma_1)\sqrt{2\lambda(\mathbf{x}_0)}}{2\lambda(\mathbf{x}_0)\sqrt{\gamma_3 p_0 \left| \frac{\partial \mathbf{p}_0}{\partial \gamma_1} \right|}} \\
&\approx \frac{K_{2D}(\gamma_1)}{\sqrt{2\lambda(\mathbf{x}_0)\gamma_3 p_0 \left| \frac{\partial \mathbf{p}_0}{\partial \gamma_1} \right|}}
\end{aligned} \tag{36}$$

Here, the last expression in the first line follows from (34). The next line arises from using (35) in (33). The last line is just a simplification of the previous line. By comparing the second and fourth expressions on the right side here, we conclude that

$$K_{2D}(\gamma_1) = \frac{\sqrt{c(\mathbf{x}_0) \left| \frac{\partial \mathbf{p}_0}{\partial \gamma_1} \right|}}{2\sqrt{2\pi}} \tag{37}$$

and

$$A_{2D}(\mathbf{x}) = \frac{\sqrt{2\lambda(\mathbf{x}(\gamma))c(\mathbf{x}_0) \left| \frac{\partial \mathbf{p}_0}{\partial \gamma_1} \right|}}{2\sqrt{2\pi J_{2D}(\gamma)}} \tag{38}$$

As in the previous cases, we see here that  $K_{2D}(\gamma_1)$  is independent of  $\lambda$ .

## 5 Summary and Conclusions.

We have derived the constants of acoustic 3D Green's functions for arbitrary choice of the running parameter along the ray. Surprisingly, we found that in all cases, the constant of the amplitude was independent of the scaling— $\lambda$ , in this discussion. Further, from the nature of the derivation, it seems clear that the same is true for isotropic and even anisotropic Green's function amplitudes. For the purpose of this text, however, these are the results that we require.

## References

Bleistein, N., J. K. Cohen and J. W. Stockwell, Jr., 2001, *Mathematics of Multidimensional Seismic Imaging, Migration and Inversion*: Springer-Verlag, New York.