Simple memory models of attenuation in complex viscoporous media

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ABSTRACT

Simple models of wave propagation in porous media depending on a small number of parameters directly related to wave propagation are presented. Based on theoretical and experimental evidence, viscous phenomena are represented in terms of singular convolution operators. The equations of motion are hyperbolic but their solutions are infinitely smooth.

Asymptotic, exact and numerical solution methods are discussed.

Introduction

Mathematical complexity of wave propagation problems for realistic porous media entails the necessity of a flexible approach to solution of specific problems. An excess of irrelevant detail for a given problem often forces an oversimplification in the description of more relevant features of the medium. In poroelasticity reliance on the original system of equations obtained by Biot leads to unfortunate compromises such as neglecting the frequency dependence of dynamic permeability. This approximation, justified for porous media whose parameters do not vary excessively over the distance one wavelength, prevents an extension of seismic modelling to more realistic micro-heterogeneous porous media (Gurevich & Lopatnikov, 1995; Gelinsky *et al.*, 1998).

In view of significant energy conversion among various wave modes, use of a system of equations is necessary in problems with interfaces, high gradients of medium parameters and scattering at a subwavelength scale. In other problems, the effective parameters vary smoothly, and it is more convenient to model the selected wave mode by a scalar equation with a realistic attenuation model. Limitations of mathematical apparatus impose a compromise: neglecting memory in the first case and neglecting polarization and wave conversion in the second case. In this paper we shall explore the second case.

In poroelastic media the influence of memory appears through the viscodynamic operator, which accounts for the friction between the pore fluid and pore walls. The resulting frequency-dependent factor $F(\omega)$ of the fluid filtration velocity in Biot's equations of motion has been determined for some idealized pore geometries such as slits, aligned pores with circular, triangular or rectangular cross-section, (Norris (1986) and Stinson & Champoux (1992)). This factor depends on the shape of the pore cross-section, but it has certain general characteristics independent of the pore geometry. Since in all these cases a boundary layer of thickness $C\omega^{-1/2}$ is present with a constant C of dimension $[L/T^{1/2}]$, the factor $F(\omega)$ depends on a dimensionless parameter $(C/R)\,\omega^{-1/2}$, where R is a characteristic length associated with the pore cross-section. The singularity $t^{-1/2}$ of the convolution kernel is therefore a universal feature of the high-frequency behavior (or wavefront features in the time domain) of idealized models of poroelastic media. For more realistic physical models (variable pore cross-section, direction, dimension and alignment), the parameters of $F(\omega)$ can be determined experimentally or by seismic inversion. This also includes a heuristic transfer function for fractal pore cross section (Wilson, 1992), while fractal models of pores (Johnson *et al.*, 1987; Ruffet *et al.*, 1991) lead to a more general singularity $t^{-\beta}$, $0 < \beta < 1$, dependent on the fractal dimension d of the pore surface.

Taken too literally, the idealized physical models are misleading. As pointed out by Champoux & Stinson (1992) and Norris (1986), these models are often incompatible (notably Zwikker-Kosten-Attenborough and Biot-Allard) and inconsistent with experiments, e.g., for artificial porous media with pores of variable cross-section. Furthermore, the shape factors appearing in such models are too complicated and henceforth inadequate for numerical applications in heterogeneous media.

As noted by Wilson (1992; 1997), Johnson et al. (1987) and Berryman et al. (1988), it is possible to construct universally valid models of the viscodynamic operator that involve only simple algebraic functions. Such models can provide a good match for various specific physical models over a wide range of frequencies. Wilson, in particular, noted that the approximation is uniform with respect to the frequency provided that the parameters of the approximating function are determined in such a way that they fit the physical models in the transition region of their transfer function. Minor variations accommodate the models of Zwikker-Kosten (Zwikker & Kosten, 1949; Attenborough, 1983) and Biot-Allard (Biot, 1962; Allard *et al.*, 1983).

At least one physical model of a micro-heterogeneous porous medium (Gurevich & Lopatnikov, 1995) can be represented by the above class of models. Indeed, the Gurevich-Lopatnikov model is well approximated by even simpler universal functions corresponding to the high-frequency limit of the Wilson model (Hanyga & Rok, 1999).

General hereditary models of attenuation

Attenuation in seismic wave propagation can be linked to a number of specific physical mechanisms (viscosity, multiple scattering, viscous friction of the pore fluid, etc). A general method of accommodating such phenomena in the equations of motion consists in allowing for memory effects in the stress-strain relations and in the inertial term. In viscoelasticity the elastic stiffness coefficients are replaced by convolution operators,

$$\sigma_{ij}(t,\mathbf{x}) = c_{ijkl} * e_{kl} \equiv \int_0^t c_{ijkl}(\tau,\mathbf{x})\, e_{kl}(t-\tau,\mathbf{x})\,\mathrm{d}\tau, \tag{1}$$

while in poroelasticity the inertial force associated with the fluid includes a viscous drag force

$$N_{ij} * u_{j,tt} \equiv \int_0^\infty N_{ij}(\tau,\mathbf{x})\, u_{j,tt}(t-\tau,\mathbf{x})\,\mathrm{d}\tau. \tag{2}$$

For specific idealized pore geometries (parallel cylidrical or triangular pores, slits) explicit expressions can be derived for N_{ij} (Biot, 1956; Norris, 1986; Champoux & Stinson, 1992). All these specific expressions share one important property:

$$N_{ij} \sim \text{const} \times t^{-1/2} \quad \text{for } t \to 0. \tag{3}$$

Singularity (3) is independent of pore geometry and is a consequence of the viscous boundary layer associated with pore fluid motion.

An alternative and often more important mechanism of attenuation in rocks is scattering and P-to-slow wave conversion on small-scale heterogeneities. A good example is the effective dispersion-attenuation law for the longitudinal wave, incorporating its conversion to the slow wave (Gurevich & Lopatnikov, 1995). It is well matched by a causal power law $k = \omega/c_\infty + \lambda \mathrm{i}(-\mathrm{i}\omega)^\alpha$, $\alpha = 0.66$, where k denotes the wavenumber (Hanyga & Rok, 1999). In one-way wave propagation this corresponds to an operator $\{1/c_\infty + [\lambda/\Gamma(1-\alpha)]\; t^{-\alpha}*\}\,\partial_t \pm \partial_x$, very close to an equation for which there are explicit closed-form solutions (see next section). The symbol $\Gamma(z)$ denotes the Gamma function (Abramowitz & Stegun, 1970).

Memory models of viscoelasticity involve a variety of singularities. Singularities are either deduced from theoretical models or from a careful interpretation of experimental data. A good example from the second group is the Bagley-Torvik model of a polymer under uniaxial tension (Bagley & Torvik, 1986), with the Young modulus given by the formula

$$\hat{E}(-\mathrm{i}\omega) = \frac{E_\mathrm{R} + E_\mathrm{G}\,(-\mathrm{i}\tau\omega)^\alpha}{1+(-\mathrm{i}\tau\omega)^\alpha}, \tag{4}$$

where $0 < \alpha < 1$, E_R and E_G denote the rubbery and glassy Young modulus, while $1/\tau$ is a characteristic frequency which determines the location of the transition region between the regions characterized by the glassy and rubbery behavior. There are two important reasons for the fractional power of frequency in (4): it allows a good matching of experimental data over a wide range of frequencies, and it requires only three parameters, in contrast to models based on rational approximations. A high-frequency asymptotic expansion of the right-hand side of equation (4) involves a term const $\times\, (-\mathrm{i}\omega)^{-\alpha}$, which in the time domain corresponds to the convolution operator const $\times\, t^{\alpha-1}/\Gamma(\alpha)$.

The general form of transfer function (4) applies to all the cases with a single transition region. In particular the the same transfer function is well known as the Cole-Cole model of dielectric permittivity (Cole & Cole, 1941). Extensive bibliographies of related results can be found in the paper of Hanyga & Seredyńska (1998) and in the book of Prüss (1993).

It has been proved that hyperbolic systems with a weakly singular convolution operator have $\mathcal{C}^\infty$-smooth solutions (Renardy, 1982; Narain & Joseph, 1982; Prüss, 1993). Due to unboundedness of the attenuation in the high-frequency limit, any initial singularity is immediately smoothed out (Hanyga, 1999a). A general pulse can be decomposed into the delta singularities, which are smoothed out in the course of propagation. In consequence, the wavefront does not carry any energy and the pulse gradually builds up *after* the passage of the wavefront. An important practical consequence is the pulse delay with respect to the wavefront.

In singular hereditary models the singular part of the convolution kernel plays a far more important role than does the regular part. The singular part determines the rate of the build-up of the pulse behind the wavefront and its gradual widening. The regular part controls the exponential decay of the amplitude and the shape of the tail.

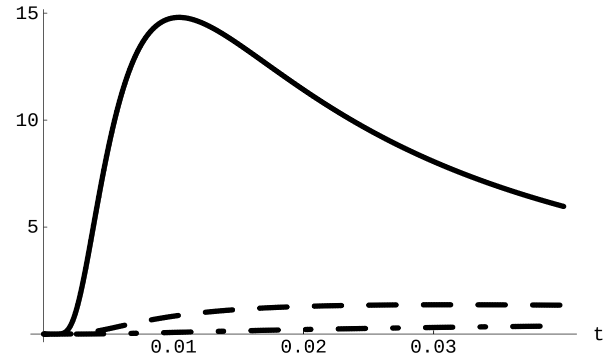

Figure 1. Basis function $f_r^{(2)}(t,\sigma), r = 0, 1, 2$ for $\sigma = 0.1$.

Scalar equations with singular convolution kernels

A general hyperbolic scalar wave equation with a singular memory can be written in the form

$$u_{,tt} + K(t, \mathbf{x}) * u_{,tt} - \nabla \cdot (L(t, \mathbf{x}) * \nabla u) = 0, \qquad (5)$$

where $L(t, \mathbf{x}) = n(\mathbf{x})^2\, \delta(t) + M(t, \mathbf{x})$, $n(\mathbf{x})$ is the refractive index and $K(t, \mathbf{x}), M(t, \mathbf{x}) \sim \text{const} \times t^{\alpha-1}$ for $t \to 0$, where $\delta(x)$ denotes the Dirac delta function.

A point source can be incorporated in equation (5), either as a delta-spiked source term on the right-hand side with zero initial conditions or through the initial conditions

$$u(0, \mathbf{x}) = 0, \quad u_{,t}(0, \mathbf{x}) = \delta(\mathbf{x}) \qquad (6)$$

In either case it is assumed that the wavefield is at rest ($u = 0$) for $t < 0$. The solutions of the two problems are related by Duhamel's principle (Courant, 1962).

Equation (5) has explicit solutions if $n(\mathbf{x})^2 \equiv 1$ and either

$$K(t, \mathbf{x}) = 2\, a\, t^{-1/2}/\sqrt{\pi} + b\, \theta(t), \qquad (7)$$

with

$$b = a^2, \quad a = \text{const}, \qquad (8)$$

or

$$K(t, \mathbf{x}) = A\, t^{-2/3}/\Gamma(1/3) + B\, t^{-1/3}/\Gamma(2/3) + C\, \theta(t) + D\, t^{1/3}/\Gamma(4/3) \qquad (9)$$

with

$$A = 2c,\ B = 2d + c^2,\ C = 2c\, d,\ D = d^2, \quad c, d = \text{const}. \qquad (10)$$

In both cases, in 1D the operator in equation (5) factors into two operators governing the left- and right-going waves. In the 1D homogeneous case equation (5) can be expressed in the form

$$Q_+ Q_- u = 0, \qquad (11)$$

with

$$Q_\pm = P(t) * \partial_t \mp \partial_x \qquad (12)$$

and $P(t) = \delta(t) + a\, t^{-1/2}/\sqrt{\pi}$ or $P(t) = \delta(t) + a\, t^{-2/3}/\Gamma(1/3) + b\, t^{-1/3}/\Gamma(2/3)$ for $\alpha = 1/2$ or $1/3$.

The solutions of the initial-value problem (5,6) are particularly simple:

$$u(t, \mathbf{x}) = (1/4\pi r)\left[f_0^{(2)}(t - r, ar) + 2a\, f_1^{(1)}(t - r, ar) + a^2\, f_2^{(2)}(t - r, ar)\right] \qquad (13)$$

for the the kernel (7) satisfying (8) and

$$u(t, \mathbf{x}) = (1/4\pi r)\left[f_0^{(3)}(t - r, ar, br) + 2a\, f_1^{(3)}(t - r, ar, br) + (2b + a^2)\, f_2^{(3)}(t - r, ar, br) + 2a\, b\, f_3^{(1)}(t - r, ar, br) + b^2\, f_4^{(2)}(t - r, ar, br)\right] \qquad (14)$$

for the kernel (9) satisfying (10). The *basis functions* $f_j^{(n)}(t, \lambda_1, \dots, \lambda_n)$ are defined by their Laplace transforms $s^{-j/n} \exp\left(-\sum_{r=1}^{n} \lambda_r s^{r/n}\right)$ for $\lambda_1 > 0$. Explicit expressions for the basis functions $f_r^{(2)}$ and $f_r^{(3)}$ can be found in the papers of Hanyga & Seredyńska ((1999b) and (1998)), resp.

In particular, the basis function

$$f_0^{(2)}(t, \sigma) = \frac{\sigma}{2\sqrt{\pi}}\, t^{-3/2}\, \mathrm{e}^{-\sigma^2/(4t)} \qquad (15)$$

has a unique maximum at $t = \sigma^2/6$ and tends to $\delta(t)$ when $\sigma \to 0$ (Figure 1). This implies that the original delta-spiked signal widens into an asymmetric pulse, with the peak arriving with a delay of $\Delta T = (ar)^2/6$ with respect to the wavefront $r = t$. The rate of decrease of the maximum value of the pulse is algebraic: $\propto (ar)^{-2}$. Depending on the value of ar, the signal can have the appearance of being sharp, with the build-up phase invisible without zooming, or being nearly diffusive in form, without any apparent connection with the

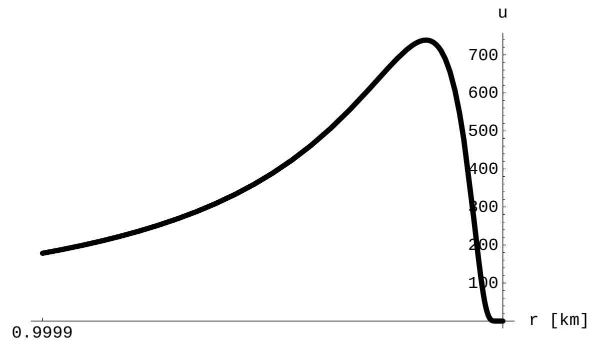

Figure 2. A snapshot of the wavefront part of the solution $u(t,\mathbf{x})$

wavefront typical of the slow wave in Biot's theory. A snapshot of the solution (13) is shown in Figure 2.

In the context of equation (5), with the memory function (7), the coefficient b is Biot's frequency, while a can be expressed in terms of b for specific idealized pore geometries (Hanyga, 1999a). For seismic waves in a poroelastic medium with frequencies $\sim 100\,\mathrm{Hz}$, $b \sim 10^5\,\mathrm{Hz}$ the attenuation is commonly assumed to be adequately represented by the term $b\,u_{,t}$. In the context of the memory function (7) this amounts to neglecting the term $2a\,t^{-1/2}/\sqrt{\pi}$. The resulting equation can be reduced to the telegraph equation with independent variables r, t. The initial-value problem (6) has an explicit solution that consists of a delta-spiked signal with an exponentially decaying amplitude followed by a negligible distorting signal (Hanyga & Seredyńska, 1999b). This is a qualitatively different behavior from solution (13).

When constraints (8) or (10) are violated, asymptotic methods described in Hanyga & Seredyńska (1999a) and Hanyga & Seredyńska (1998) yield approximate solutions in terms of the basis functions. The solution of eq. (5) with the kernel (7) and the initial conditions (6) is similar in form to (13) and reduces to it in the limit of $b \to a^2$:

$$u(t,\mathbf{x}) = (1/4\pi r)\,\mathrm{e}^{-dr/2}\left\{f_0^{(2)}(t-r,ar) + 2a(1+d\,r/4)f_1^{(2)}(t-r,ar) + \dots\right\}, \quad (16)$$

where $d = b - a^2$. The parameter a controls the high-frequency aspects such as the build-up of the pulse at the wavefront, while d controls the rate of exponential decay of its amplitude.

It is natural to expect that a physically reasonable model involves a vanishing memory. Replacing the kernel $K(t)$, equation (7), in equation (5) by $K(t)\,\mathrm{e}^{-t/\gamma}$ results in a negligible correction to the coefficient of $f_1^{(2)}$ of the asymptotic solution (Hanyga & Seredyńska, 1999b). On the other hand, the behavior of the exact solution at the source changes dramatically: with vanishing memory, the wavefield at the source decays exponentially; otherwise it does not.

Fractional-derivative formulation and finite-difference approximations.

Equations (5), with the kernels (7), and (9) can be expressed in terms of fractional derivatives. In particular, equation (5) with the kernel (7) can be expressed in the form

$$\left(\mathrm{D}^2 + 2a\,\mathrm{D}^{3/2} + b\,\mathrm{D}\right)u - \nabla\cdot\left(n^2\,\nabla u\right) = 0, \quad (17)$$

where the $\mathrm{D} = \partial/\partial t$ and the fractional derivative $\mathrm{D}^{3/2}$ is defined in the sense of Caputo (Appendix A).

The Caputo fractional derivative can be expressed in terms of the Riemann-Liouville derivative, which in turn can be expressed in terms of finite differences (the Grünwald-Letnikow derivative). This immediately leads to finite-difference (FD) schemes of various types, in particular the leapfrog. The leapfrog scheme can be obtained by taking advantage of the identity $\mathrm{D}^{3/2} = \mathrm{D}^{1/2}\,\mathrm{D}$, which holds for the Caputo derivative. Introducing a new variable $v = \mathrm{D}\,u$, equation (17) can be recast in the form

$$\mathrm{D}\,u = v \quad (18)$$

$$\left(\mathrm{D} + 2a\,\mathrm{D}^{1/2} + b\right)v = \nabla\cdot\left(n^2\,\nabla u\right) \quad (19)$$

A leapfrog FD scheme is obtained by expressing the Caputo derivative $\mathrm{D}^{1/2}$ in terms of the Grünwald-Letnikow approximate formula and solving for the value of v at the last step. The operator on the left-hand side of equation (19) has a causal continuous inverse for $b, a > 0$. Stability of the numerical scheme based on this formulation is examined in Hanyga (1999c).

Wilson models of porous media.

The main manifestation of singular memory is associated with the singular part of the convolution kernel. Asymptotic analysis shows that the regular part, responsible for low-frequency behavior, modifies the overall attenuation and higher-order amplitudes.

In poroelasticity one can distinguish between high- and low-frequency regimes, separated by a single transition region. The low-frequency action of the memory can be represented by an additional parameter that corrects the theory for low frequencies as well as for a transition region.

A simple model of a poroelastic medium with these properties can be constructed by replacing the viscodynamic operator with a simple algebraic function of frequency (Berryman *et al.*, 1988; Johnson *et al.*, 1987). Alternatively, an acoustic model with a frequency-dependent density and bulk modulus can be used (Deane, (1997) for seabeds; Allard & Champoux (1992), Wilson (1992; 1997) for synthetic porous air-saturated materials). In the second group of models, the density and the bulk modulus become convolution operators

$$\rho * u_{,tt} - \nabla \cdot [K * \nabla u] = 0 \tag{20}$$

In order to match all the frequencies it is important to choose the parameters of the simplified model in such a way that the matching in the transition region is optimal (Wilson, 1992; Wilson, 1997). The resulting equations of motion are well adapted for numerical solution by generalized FD schemes.

The models developed by Wilson involve viscosity in the stress-strain constitutive relations, due to thermal relaxation in the pore fluid, and in the inertial part, due to viscous relaxation. The effective frequency dependent density $\hat{\rho}(\omega)$ and bulk modulus $\hat{K}(\omega)$ are given by the equations

$$\begin{aligned} \hat{K}(\omega) &= \frac{K_\infty}{1 + \Gamma_K \, \tilde{R}(-\mathrm{i}\,\omega, \gamma_K, \epsilon_K)} \\ \hat{\rho}(\omega) &= \frac{\rho_\infty}{1 + \Gamma_\rho \, \tilde{R}(-\mathrm{i}\,\omega, \gamma_\rho, \epsilon_\rho)}, \end{aligned} \tag{21}$$

where the relaxation function $\tilde{R}(s, \gamma, \epsilon)$ is defined by one of the following formulas:

$$\tilde{R}(s, \gamma, \epsilon) = \tilde{R}_0(s, \gamma) \equiv (1 + \gamma\, s)^{-\alpha} \tag{22}$$

(Wilson, 1992), or

$$\tilde{R}(s, \gamma, \epsilon) = \tilde{R}_1(s, \gamma, \epsilon) \equiv \{1 + \epsilon\, [(1 + \gamma\, s)^\alpha - 1]\}^{-1} \tag{23}$$

(Wilson, 1997), with $0 < \alpha < 1$ and $s = -\mathrm{i}\omega$.

In the time domain, the relaxation function (22) combines the singularity $t^{\alpha-1}$ with a vanishing memory factor $\mathrm{e}^{-t/\gamma}$

$$R_0(t) = (1/\gamma)^\alpha \, \mathrm{e}^{-t/\gamma} \, t^{\alpha-1} / \Gamma(\alpha). \tag{24}$$

The relaxation function (23) can be explicitly calculated in the time domain for $\alpha = 1/2$. It is, however, more instructive to examine its high- and low-frequency asymptotics in the frequency domain:

$$\tilde{R}_1(s) \sim \begin{cases} (\tau_{\mathrm{H}}\, s)^{-\alpha} & \text{for } \omega \to \infty \\ 1 - \tau_{\mathrm{L}}\, s & \text{for } \omega \to 0, \end{cases} \tag{25}$$

where $\tau_{\mathrm{H}} = \epsilon^{1/\alpha}\, \gamma$ and $\tau_{\mathrm{L}} = \alpha\, \epsilon\, \gamma$. It follows that the relaxation function (23) has two separate scales for the short-time and long-time response. Wilson (1997) shows that the model based on the density and bulk modulus (21) with the relaxation function (23) closely matches the theoretical prediction and experimental results for dispersion and attenuation in a synthetic porous material with cylindrical pores with a variable radius assuming two different values (Stinson & Champoux, 1992). For this model, the Biot equations with the improvements suggested in Allard *et al.* (1983) give erroneous predictions. On the other hand, in a different study involving a ceramic material the Biot-Allard model and the simpler Wilson model involving the relaxation function (22) agree very well with experimental data.

In Wilson's paper, the special value of the exponent $\alpha = 1/2$ is justified by the diffusive mechanism of pore fluid relaxations. More general exponents are expected to apply if attenuation is mostly due to multiple scattering in a micro-heterogeneous porous medium (Hanyga & Rok, 1999).

In the high-frequency limit or, equivalently, for $\gamma_\rho, \gamma_K \to \infty$, the Wilson models reduce to the simpler models considered in the preceding section. Finite γ_K, γ_ρ affects the tail of the pulse, in particular, by a theorem of Lokshin & Suvorova (1982), causing the disturbance at the source to fade away with time A comparison of the fundamental solution

$$\begin{aligned} u(t, \mathbf{x}) = \big[a^2\, f_0(t - ar, br, \gamma) - 2a\, b\, f_{-1}(t - ar, br, \gamma) + \\ b^2\, f_{-2}(t - ar, br, \gamma)\big] / (4\,\pi\, r) \end{aligned} \tag{26}$$

$$f_0(t, y, \gamma) = \frac{y}{2\sqrt{\pi}}\, \theta(t)\, t^{-3/2}\, \mathrm{e}^{-\gamma\, t}\, \mathrm{e}^{-y^2/(4\,t)} \tag{27}$$

$$f_{-1}(t, y, \gamma) = \mathrm{D}^{-1} \frac{\partial}{\partial y} f_0(t, y, \gamma) \tag{28}$$

$$f_{-2}(t, y, \gamma) = \mathrm{D}^{-2} \frac{\partial^2}{\partial y^2} f_0(t, y, \gamma) \tag{29}$$

of equation (20) with

$$\begin{aligned} \tilde{\rho} &= \rho_\infty \left[a + b\, s^{-1}\, (\gamma + s)^{1/2}\right] \\ \tilde{K} &= K_\infty \left[a + b\, s^{-1}\, (\gamma + s)^{1/2}\right]^{-1} \end{aligned} \tag{30}$$

is shown in Figure 3 (Hanyga, 1999b). The model defined by equation (30) is formally similar to the Wilson model but it allows explicit analytic solutions.

Fractional calculus and numerical implementation of Wilson models

Notwithstanding their complexity the Wilson models are amenable to finite-difference approximations. In the first step the Laplace domain expressions $(1 + \tau\, s)^{\pm\alpha}$ have to be replaced by the time-domain operators $(1 + \tau\, \mathrm{D})^{\pm\alpha}$. The latter can be defined as follows:

$$(1 + \tau\, \mathrm{D})^{\pm\alpha}\, \phi(t) = \tau^{-\alpha} \mathrm{e}^{-t/\tau} \mathrm{D}^{\pm\alpha} \left[\mathrm{e}^{t/\tau}\, \phi(t)\right], \tag{31}$$

where $\mathrm{D}^{-\alpha}$ is a fractional integral operator and D^α is a Caputo fractional derivative (Appendix A).

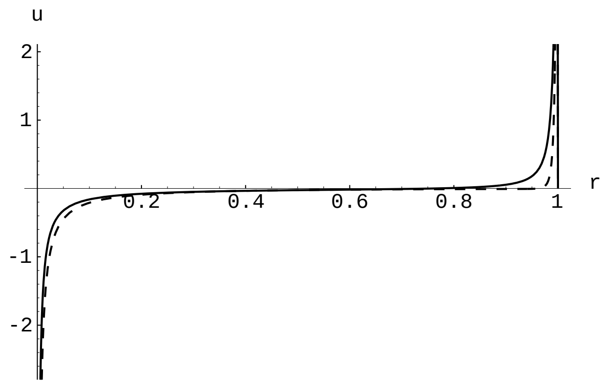

Figure 3. Comparison of the snapshots s of $u(t, \mathbf{x})$, (26) at $t = 1.0$ for $\gamma = 100$ (solid line) and $\gamma = 0$ (dashed line).

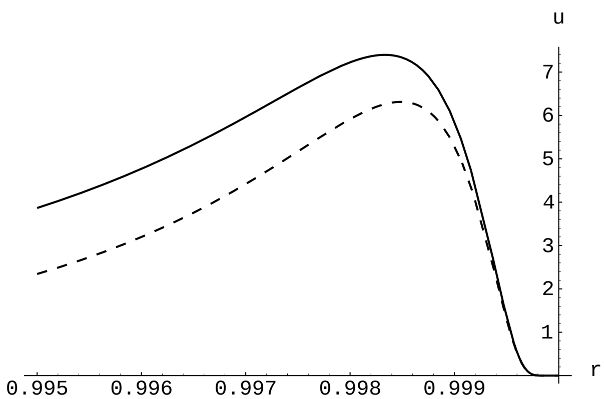

Figure 4. Comparison of the snapshots of $u(t, \mathbf{x})$, (26), at $t = 1.0$ for $\gamma = 100$ (solid line) and $\gamma = 0$ (dashed line). Wavefront pulse.

The operator (31) can be readily discretized applying the definition (31) and equation (A7) or (A5).

A time-domain formulation of the Wilson equation can be obtained by introducing an auxiliary variable $\mathbf{v} = K * \nabla\theta$:

$$\begin{aligned} \left[1 + \Gamma_\rho\ (1 + \gamma_\rho\,\mathrm{D})^{-\alpha}\right]\ \mathrm{div}\,\mathbf{v} \\ = \rho_\infty\,\mathrm{D}^2\theta \\ \left[1 + \Gamma_K\ (1 + \gamma_K\,\mathrm{D})^{-\alpha}\right]\ \mathbf{v} \\ = K_\infty \nabla\theta \end{aligned} \tag{32}$$

for the relaxation function (22), and

$$\begin{aligned} \left[\Gamma'_\rho + \epsilon_\rho\ (1 + \gamma_\rho\,\mathrm{D})^{\alpha}\right]\ \mathrm{div}\,\mathbf{v} = \\ \rho_\infty\ \left[1 - \epsilon_\rho + \epsilon_\rho\,(1 + \gamma_\rho\,\mathrm{D})^{\alpha}\right]\mathrm{D}^2\theta \\ \left[\Gamma'_K + \epsilon_K\ (1 + \gamma_K\,\mathrm{D})^{\alpha}\right]\ \mathbf{v} = \\ K_\infty\ \left[1 - \epsilon_K + \epsilon_K\,(1 + \gamma_K\,\mathrm{D})^{\alpha}\right]\ \nabla\theta \end{aligned} \tag{33}$$

with $\Gamma'_\rho = \Gamma_\rho + 1 - \epsilon_\rho, \Gamma'_K = \Gamma_K + 1 - \epsilon_K$ for the relaxation function (23). Equations (32) and (33) can readily be approximated by finite difference schemes.

Conclusions

In the context of wave propagation problems complex physical models can be replaced by simple models depending on a small number of relevant parameters. Due to their simple structure the resulting equations are amenable to rigorous mathematical analysis. The associated initial-value problems can be solved by asymptotic, numerical and analytical methods.

Notwithstanding their simplicity these models can account for the singularity of the convolution kernels describing the viscous drag, thermal relaxation and multiple scattering. An important practical consequence of these phenomena is pulse delay with respect to the ray-theoretical travel time.

Singular convolution kernels cannot be approximated by a discrete set of relaxation mechanims of the Maxwell-Voigt or Zener type. This follows from the fact that the relaxation spectra (Christensen, 1971) of singular convolution kernels have a singularity at the zero relaxation time. Consequently the numerical methods commonly used in visco-elastic modeling are not applicable for singular memory models.

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APPENDIX A: Elements of Fractional Calculus

The Dirichlet formula for repeated indefinite integral

$$(\mathrm{I}^n) f(t) \equiv \int_0^t \mathrm{d}\tau_1 \int_0^{\tau_1} \dots \int_0^{\tau_{n-1}} \mathrm{d}\tau_n \, f(\tau_n)$$
$$= \int_0^t \mathrm{d}\tau \, (t-\tau)^{n-1} f(\tau)/(n-1)! \equiv [1/(n-1)!]\, t^{n-1} * f(t) \qquad \text{(A1)}$$

has a natural generalization to positive real values of the exponent:

$$(\mathrm{I}^\alpha f)(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} * f(t) \qquad \text{(A2)}$$

An alternative notation for *fractional integrals* is $\mathrm{D}^{-\alpha} f \equiv \mathrm{I}^\alpha f$.

The *Riemann-Liouville derivative* $\mathrm{D}^\alpha_{\mathrm{RL}}$ (Samko *et al.*, 1993; Miller & Ross, 1993) and the *Caputo derivative* (Caputo, 1969; Gorenflo & Mainardi, 1998), are defined as the left and right inverse of I^α, $\alpha > 0$, respectively. More specifically,

$$\mathrm{D}^\alpha_{\mathrm{RL}} = \mathrm{D}^m \, \mathrm{I}^{m-\alpha}, \quad \mathrm{D}^\alpha = \mathrm{I}^{m-\alpha} \, \mathrm{D}^m$$

for $m > \alpha \geq m - 1$.

The Riemann-Liouville fractional derivative is inappropriate for modelling dissipation because $\mathrm{D}^\alpha_{\mathrm{RL}} 1 \neq 0$, which would entail dissipation in a system at rest. In contrast the Caputo derivative of a constant is zero. The Caputo derivative is also more convenient in the context of conventional initial-value problems because its Laplace transform depends on the initial values of the derivatives of integer order:

$$(\mathrm{D}^\alpha f)\tilde{}(s) = s^\alpha \tilde{f}(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0+) \qquad \text{(A3)}$$

for $m-1 < \alpha \leq m$ (Gorenflo & Mainardi, 1998). Finally, the Caputo derivative satisfies the exponent law at least in one case

$$\mathrm{D}^{\alpha+1} = \mathrm{D}^\alpha \, \mathrm{D} \quad \text{for } 0 < \alpha < 1. \qquad \text{(A4)}$$

On the other hand, the Riemann-Liouville derivative has its equivalent finite-difference definition, known as the Grünwald-Letnikow derivative (Samko *et al.*, 1993; Miller & Ross, 1993):

$$\mathrm{D}^\alpha_{\mathrm{RL}} f(t) = \lim_{h \to 0+} h^{-\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(t - kh) \qquad \text{(A5)}$$

Note that, for a causal function $f(t)$ ($f(t) = 0$ for $t < 0$), the summation terminates at $k = [t/h]$ (the integer part of t/h).

The Caputo derivative can be expressed in terms of the Riemann-Liouville derivative

$$\mathrm{D}^\alpha f(t) = \mathrm{D}^\alpha_{\mathrm{RL}} f(t) + \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0+) \qquad \text{(A6)}$$

for $m > \alpha \geq m - 1$.

Fractional integral operators can also be approximated by finite-difference operators (Gorenflo, 1997; Lubich, 1986):

$$\mathrm{D}^{-\alpha}_{\mathrm{RL}} f(t) = \lim_{h \to 0} h^\alpha \sum_{k=0}^{\infty} (-1)^k \binom{-\alpha}{k} f(t - k\,h) \qquad \text{(A7)}$$