

Lyapunov exponents and localization in randomly layered elastic media

John A. Scales

*Department of Geophysics and Center for Wave Phenomena
Colorado School of Mines, Golden, CO 80401, USA*

Erik S. Van Vleck

*Department of Mathematical and Computer Sciences
Colorado School of Mines, Golden, CO 80401, USA*

ABSTRACT

Wave propagation in a randomly heterogeneous elastic medium is often modeled as a random dynamical system associated with a lattice of coupled springs and masses. The scattering effects of the heterogeneities on a propagating pulse are characterized by the frequency dependent localization length—effectively the “skin depth” for multiple scattering attenuation. On a one-dimensional (1D) lattice of length N , the localization length is the reciprocal of the positive Lyapunov exponent of an N -fold product of 2×2 random matrices. In this case, all propagating solutions are either exponentially growing or decaying. In higher dimensions the situation is more complicated, but assuming quasi-1D propagation, the localization length can still be characterized in terms of the Lyapunov spectrum of products of random matrices. We describe a robust numerical procedure for estimating these exponents for 1D or quasi-1D propagation in randomly layered media. In addition, we provide uncertainty estimates for the exponents.

Key words: Localization, Lyapunov Exponents, Random Media

Background: Wave Propagation in Disordered Media and Localization

Broadly speaking, the effects of multiple scattering on a wave propagating in a disordered medium are:

- The path length is increased.
- The pulse is dispersed.
- The pulse is attenuated as energy is shifted from the direct arrival into the multiple scattering coda.

These ideas are well understood and can be explained via perturbation theory for weakly disordered media. (For an overview of the mathematical results see (Asch *et al.*, 1991).) Randomly layered media are common in a number of applications including seismology, where the layering is due to the natural process of sedimentation, and non-destructive evaluation, where the layering could be the result of lamination of composite materials. As an example, consider the model shown in Figure 1. Here we have chosen a pseudo-random sequence

(to be described later) of layer thicknesses and spring constants (i.e., elastic stiffnesses). We can scale these to reflect typical values for different applications. For example, if we take the minimum layer thickness to be 1 m, assume a constant density, and scale the spring constants to correspond to acoustic speeds in the range of 2400-4000 m/s, we can model the lithology of the upper crust of the earth. Figure 2 shows a 2D finite difference wave propagation simulation for such an earth model; at a given depth, the plot shows the energy recorded at that depth as a function of time. The first energy seen at any given depth is the primary downgoing pulse. Coherent events sloping downward to the right (such as the primary arrival) correspond to downgoing waves, while events sloping downward to the left correspond to upgoing waves. In a simulation such as this it is possible to follow the envelope of the pulse as it propagates into the medium. Figure 3 shows a log-linear plot of the amplitude of this pulse. Since the medium is perfectly elastic, the

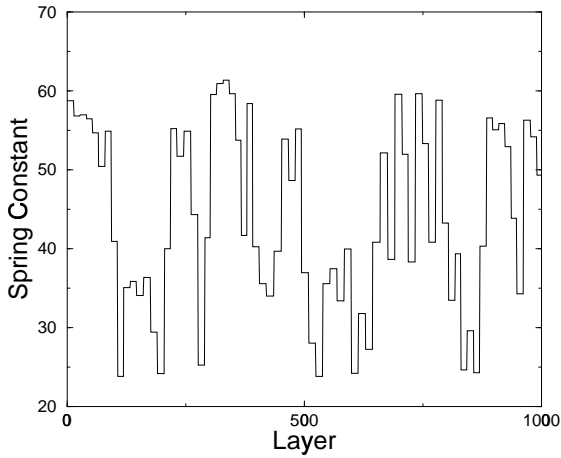


Figure 1. A model of a randomly laminated medium obtained by choosing pseudo-random sequences of spring constants and layer thicknesses. These spring constants and layer thicknesses can then be scaled to reflect typical elastic wave speeds for a given application.

decay shown in this figure is due to multiple scattering and geometrical spreading.

There is a close connection between multiple-scattering attenuation and Anderson localization (Anderson, 1958). The Schrödinger equation for a lattice with random potential is closely related to the frequency-domain equations of motion for an elastic lattice with randomly varying spring constants and masses. It was shown by Ishii (1973) that in 1D any degree of uncorrelated random disorder is sufficient to cause exponential localization of the eigenfunctions of the quantum mechanical or elastic disordered system. (An eigenfunction ψ_n is exponentially localized if it decays exponentially at large distances from its maximum: $|\psi_n| < |\psi_0| \exp(-\alpha|n|)$, where α is a positive constant and the maximum of ψ has been taken to be at $n = 0$.) As we will describe in more detail below, for independent random disorder this exponential localization follows from a theorem of Furstenberg (Furstenberg, 1963) once the underlying equations have been rewritten in terms of transfer matrices.

The localization length can be thought of as the skin depth associated with multiple scattering attenuation. Consider a vertically incident plane wave in a layered medium. If the envelope of a plane wave decays exponentially with distance $e^{-z/\ell(f)}$, then $\ell(f)$ is the frequency dependent localization length. Thus

$$\ell^{-1} = \lim_{L \rightarrow \infty} \left(-\frac{1}{L} \ln |T| \right) \quad (1)$$

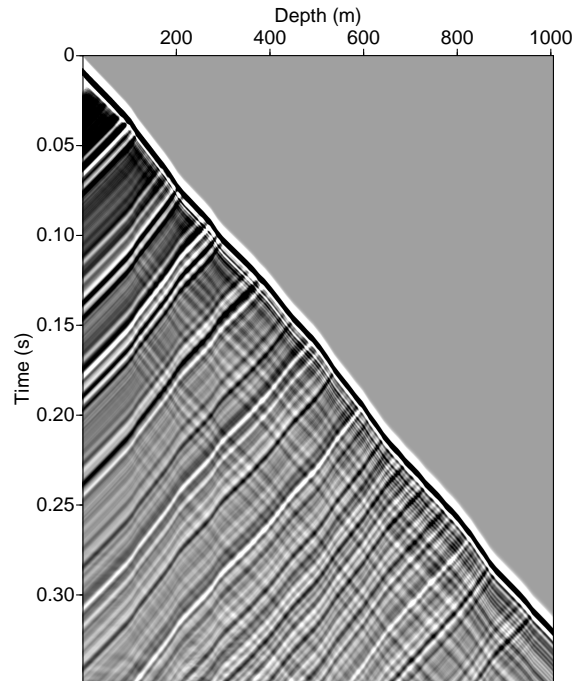


Figure 2. We take the model shown in Figure 1 and scale the spring constants and layer thicknesses to reflect values typical of the earth's near surface. Here is shown a 2D acoustic finite-difference simulation of a pulse propagating into such a medium. The pulse is generated by applying a band-limited point source (with a central frequency of 60 Hz) just below the upper surface. This figure shows the acoustic response at each layer as a function of time. The first event seen at each depth is the direct wave, with all the other events being upgoing or downgoing multiply scattered energy.

where T is the transmission coefficient and L is the propagation distance. An approximation to the localization length for finite L is

$$\ell(f) \approx \frac{-L}{\ln |T|}. \quad (2)$$

The frequency dependent localization length would be an extremely useful quantity to have in many applications involving wave propagation in highly heterogeneous media since it provides the means to scale the effects of complex microstructure. For example, given the exponential nature of the decay, it is natural to think of the multiple scattering as a relaxation mechanism. Hence there is a Q (quality factor) equal to $\pi f \ell / v(f)$, where v is the phase velocity (Futterman, 1962). Then from Kramers-Krönig, the velocity dispersion is

$$\frac{1}{v(f)} - \frac{1}{v(\infty)} = \mathcal{H} \left(\frac{1}{2\pi f \ell} \right) \quad (3)$$

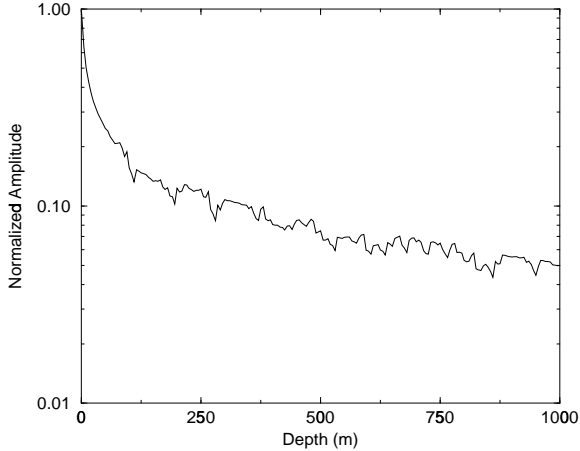


Figure 3. Log-linear plot of amplitude of the envelope of the downgoing pulse as a function of depth. Since the medium is perfectly elastic, this decay is due to multiple scattering and geometrical spreading.

where \mathcal{H} is the Hilbert transform, v_i are the layer velocities, $v(\infty) \equiv v_\infty = \langle v_i^{-1} \rangle^{-1}$ is the infinite frequency (ray theoretic) effective velocity, and the angled brackets denote averaging. (The zero frequency effective velocity, the so-called Backus velocity, is $v_0 = \langle v_i^{-2} \rangle^{-1/2}$.)

A dispersion formula such as this tells us how to relate observations made at one length scale with those made at another. Sheng *et al.* ((1986a) and (1986b)) argue that $\ell(f)$ has the form $c_1 + c_2/f^2$, in which case it is easy to show that the phase velocity dispersion formula is

$$v(f) = v_\infty \left(1 - \frac{v_\infty}{2\pi} \frac{\sqrt{c_2/c_1}}{c_1 f^2 + c_2} \right). \quad (4)$$

Rather than assuming a particular functional form for $\ell(f)$ we seek a direct method of calculating it. Returning to the finite difference simulation shown above, it would appear that in principle one could do a spectral analysis of the observed decay of the pulse and from this estimate the frequency dependent localization length. However, our goal is to develop robust numerical procedures for characterizing this multiple-scattering attenuation directly from in-situ measurements of the elastic properties of the medium, which we regard as a realization of some underlying stochastic process. We will achieve this by casting the wave propagation problem in terms of the products of random matrices. This is possible for 1D systems or quasi-1D systems such as the laminated medium described above, in which the wave propagation reduces to the application of transfer

matrices. We will show later that the decay of energy in these quasi-1D systems, which we characterize in terms of the localization length, can be computed from the Lyapunov exponents (LEs) of the product of random transfer matrices. Further, we will describe a numerical algorithm based on the repeated application of the discrete QR algorithm for computing LEs, which allows us to compute just the positive LEs along with error estimates.

Equations of Motion

Let us begin with the simplest case, waves propagating in a heterogeneous 1D medium. Later, we will show how to generalize these results to higher dimensional systems. The 1D wave equation is

$$\rho(z) \frac{\partial^2 U}{\partial t^2} = \frac{\partial}{\partial z} \left(K(z) \frac{\partial U}{\partial z} \right) \quad (5)$$

where U is the displacement, ρ the density and K the elastic modulus. To obtain a discrete approximation to the wave equation we could either apply a finite difference approximation to the spatial derivatives in Equation 5, or replace the continuum with a discrete spring/mass system and appeal to Newton's second law of motion.

The discrete approach has the advantage that the problem can be characterized in terms of the eigenmodes of the lattice. The localization itself is seen to arise from the effects of individual heterogeneities (lattice defects), which give rise to eigenmodes whose envelopes decay exponentially away from the defect sites. Further, the analysis of localization can be studied via the asymptotic properties of products of (small) random matrices. We will see, for example, that the localization length ℓ can be identified with the reciprocal of the smallest positive Lyapunov exponent of this product of random matrices. In 1D this is straightforward since there is only one positive Lyapunov exponent. In the more general quasi-1D case this exponent will control the dominant features of decay, but the other positive exponents could play a role in understanding the fine details.

There are N point masses m_ℓ connected by $N - 1$ Hooke's law springs of stiffness k_ℓ . The end masses are fixed, as shown in Figure 4 and the longitudinal displacement of the ℓ -th mass x_ℓ is measured relative to its equilibrium position.

The kinetic energy of the system is $\frac{1}{2} \sum_{\ell=0}^N m_\ell \dot{x}_\ell^2$ and the potential energy (assuming only nearest neighbor interactions) is $\frac{1}{2} \sum_{\ell=0}^N k_{\ell+1} (x_{\ell+1} - x_\ell)^2$. The lattice equations of motion are

$$m_\ell \ddot{x}_\ell - k_{\ell+1} (x_{\ell+1} - x_\ell) - k_\ell (x_{\ell-1} - x_\ell) = 0. \quad (6)$$

Assuming a sinusoidal time dependence, the equation for the spatial part of the motion is

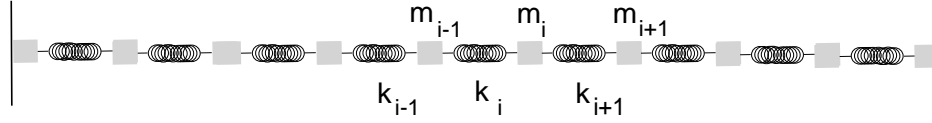


Figure 4. A simple mechanical system exhibiting localization of energy. Masses m_i are connected via springs of stiffness k_i . The end masses are fixed. Longitudinal displacement of the i -th mass x_i is measured relative to its equilibrium position.

$$-m_\ell \omega^2 z_\ell = k_{\ell+1} z_{\ell+1} - (k_{\ell+1} + k_\ell) z_\ell + k_\ell z_{\ell-1} \quad (7)$$

where $x_\ell \equiv z_\ell e^{i\omega t}$, or

$$(T + \omega^2 M) \mathbf{z} = 0 \quad (8)$$

where M is the mass matrix $M = \text{diag}(m_\ell)$ and T is the tridiagonal matrix of spring constants: $T(\ell, \ell) = -(k_{\ell+1} + k_\ell)$, $T(\ell, \ell + 1) = k_{\ell+1}$, and $T(\ell + 1, \ell) = k_\ell$.

Free oscillations of the 1D lattice

For the ordered monoatomic 1D lattice: $m_\ell = m$, $k_\ell = k$ for all ℓ , so the equation for the free oscillations reduces to

$$(T + \omega^2 I) \mathbf{z} = 0. \quad (9)$$

Since the spring constants are all equal to k , T reduces to k times

$$\begin{bmatrix} -2 & 1 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & \dots \\ 0 & 1 & -2 & 1 & \dots \\ \vdots & & & \ddots & \\ 0 & \dots & 0 & 1 & -2 \end{bmatrix}. \quad (10)$$

The eigenvalues of this matrix are

$$\omega_\ell^2 = -2 + 2 \cos\left(\frac{\ell\pi}{N}\right) \quad (11)$$

while the eigenvectors are

$$\mathbf{z}_\ell = \{\sin(\ell\pi/N), \sin(2\ell\pi/N), \dots, \sin((N-1)\ell\pi/N)\}. \quad (12)$$

Another way of looking at the free oscillations of the lattice is to recast the eigenvalue problem in Equation 7 as a recursion relation:

$$z_{\ell+1} = \left(2 - \frac{m\omega^2}{k}\right) z_\ell - z_{\ell-1} \quad (13)$$

which can be re-written as a one-step mapping

$$\begin{pmatrix} z_\ell \\ z_{\ell+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 - \frac{m\omega^2}{k} \end{pmatrix} \begin{pmatrix} z_{\ell-1} \\ z_\ell \end{pmatrix}. \quad (14)$$

Defining the matrix

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 2a \end{pmatrix} \quad (15)$$

where $a = 1 - \frac{m\omega^2}{2k}$, it follows by induction that

$$\begin{pmatrix} z_\ell \\ z_{\ell+1} \end{pmatrix} = B^\ell \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \quad (16)$$

where B^ℓ is the ℓ -th power of the matrix B .

The eigenvalues of B are $\lambda_{1,2} = a \pm \sqrt{a^2 - 1}$. If $|a| > 1$ ($\omega^2 > 4k/m$) then the z_ℓ solutions are exponentially growing or decaying and cannot satisfy the zero displacement boundary conditions. If $|a| < 1$ ($\omega^2 < 4k/m$), then the solutions are oscillatory and can be made to satisfy the boundary conditions. Taking $z_0 = 0$, then

$$\begin{pmatrix} z_{N-1} \\ z_N \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2a \end{pmatrix}^N \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (17)$$

Obviously this equation cannot produce an eigenvector for arbitrary values of a . For example, if we are trying to compute normal modes whose displacement vanishes at both ends of the lattice, then only using a associated with an eigenvalue will result in z_N being zero.

Localized States on Lattices

For the homogeneous lattice all the eigenmodes are nonzero everywhere except at the nodal points. In that sense they have global extent. Suppose we disturb the regularity of the lattice by perturbing either a spring constant or a mass. Rayleigh's principle ((Rayleigh, 1945), Volume I, Sec. 88) says that if a single mass is reduced (or a spring constant increased) then all the frequencies are unchanged or increased, but not by more than the distance to the nearest unperturbed frequency. Similarly, if a single mass is increased (or a spring constant decreased) then all the frequencies are unchanged or decreased, but not by more than the distance to the nearest unperturbed frequency.

In spite of this apparent symmetry between the effects of increasing versus decreasing masses (or decreasing versus increasing spring constants), there is a profound difference in practice. Because the low frequency limit is zero (at least for the monoatomic chain), decreasing a spring constant, and hence one or more frequencies, cannot result in new frequencies outside the band of frequencies allowed by the dispersion relation for the homogeneous or perfect lattice. On the other hand, increasing a spring constant can result in perturbing an eigenfrequency beyond the maximum allowed by the homo-

geneous dispersion relation ($2\sqrt{k/m}$ for the monoatomic chain). If this happens, the resulting eigenmode must be exponentially damped. One way to see that this is so is to observe that for complex wavenumbers the sine in the dispersion relation becomes a sinh, which can match any frequency whatsoever. The result also follows from the discussion following Equation 16.

This effect is illustrated in Figure 5 which shows the two highest frequency eigenmodes for systems of point masses connected by 50 springs (spring constant equal to one) with, respectively, one and two perturbed spring constants. In the first case, a single frequency is pushed into the forbidden band beyond $2\sqrt{k/m}$, resulting in a single localized eigenmode. Increasing two spring constants sufficiently results in two eigenfrequencies in the forbidden (or *impurity*) band and therefore two localized eigenmodes. (It is possible, by making a sufficiently large change in a single spring constant, to generate more than one localized mode centered on the impurity.) As a single spring constant is perturbed by ϵ , then the envelope of the highest frequency mode switches from convex to concave (i.e., becomes exponentially damped) at precisely that value of ϵ which pushes the first perturbed eigenfrequency into the impurity band.

The corresponding experiment, that of decreasing a single spring constant, does not result in a localized mode because the minimum frequency for the monoatomic chain is zero. For a chain with more than one type of “atom”, the dispersion curve is multi-valued, so we should expect to see localized modes associated with decreasing a spring constant since the higher frequency branches of the dispersion curve will have nonzero minimum frequency; pushing a frequency off the bottom of an upper branch would necessarily result in an exponentially damped eigenmode, just as pushing a frequency off the top of the acoustical branch does.

The physics of randomly disordered lattices began to be studied intensively in the 1940s and 1950s. A pioneering series of papers by Lifshitz ((Lifshitz, 1943a), (Lifshitz, 1943b), (Lifshitz, 1944)) was published in Russia during the Second World War. This phase of research culminated in the seminal paper by Anderson (1958). Anderson’s model was of lattices—regular or irregular—of electron spins (or other entities) each of whose energy was a random variable. Then, provided the interparticle potential decayed sufficiently fast as a function of distance (faster than r^{-3}), and the disorder was stronger than some critical value, Anderson was able to show that the electron wave function was localized in space asymptotically with time. For a recent review of localization theory and experiment see Kramer & MacKinnon (1993).

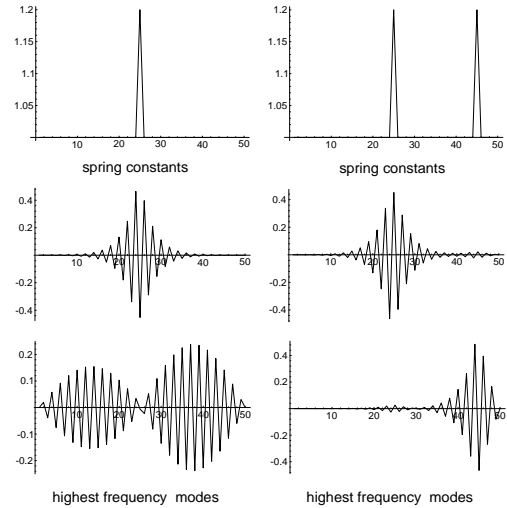


Figure 5. Perturb a single site by increasing its spring constant (or decreasing its mass). Rayleigh’s principle says that this will result in increasing (or leaving unchanged) the frequencies. Increasing a single spring constant sufficiently results in a single eigenvalue being pushed out of the band of frequencies allowed by the homogeneous dispersion relation. This must therefore be associated with a localized mode. Perturbing two lattice sites in this case pushes two frequencies outside the allowed band, resulting in two localized modes.

Products of Random Matrices

The first systematic study of the randomly disordered 1D chain was made by Freeman Dyson (1952). Dyson developed a method for calculating the distribution of eigenfrequencies in the $N \rightarrow \infty$ limit as a continued fraction. If the oscillators themselves are distributed according to an exponential law, Dyson’s result is analytic. This work was extended by Helmut Schmidt (1957) who developed the recursive method of computing the eigenfunctions.

The eigenvector recursion formula extends readily to arbitrarily disordered chains. In this case the equations of motion are

$$z_{\ell+1} = \frac{k_{\ell+1} + k_{\ell} - m_{\ell}\omega^2}{k_{\ell+1}}z_{\ell} - \frac{k_{\ell}}{k_{\ell+1}}z_{\ell-1}, \quad (18)$$

which we can rewrite as

$$\begin{pmatrix} z_{\ell} \\ z_{\ell+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k_{\ell}}{k_{\ell+1}} & \frac{k_{\ell+1} + k_{\ell} - m_{\ell}\omega^2}{k_{\ell+1}} \end{pmatrix} \begin{pmatrix} z_{\ell-1} \\ z_{\ell} \end{pmatrix}. \quad (19)$$

Defining the matrix

$$B_{\ell} = \begin{pmatrix} 0 & 1 \\ -\frac{k_{\ell}}{k_{\ell+1}} & \frac{k_{\ell+1} + k_{\ell} - m_{\ell}\omega^2}{k_{\ell+1}} \end{pmatrix} \quad (20)$$

it follows by induction that

$$\begin{pmatrix} z_{\ell} \\ z_{\ell+1} \end{pmatrix} = B_{\ell} \cdot B_{\ell-1} \cdots B_1 \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \equiv P_{\ell} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}. \quad (21)$$

In this way we have managed to reduce the study of wave propagation in disordered 1D media to application of the theory of products of random matrices (PRM). We note that for constant spring constants and varying masses the coefficient matrix is symplectic, i.e. $J^T B_\ell J = B_\ell$ for all ℓ where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (22)$$

Also in this case, the determinant of each transfer matrix is 1 and hence the determinant of the product P_N is also 1. That the product matrices are unimodular assures that the sum of the two LEs is zero. In general, for varying spring constants, the matrices B_ℓ are not symplectic, nor are the determinants equal to 1. But since the determinant of the product P_N is the product of the determinants of the individual B_ℓ , the determinant of P_N equals k_1/k_N . In the 1D numerical experiments described below, with pseudo-random k_i , we have always found there to be one positive and one negative LE.

Using the PRM approach it is clear that the growth or decay of solutions to the equations of motion is governed by exponential growth and decay rates of the product matrix P_N . The logarithm of the eigenvalues of a symmetrized version of this matrix (described below) are the Lyapunov exponents. Now the localization length is defined in terms of the exponential decay of eigenfunctions. But it seems reasonable to associate this localization length with the reciprocal of the positive (1D) or smallest positive (quasi-1D) LE. This conjecture is well supported by numerical evidence (cf. (Crisanti *et al.*, 1993), (Kramer & MacKinnon, 1993), and the simulations below).

So there remain two issues that must be addressed. First, what can be said about asymptotic properties of products of random matrices, such as occur in Equation 21? Secondly, how can the Lyapunov spectrum of P_N be accurately and efficiently computed?

Asymptotic Properties of Products of Random Matrices

The first basic result is due to Furstenberg (1963), who showed that if the matrices are independent and non-singular in a certain sense, the limit

$$\lambda_1 = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \|P_N\| \quad (23)$$

exists almost surely. Further, the maximum Lyapunov exponent λ_1 is a non-random quantity; in other words

$$\lambda_1 = \lim_{N \rightarrow \infty} \frac{1}{N} \langle \ln \|P_N\| \rangle \quad (24)$$

where the angled brackets refer to the average under the distribution associated with the B_ℓ , which are assumed to be independent and identically distributed. The Lyapunov exponent measures the growth rate associated with *typical* vectors z

$$\lambda_1 = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \|P_N z\| \quad (25)$$

although this does not preclude different growth rates for improbable choices of z .

These results were generalized by Virster (1979), who showed that if the matrices B_ℓ were of the form $B_\ell = B(\xi_\ell)$, where ξ_ℓ is a stationary ergodic Markov chain and B is a matrix function on the state space of the chain, then λ_1 exists and is positive almost surely. Virster's results, which are equivalent to Furstenberg's in the special case of independent matrices, thus apply to a rather large class of disordered systems including, for example, exponentially correlated Gaussian systems.

Calculation of the Lyapunov Spectrum

To determine the Lyapunov exponents of our linear mapping we will employ a method based upon performing a QR decomposition at each iteration. The discrete QR algorithm to be described below is the most widely used technique to approximate LEs. Other methods based upon performing a singular value decomposition are also possible (see Abarbanel *et al.* (1992) and Geist *et al.* (1990)). We consider the rates of growth and decay for the linear mapping

$$P_{N+1} = B_N P_N, \quad P_N \in \mathbb{R}^{n \times n}, \quad N = 0, 1, \dots, \quad Y_0 = I. \quad (26)$$

Definition (Oseledec, 1968): Let

$$P_{N+1} = B_N \cdots B_0, \quad N = 1, \dots,$$

be a fundamental solution of (26) (with $(P^0)^T P^0 = I$). Then, the following symmetric positive definite matrix exists

$$\Lambda = \lim_{N \rightarrow \infty} (P_N^T P_N)^{\frac{1}{2N}},$$

the logarithms of the eigenvalues of which are called **Lyapunov exponents**, and are denoted by $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. The λ_i 's do not depend on the initial condition matrix P^0 almost surely.

The theorem of Oseledec (Oseledec, 1968) leads to an equivalent characterization of LEs. Let $\lambda^{(1)} > \lambda^{(2)} > \dots$ be the LEs of (26) not repeated by multiplicity. Let $E^{(i)}$ be the invariant subspace of \mathbb{R}^n corresponding to the eigenvalues of Λ whose logarithm is less than or equal to $\lambda^{(i)}$, so that $\mathbb{R}^n = E^{(1)} \supset E^{(2)} \supset \dots$. Let $\mathbf{p}_k \in E^{(k)} \setminus E^{(k+1)}$, then one has

$$\lambda^{(k)} = \limsup_{N \rightarrow \infty} \frac{1}{N} \log \|P_N \mathbf{p}_k\|$$

The system (26) is called **regular** (Lyapunov, 1949) if

$$\begin{aligned} \sum_{k=1}^p \lambda_k &= \lim_{N \rightarrow \infty} \frac{1}{N} \log(|\det(B_{N-1} \cdots B_0)|) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \log(|\det(P_N)|). \end{aligned} \quad (27)$$

If (26) is regular and upper triangular, one has (Lyapunov, 1949)

$$\lambda_k = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} \log |(B_j)_{kk}|, \quad k = 1, \dots, p.$$

When the B_j are not upper triangular, we will successively compute QR factorizations of the transfer matrices. Given orthogonal $P_0 : P_0^T P_0 = I$, let $Q_0 = P_0$. Set $Z_{N+1} = B_N Q_N$, $N = 0, 1, \dots$ and then decompose $Z_{N+1} = Q_{N+1} R_{N+1}$, where R_{N+1} is upper triangular with positive diagonal entries. Since we obtain $Q_{N+1} R_{N+1} = B_N Q_N$, then $Q_{N+1}^T B_N Q_N = R_{N+1}$ is upper triangular. Thus, we can obtain the LEs as

$$\begin{aligned} \lambda_k &= \lim_{N \rightarrow \infty} \frac{1}{N} \log((R_N)_{kk} \cdots (R_1)_{kk}) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N \log((R_j)_{kk}) \end{aligned} \quad (28)$$

Following the work in (Dieci *et al.*, 1995) and (Dieci & Van Vleck, 1995) we will estimate the error in our calculation by fitting the error as $\frac{K}{N} + \epsilon$. If $\lambda_k(N)$ is the k^{th} finite time Lyapunov exponent at iterate N , then simple linear regression gives us the following values for ϵ and K :

$$K = \frac{\sum_{n=1}^N \frac{\lambda_k(n)}{n} - \frac{1}{N} \left(\sum_{n=1}^N \frac{1}{n} \right) \left(\sum_{n=1}^N \lambda_k(n) \right)}{\sum_{n=1}^N \frac{1}{n^2} - \frac{1}{N} \left(\sum_{n=1}^N \frac{1}{n} \right)^2}$$

and

$$\epsilon = \frac{1}{N} \left(\sum_{n=1}^N \lambda_k(n) - K \sum_{n=1}^N \frac{1}{n} \right) - \lambda_k^*$$

where λ_k^* is the exact Lyapunov exponent. Since we do not know the exact Lyapunov exponent, we use the best available approximation, $\lambda_k(N)$, instead of λ_k^* .

To recap, we have described a technique for estimating the exponential growth or decay of products of random transfer matrices by computing the Lyapunov spectrum of this matrix product via a discrete QR method. Error estimates follow from linear regression of the finite time exponents. Now we will illustrate applications of these methods to wave propagation problems of the sort described in the first section.

Simulations - One Dimension

To do the simulations we generate pseudo-random sequences of spring constants and layer thicknesses. These are chosen in an uncorrelated fashion from uniform distributions and then correlated by applying a running average of a given length.

Each layer is described by its thickness and spring constant and the medium is described by a sequence of thicknesses and the corresponding spring constants in each layer. Thus, the statistical properties of our simulated medium are described in terms of maximum and minimum values and correlation lengths for both the thicknesses of the layers and the spring constants.

If the layer thicknesses were all equal to 1, say, then the maximum frequency of propagation on the lattice would correspond to a wavelength of 1. By putting groups of homogeneous spring constants between the “scatterers” we can simulate wavelengths smaller than the distance between the scatterers.

Once the medium is determined, the task is to compute the positive Lyapunov exponents. The Lyapunov exponents are defined as limits, although our medium will have a finite thickness. To obtain better approximations one could average over several starting vectors or, more easily, simply consider the medium as periodic and average over several periods.

In Figures 6 – 12 we illustrate some of the numerical results we have obtained. All plots are in log-log scale and plot the frequency $\omega/(2\pi)$ on the horizontal axis against the computed localization length on the vertical axis.

We use the computed upper bound for the Lyapunov exponent to compute a lower bound on the localization length; this is denoted by *LEerrorp* in the figures. The simulations are for media with a total thickness of 1000 and the Lyapunov exponent was obtained by averaging over 10 “periods.” In all of the simulations the spring constants had a maximum value of 100 and a minimum value of 1. The correlation length for the spring constants and the thicknesses was allowed to vary, as were the maximum and minimum thickness of the layers.

The maximum localization length computable in this way is the length of the lattice, in this case 1000. Thus, values of $1/\lambda_1$ greater than 1000 correspond to weak attenuation. On the other hand, the minimum meaningful localization length is the lattice spacing. We see that at sufficiently high frequencies the waves are localized. However, as the medium becomes more and more smooth, the range of frequencies that are localized goes to zero.

Figures 6–8 illustrate the lack of monotonicity that occurs for layers of unit thickness and small correlation lengths for the spring constants. All figures provide com-

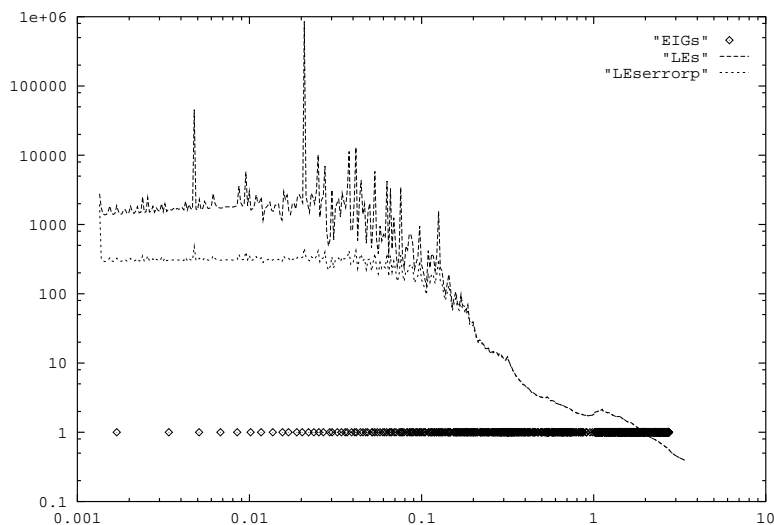


Figure 6. $\frac{1}{\lambda_1} vs. \frac{\omega}{2\pi}$, where λ_1 is the positive Lyapunov exponent, the spring correlation length = 1, thickness correlation length = 1, maximum thickness = 1, minimum thickness = 1

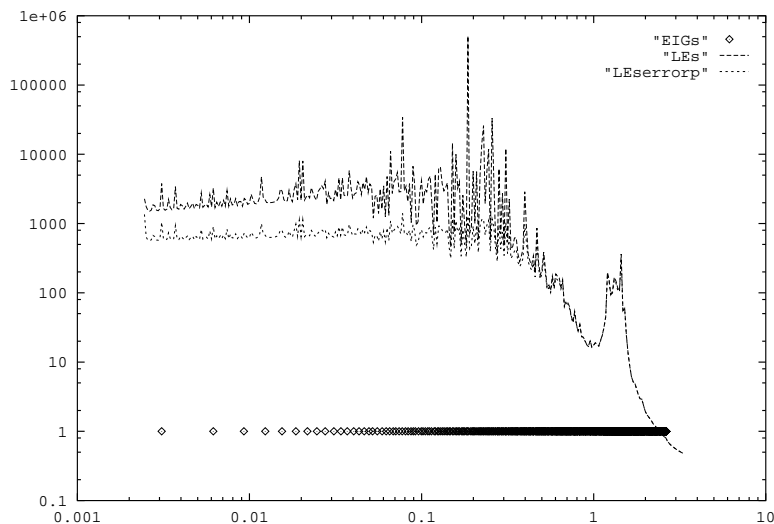


Figure 7. $\frac{1}{\lambda_1} vs. \frac{\omega}{2\pi}$, spring correlation length = 2, thickness correlation length = 1, maximum thickness = 1, minimum thickness = 1

Figure 16 was computed for the model shown in Figure 1 and therefore affords a direct comparison with the 1D LE calculation in Figure 12 and the time-domain finite difference results in Figures 3 (2D) and 13 (1D). In 2D, as we would have expected from the finite difference results, we see localization lengths less than 1000 for nearly all frequencies. In addition we see clear evidence

of nonmonotonicity of the frequency dependence of the localization length.

That the Lyapunov exponents are nonrandom in the quasi-1D case follows under reasonable assumptions from the work of Osceledec (1968). However, in the quasi-1D case there is no guarantee that the first N_2 finite time Lyapunov exponents are positive even if $\det(P_N) \geq 1$. As a consequence there appear to be two alternative defini-

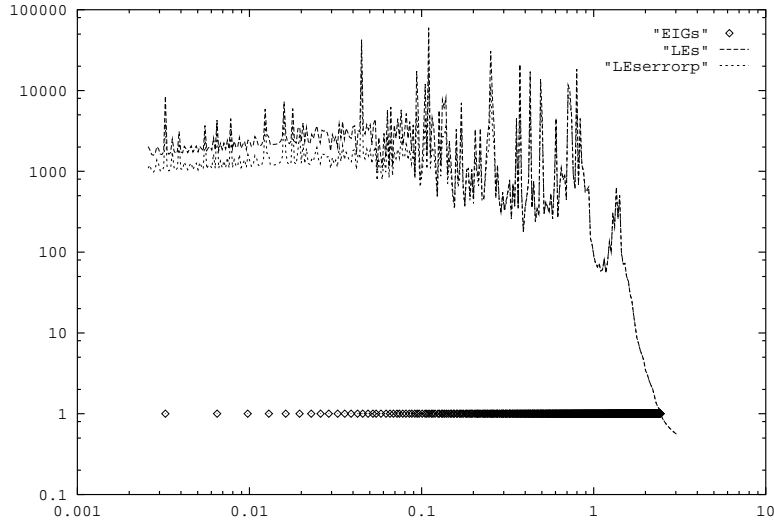


Figure 8. $\frac{1}{\lambda_1} vs. \frac{\omega}{2\pi}$, spring correlation length = 4, thickness correlation length = 1, maximum thickness = 1, minimum thickness = 1

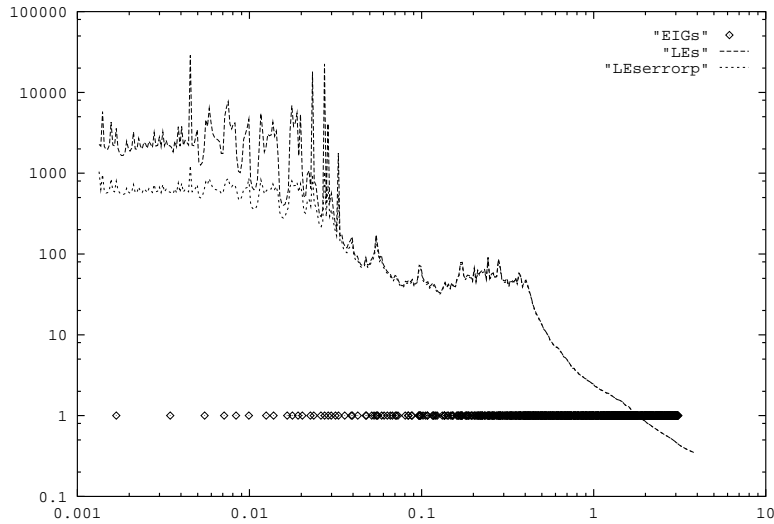


Figure 9. $\frac{1}{\lambda_1} vs. \frac{\omega}{2\pi}$, spring correlation length = 1, thickness correlation length = 1, maximum thickness = 10, minimum thickness = 5

tions of the localization length. The first is to define the localization length as the reciprocal of the smallest positive exponent. The second is to define the localization length as the $\frac{1}{\lambda_{N_2}}$ if λ_{N_2} is positive and infinite otherwise. We have employed the second definition in our simulations.

Finally, to decrease the computation time we compute only the first N_2 exponents and not all $2N_2$ expo-

nents (see Dieci & Van Vleck (1995)). Note in Figures 14 and 15 that it appears that the localization lengths are converging as the width N_2 is being increased.

Conclusions

Classical waves scattering from random heterogeneities in elastic solids may excite vibrational modes that are loc-

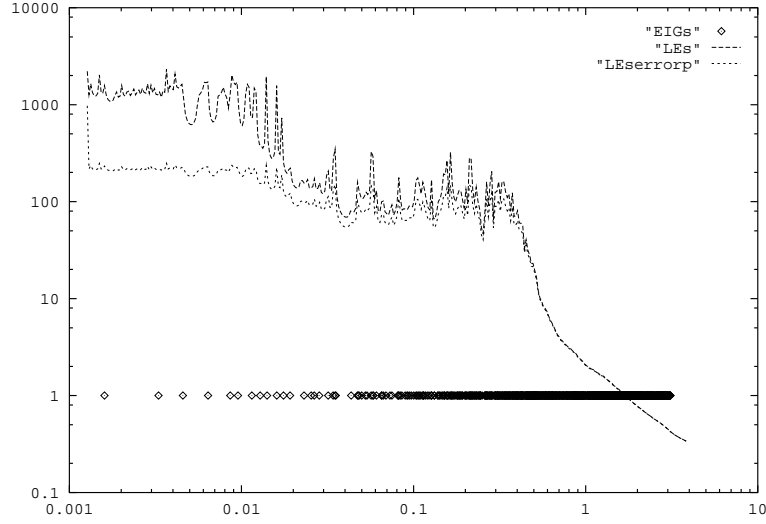


Figure 10. $\frac{1}{\lambda_1}$ vs. $\frac{\omega}{2\pi}$, spring correlation length = 1, thickness correlation length = 1, maximum thickness = 20, minimum thickness = 10

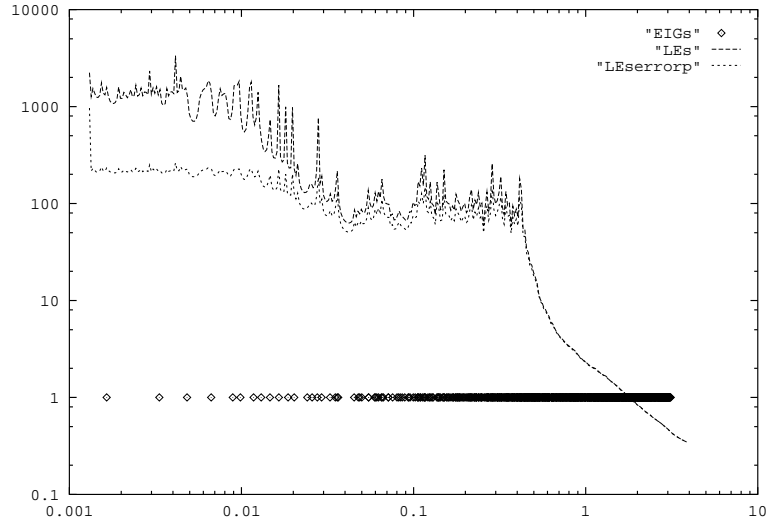


Figure 11. $\frac{1}{\lambda_1}$ vs. $\frac{\omega}{2\pi}$, spring correlation length = 1, thickness correlation length = 4, maximum thickness = 20, minimum thickness = 10

alized about the heterogeneities. This causes propagating energy to be converted into localized fluctuations and is manifested in the dispersion and attenuation of the waves. With a sufficiently strong random distribution of heterogeneities, it is possible for a propagating wave to become trapped altogether. The frequency-dependent length scale on which this attenuation occurs—the localization length—is the fundamental quantity character-

izing this phenomenon. For quasi-1D wave propagation, i.e. propagation that can be described via products of transfer matrices, the localization length can be estimated from the Lyapunov spectrum of this product of random matrices. We have developed an algorithm for computing the positive, finite-time Lyapunov exponents (along with error estimates) associated with layered elastic media and applied this algorithm to a number

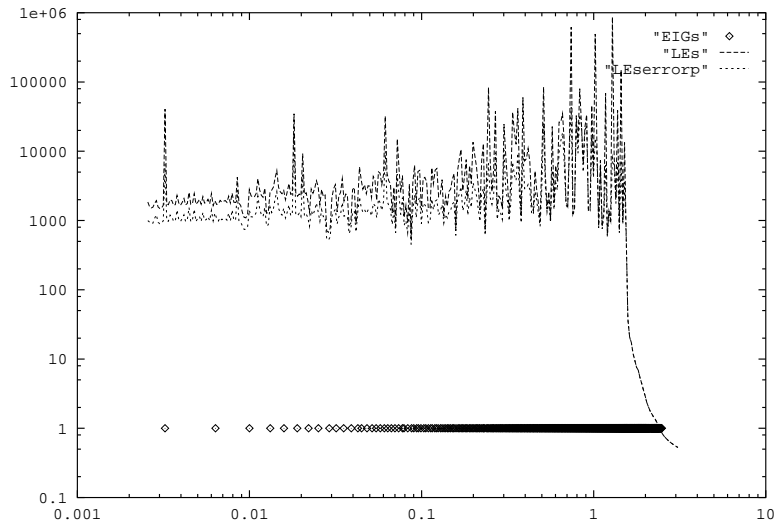


Figure 12. $\frac{1}{\lambda_1}$ vs. $\frac{\omega}{2\pi}$, spring correlation length = 4, thickness correlation length = 4, maximum thickness = 20, minimum thickness = 10

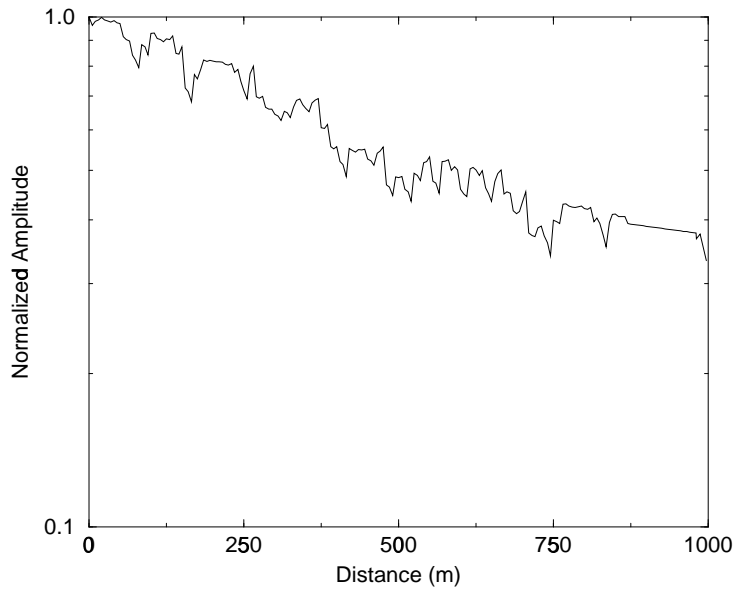


Figure 13. Log-linear plot of amplitude of the envelope of the downgoing pulse as a function of depth in a purely 1D medium.

of pseudo-randomly generated models. The results show overall agreement with coarse estimates of localization made from time-domain finite difference calculations, but provide a more detailed picture of scattering attenuation.

Acknowledgements

This work was partially supported by the sponsors of the Consortium Project on Seismic Inverse Methods for Complex Structures at the Center for Wave Phenomena and the National Science Foundation under grant DMS-

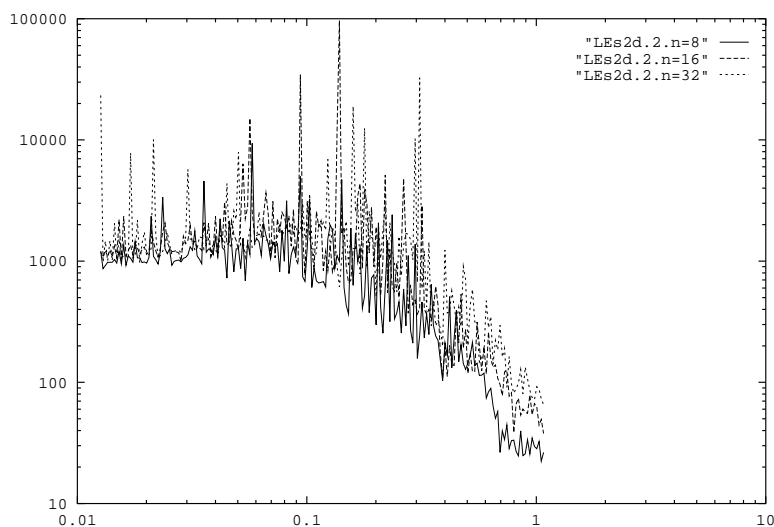


Figure 14. $\frac{1}{\lambda_{N_2}} vs. \frac{\omega}{2\pi}$, spring correlation length = 2, thickness correlation length = 1, maximum thickness = 1, minimum thickness = 1

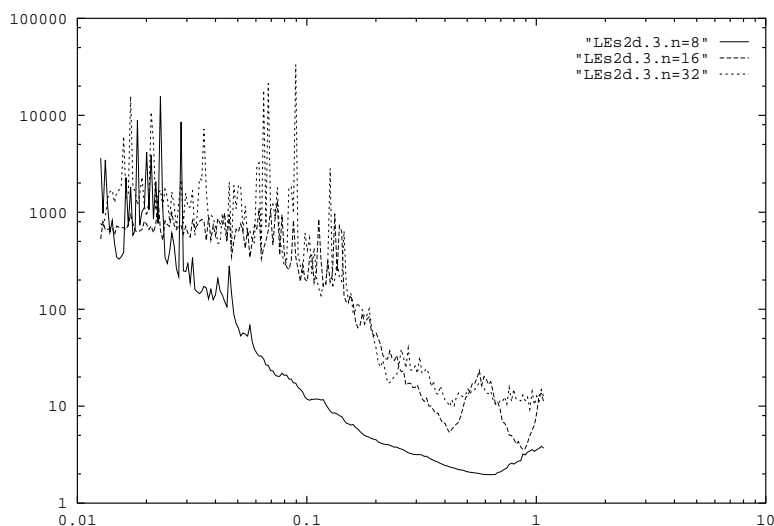


Figure 15. $\frac{1}{\lambda_{N_2}} vs. \frac{\omega}{2\pi}$, spring correlation length = 4, thickness correlation length = 1, maximum thickness = 1, minimum thickness = 1

9506603. In addition JS acknowledges the support of the Shell Foundation and EVV acknowledges the support of the National Science Foundation under grant DMS-9505049.

References

- Abarbanel, H., Brown, R., & Kennel, M. 1992. Local Lyapunov Exponents from Observed Data. *Journal of Nonlinear Science*, **2**, 343–365.
- Anderson, P. W. 1958. Absence of diffusion in certain random lattices. *Physical Review*, **109**, 1492–1505.
- Asch, M., Kohler, W., Papanicolaou, G., Postel, M., & White, B. 1991. Frequency content of randomly scattered signals. *SIAM Review*, **33**, 519–625.
- Crisanti, A., Paladin, G., & Vulpiani, A. 1993. *Products*

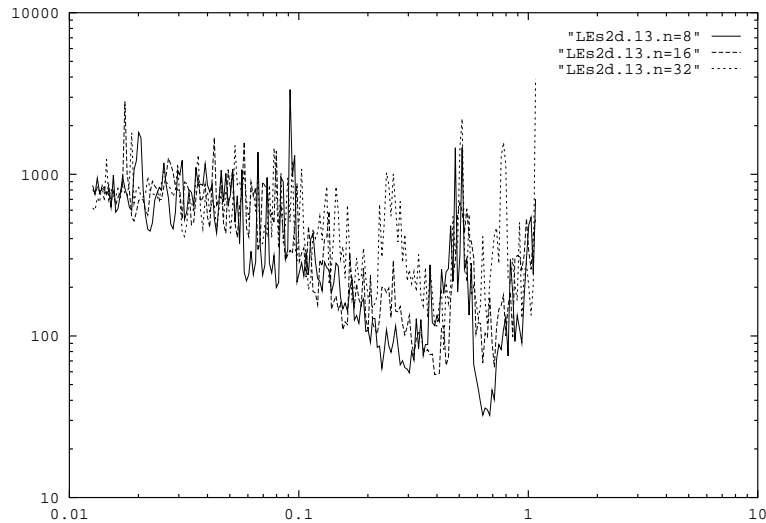


Figure 16. $\frac{1}{\lambda_{N_2}}$ vs. $\frac{\omega}{2\pi}$, spring correlation length = 4, thickness correlation length = 4, maximum thickness = 20, minimum thickness = 10

- of random matrices.* Springer.
- Dieci, L., & Van Vleck, E.S. 1995. Computation of a Few Lyapunov Exponents for Continuous and Discrete Dynamical Systems. *Appl. Numer. Math.*, **17**, 275–291.
- Dieci, L., Russell, R.D., & Van Vleck, E.S. 1995. On the Computation of Lyapunov Exponents for Continuous Dynamical Systems. *SIAM J. Numer. Anal.* (to appear).
- Dyson, F. J. 1952. The dynamics of a disordered linear chain. *Physical Review*, **92**, 1331–1338.
- Furstenberg, H. 1963. Noncommuting random products. *Transactions American Mathematical Society*, **108**, 377.
- Futterman, Walter I. 1962. Dispersive body waves. *Journal of Geophysical Research*, **67**, 5279–5291.
- Geist, K., Parlitz, U., & Lauterborn, W. 1990. Comparison of Different Methods for Computing Lyapunov Exponents. *Prog. Theor. Phys.*, **83**, 875–893.
- Ishii, K. 1973. Localization of eigenstates and transport phenomena in the one-dimensional disordered system. *Suppl. Progress of Theoretical Physics*, **53**, 77–138.
- Kramer, B., & MacKinnon, A. 1993. Localization: theory and experiment. *Reports on Progress in Physics*, **56**, 1469–1564.
- Legendijk, A., Van Albada, M., & Van Der Mark, M. 1986. Localization of light: the quest for the white hole. *Physica*, **140A**, 183–190.
- Lifshitz, I. M. 1943a. Optical behaviour of non-ideal crystal lattices in the infra-red. I. *Journal of Physics*, **7**, 215–228.
- Lifshitz, I. M. 1943b. Optical behaviour of non-ideal crystal lattices in the infra-red. II. *Journal of Physics*, **7**, 249–261.
- Lifshitz, I. M. 1944. Optical behaviour of non-ideal crystal lattices in the infra-red. III. *Journal of Physics*, **8**, 89–105.
- Lyapunov, A. 1949. Problém Général de la Stabilité du Mouvement. *Annals of Mathematics Studies*, **17**.
- Oseledec, V. I. 1968. A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems. *Transactions Moscow Mathematical Society*, **19**, 197–237.
- Rayleigh, J. W. S. 1945. *Theory of sound.* Dover.
- Schmidt, H. 1957. Disordered one-dimensional crystals. *Physical Review*, **105**, 425–441.
- Sheng, P., Zhang, Z.-Q., White, B., & Papanicolaou, G. 1986a. Minimum wave-localization length in a one-dimensional random medium. *Physical Review*, **B34**, 4757–4761.
- Sheng, P., Zhang, Z.-Q., White, B., & Papanicolaou, G. 1986b. Multiple scattering noise in one dimension: universality through localization length scaling. *Physical Review Letters*, **57**, 1000–1003.
- Virster, A. D. 1979. On the products of random matrices and operators. *Theory of Probability and its Applications*, **24**, 367–377.