Applicability of the Gaussian Beam Approach for Modeling of Seismic Data in Triangulated Subsurface Models

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ABSTRACT

Part of the challenge in efforts to improve seismic imaging is to develop software to generate computer models of complex geology. One approach uses a triangulation technique designed to support efficient and accurate computation of seismic wavefields for two-dimensional models of the Earth's interior. This triangulation approach offers the opportunity to perform dynamic ray tracing and create synthetic seismograms based on the method of Gaussian beams.

The combination of the Gaussian beam method (GBM) and the triangulation technique is especially promising if analytic velocity profiles are assumed in each triangle. A constant gradient of sloth, the inverse of the velocity squared, allows analytic solutions of both kinematic and dynamic ray tracing equations. In this way, the attractive feature of computational efficiency is maintained even in models of complex geology.

Here, I give a brief introduction to the triangulation principle. My main intention, however, is to enhance the understanding of the Gaussian beam approach. More specifically, the basics of the GBM are illustrated with emphasis on inherent approximations and assumptions. I show that Gaussian beam modeling is influenced by the choice of initial beam parameter even in simple two-layer or slightly inhomoge-

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neous media. The problem is compounded in attempts to compute accurate synthetic seismograms where strong lateral velocity inhomogeneities are present. I discuss the severity of these problems and outline some practical ways to reduce them to a certain extent. Example computations of high-frequency seismograms based on the method of Gaussian beams exhibit advantages and disadvantages of this increasingly more popular method for the modeling and migration of seismic data.

INTRODUCTION

Triangulation methods for representing subsurface models in computers have helped to overcome limitations inherent in previous computer models of the Earth's interior. A mesh of triangles can flexibly characterize complex subsurface models that can, for example, include velocity lenses and overhanging dome structures. Additionally, Hale and Cohen (1991) designed spatial data structures that contain the adjacency topology (Weiler, 1988) of the model, minimizing computing time used to search for the triangles and model parameters in the vicinity of an arbitrary point in the model. Following the work of Hale and Cohen, Wiggins et al., (1993) recently extended Weiler's topological structures to build three-dimensional depth-migration velocity models.

Applicability to complex two- and three-dimensional subsurface structures and computational efficiency motivate the use of triangulated models for seismic ray tracing. The combination of dynamic ray tracing (Červený, 1985a) and the triangulation technique would seem especially promising. Dynamic ray tracing is an important extension of classical kinematic ray tracing that provides useful dynamic properties such as geometrical spreading and wavefront-curvature information along the rays. For models composed of triangles in which the inverse of the squared velocity varies linearly, this additional information can be computed analytically, without computer-time consuming numerical integration along the rays. Hale (1991) took advantage
of the dynamic ray tracing results to generate synthetic seismograms based on the method of Gaussian beams. Based on Hale's research, Rüger (1993) and Rüger and Alkhalifah (1995) extended this methodology to allow for an increased variety of ray-theoretical experiments, including an option to perform Fresnel-volume ray tracing (Červený and Soares, 1992) and anisotropic ray tracing.

Here, I present the theory of isotropic dynamic ray tracing in a self-contained, tutorial form. I investigate in detail one specific application, the Gaussian beam method (GBM), and address its current limitations. The GBM for modeling of synthetic data is a well-established subject in the geophysical literature (e.g., Červený et al., 1982). More recently, this method has also been used for migration of seismic data (Hill, 1990). The growing interest in the GBM necessitates a thorough investigation of intrinsic approximations and possible pitfalls of the method. These problems, of course, are not a new subject. Discussions can be found for example in Müller (1984), Ben-Menahem and Beydoun 1985, and White et al. (1987). However, none of these papers explicitly describes Gaussian-beam-specific artifacts and how they might be avoided. Moreover, most papers are addressed primarily to readers with a strong background in ray theory. Here, I try to fill in this gap. Consequently, this paper is intended to provide the interested reader with information to enable effective and critical use of the GBM.

A look backwards to the basic theory of dynamic ray tracing and the GBM reveals inherent assumptions of the method. Three main conditions on the length scale of the medium (Bleistein, 1984), the width of the Gaussian beams, and the velocity and its derivatives will be established. The choice of beam parameter is a critically difficult decision to be made in modeling of seismic data with the GBM. I use synthetic forward modeled data to show that difficulties may arise in slightly inhomogeneous and even two-layer media. Moreover, while the choice of beam parameter may be acceptable for one class of rays (e.g., primary reflections from a given reflector), it may be poor for some other classes of rays. These and other problems are even more
severe if strong velocity gradients and discontinuities are present. The synthetic data computations are performed for acoustic media; however, the gained insight is valid for modeling in elastic models as well.

**EXPLICIT MODEL REPRESENTATION**

The computer generation of complex geologic subsurface structures is often more involved than the actual modeling of physical phenomena, such as the propagation of seismic waves. One way to overcome the problem is the uniform sampling of the subsurface properties on a rectangular grid. Unfortunately, the gained benefit of computational simplicity has to be paid by serious drawbacks, including increased memory storage and reduced computational efficiency.

Various authors have proposed alternative methods. Two- and three-dimensional models can, for example, be represented on triangular and tetrahedral meshes, respectively (e.g., Müller, 1984; Kästner and Fritsche, 1988; Wiggins et al., 1993). One example of a two-dimensional triangulated model is shown in Figure 1. taken from Hale and Cohen (1991). Layer boundaries are shown as black lines in the figure. They are obtained by cubic spline interpolation of coordinate information previously provided at isolated subsurface points. The smoothness of curved boundaries is obtained by adding vertices to the boundary representation until the difference in slope of adjacent triangle edges along the interface falls below a specified limit. The white lines are edges of auxiliary triangles used in the model building. Triangles are constructed by a method known as constrained Delaunay triangulation to discriminate against long and thin triangles. This conveniently guarantees that medium properties such as the propagation speed (indicated by the shading of the triangles in Figure 1) have a range of influence as limited as possible away from any given point of specification.

As mentioned above, the velocity field in each triangle is constrained to be constant or to have a constant gradient in sloth; the propagation speed can be either
continuous or discontinuous across defined boundaries (see, for example, the salt-sediment interface). Note how overhanging structures, interfaces of high curvature and significant velocity gradients can be easily represented by this triangulation technique. Moreover, while the overall structure is rather complex, within each triangular cell, seismic parameters (here: the slope) vary only linearly. Seismic ray tracing is efficient and accurate for this velocity profile. In fact, models of constant gradient in slope yield closed-form solutions for the raypaths. Determination of the intersection of raypath and triangle edges, which can be a difficult task for some other model representations, is simple and involves only solving a quadratic equation. Likewise, the dynamic ray tracing equations can be solved analytically.

The efficiency of model building and applications such as seismic ray tracing can be significantly improved by explicitly representing the topology of the model. Adding explicit topological information to the model data, for example, makes each element in the model aware of its neighbors. Each triangle can access information on the geometric position of the neighboring triangles and associated parameters. This property is especially valuable for efficient ray tracing: after determining the raypath in the first triangle, the velocity of the next triangle in the ray propagation can be determined without any searching through the model. In the same way, one can easily determine if the triangle edge has been defined to reflect, stop, or transmit the ray. More information on explicit topological data structures can be found in Weiler (1988).

**KINEMATIC AND DYNAMIC RAY TRACING**

Ray tracing is an important tool for obtaining mathematical and physical insight into the behavior of waves propagating in the Earth’s interior. Unlike full-waveform techniques (e.g., finite differencing and integral equation methods), ray theory is a high-frequency asymptotic method. Consequently, this method is accurate only when the wavelength of the seismic signal is much smaller than the natural physical scale
of the medium considered. Quantities that qualify for the natural length scale are the smallest values of distances between interfaces, radius of curvature of interfaces or gradients in relative velocity. Under the high-frequency assumption, only the leading-order terms of the ray series

\[ u(\omega, x_i) = e^{i\omega \tau(x_i)} \sum_{n=0}^{\infty} U^{(n)}(x_i) \left( \frac{1}{i\omega} \right)^n, \]  

(1)

are used to represent the seismic wave field. Note that the amplitude coefficients \( U^{(n)} \) and the eikonal \( \tau \) depend only on spatial coordinates \( x_i (i = 1, 2, 3) \). This ray series solution is written in inverse powers of frequency \( \omega \), so if we are interested in high frequencies, we need consider only the first few terms in the series. Specifically, in a practical application such as high-frequency modeling, only the \( n = 0 \) term is used. Then, \( U^{(0)} \) will simply be called \( U \). If higher-order waves (e.g., headwaves) are of interest, more than the \( U^{(0)} \) term is needed. Some wave-propagation effects associated with higher-order terms are described, for example, in Kiselev and Tsvankin, 1989.

Substituting equation (1) into the reduced wave equation (Helmholtz equation) \( u_{,ii} + \omega^2/c^2(x_i) u = 0 \) and rearranging the result in expressions with the highest powers of \( (i\omega) \) yields, to leading order, the eikonal equation

\[ \tau(x_j, \dot{x}_j), \tau(x_j, \dot{x}_j) = \frac{1}{\nu^2(x_j)}. \]  

(2)

As throughout this paper, the well-known Einstein summation rule for repeated indices is used. The comma in front of the indices denotes a partial derivative with respect to cartesian components \( x_i \); e.g., \( \tau, i = \partial \tau / \partial x_i \) and \( u, ii = \sum_{i=1}^{3} \partial^2 u / \partial x_i^2 \).

When we introduce the \textit{slowness vector} \( \mathbf{p}_i = \tau, i \), equation (2) can be represented as a Hamilton-Jacobi equation

\[ H(p_i, x_i) = \frac{1}{2} \left( p_i p_i - \frac{1}{\nu^2} \right) = 0. \]  

(3)

The eikonal equation is the starting point for the derivation of equations for both kinematic and dynamic ray tracing. Recasting this equation in the form (3) has two
advantages. First, standard methods (such as the method of characteristics [e.g., Bleistein, 1984]) may be applied. Characteristic coordinates, for example, can be used, with one of them identified as phase or traveltine; the physical concept of rays can be introduced as characteristics, and the solution for \( u \) along the characteristics (e.g., the solution for \( u \) along the rays) can be visualized to span the solution field \( u(x_i) \) in the entire medium. Second, the Hamilton-Jacobi equation (3) can also be derived by applying classical mechanics theory (e.g., Goldstein, 1950, Chapter 9). Considering the propagation of waves as movement of particles also leads to an expression equivalent to the eikonal equation. In this sense, ray theory is only a classical-mechanics approximation of wave propagation. Therefore, observations obtained by using the ray-theoretical approach outlined in this chapter are valid only if phenomena involving the wavelength of the signal are negligible. This is the physical basis of the restriction that the length scale of the medium must be much larger than the wavelength of the signal. Moreover, ray-theoretical results should only be applied at distances several wavelengths away from the source.

Choosing the nonphysical quantity \( \sigma \) with units of \([\text{length}^2/\text{time}]\) as independent parameter, the characteristic equations of (3) can be represented in the following form (e.g., Červený 1985a)

\[
\begin{align*}
\frac{dx_i}{d\sigma} &= p_i , \\
\frac{dp_i}{d\sigma} &= \frac{1}{2} \left( \frac{1}{v^2} \right) \dot{x} , \\
\frac{d\tau}{d\sigma} &= v^{-2} .
\end{align*}
\]

(4)

The first equation in (4) does not contain any velocity dependence. As a consequence, simple analytic expressions for rays result in some special situations. For example, if the gradient of \( v^{-2} \) in the medium is constant, we can solve for the slowness components \( p_i \) analytically. The inverse of the velocity squared, or sloth, can then be written in the form
\[ \frac{1}{v^2} \equiv s(x_i) = s_{00} + s_{i;i} x_i, \]

and the equation for the slowness vector yields

\[ p_i(\sigma) = p_i(\sigma_0) + \frac{1}{2} s_{i;\alpha} (\sigma - \sigma_0). \]

Likewise, coordinates and traveltime along the raypath are described by low-order polynomial expressions. The simplicity of the solutions and the fact that the polynomials required to compute slowness and traveltime can be evaluated efficiently on the computer make media with a constant gradient in slope particularly attractive.

The system of equations (4) can be used to evaluate slowness and traveltime along a raypath, but it also allows computations of properties of the wavefield close to the ray. The cartesian coordinate system is not very convenient for this application. Better suited are orthogonal coordinate systems that move along the rays. Such systems are described by Červený et al., (1982), for example, and are used to investigate geometrical spreading and curvature along the ray.

For each ray, we introduce an orthogonal curvilinear coordinate system \((q_1, q_2, s)\) connected to the ray. In this so-called \textit{ray-centered coordinate system}, the coordinate \(s\) represents the monotonically increasing arclength along the ray. Parameters \(q_1\) and \(q_2\) form a 2-D cartesian coordinate system in a plane orthogonal to the ray at fixed arclength. As sketched in Figure 2, the basis vectors are the unit tangent \(\hat{t}\) to the ray at given \(s\) and the unit vectors \(\hat{e}_1\) and \(\hat{e}_2\) perpendicular to the ray. Any point \(S\) near the ray can be expressed by its orthogonal projection \(O_S\) onto the ray and its coordinates \(q_1\) and \(q_2\) in the \([\hat{e}_1, \hat{e}_2]\)-plane. Several comments on this specific coordinate system are appropriate at this point:

- In general, more than one representation of \(S\) may exist in the curvilinear coordinate system attached to a specific ray. Imagine, for example, a circular raypath with \(S\) being located in its center. In this case, an infinitude of projection points \(O_S\) can be defined on the ray. A regular coordinate system is
obtained only if the radius of ray curvature is much larger than the wavelength of the seismic signal.

- In elastic modeling, unit vectors \( \hat{e}_1 \) and \( \hat{e}_2 \) can be chosen such as to coincide with the polarization vectors of the two shear waves (SV- and SH-waves). The rotation implied by the torsion of the ray conveniently decouples the SV- and SH-motion of the particles.

- The vector basis \((\hat{e}_1, \hat{e}_2, \hat{t})\) is chosen to be right-handed. The scale factors for this curvilinear coordinate system are

\[
h_{q_1} = h_{q_2} = 1 \ , \ h_s = h \ ,
\]

where \( h \) can be expressed by the derivative of the velocity field along the raypath

\[
h = 1 + q_I \left( v^{-1} \frac{\partial v}{\partial q_I} \right)_{q_K=0}, \tag{5}
\]

with capital letter indices having the values 1 and 2. The subscript \( q_K = 0 \) denotes that the derivatives are evaluated at the central ray.

- Later, when we apply the (dynamic) ray theory of Gaussian beams to two-dimensional media, and only \( \hat{e}_1 \) has to be considered.

In ray-centered coordinates the eikonal equation (2) has the following form

\[
\left( \frac{\partial \tau}{\partial q_1} \right)^2 + \left( \frac{\partial \tau}{\partial q_2} \right)^2 + \frac{1}{h^2} \left( \frac{\partial \tau}{\partial s} \right)^2 = \frac{1}{v^2(q_1, q_2, s)}. \tag{6}
\]

We seek an approximate description of the traveltime close to a selected central ray. Consequently, the solution \( \tau(q_1, q_2, s) \) can be expressed as a truncated Taylor series in \( q_1 \) and \( q_2 \).

\[
\tau(q_1, q_2, s) = \tau(0, 0, s) + \frac{1}{2} q_I q_J M_{IJ}. \tag{7}
\]

Here we denote the symmetric matrix of second derivatives of the traveltime field by

\[
M_{IJ}(s) \equiv \left[ \frac{\partial^2 \tau(q_1, q_2, s)}{\partial q_I \partial q_J} \right]_{q_K=0}. \tag{8}
\]

The linear term \( \partial / \partial q_I [\tau(q_1, q_2, s)]_{q_K=0} \) vanishes because
of orthogonality of rays and wavefronts in isotropic media. While the case of rays incident on velocity discontinuities can be handled by applying boundary conditions, strong velocity gradients and jumps near the raypath cannot be sensed by the ray. Thus, the approximate traveltime (7) will be accurate only if the velocity field changes smoothly away from the ray. Specifically, velocity discontinuities in the paraxial vicinity of the central ray will yield erroneous traveltime extrapolations. This can be a severe restriction in models with significant velocity variations in which the high-frequency criterion of ray theory (wavelength $<<$ length scale) is still well satisfied. Additionally, the second term in the Taylor series (7) has to be much smaller than the first term. Physically speaking, this implies that the radius of wavefront curvature needs to exceed the wavelength of the signal.

The derivatives of the paraxial eikonal function (7) can now be inserted into (6). Subsequently, the ratio $h^2/v^2$ is expanded to second order in $q_l$, which involves a second derivative of the velocity field. The paraxial approximation of the eikonal equation in ray-centered coordinates reduces to the simple equation

$$\frac{dM}{ds} + vM^2 + \frac{1}{v^2} V = 0,$$

(8)

where

$$V = V_{JJ}(s) = \left[ \frac{\partial^2 v(q_1, q_2, s)}{\partial q_I \partial q_J} \right]_{q_K=0}$$

is the matrix of second derivatives of the velocity field at points along the ray under investigation. If we choose $\sigma$ as the monotonically increasing parameter along the ray, (8) can be rewritten as

$$\frac{dM}{d\sigma} + M^2 + \frac{1}{v^3} V = 0,$$

(9)

Equation (9), the so-called dynamic ray-tracing equation, is a first-order Ricatti-type differential equation to be solved along a known ray. Below we will find that its solution can help to determine important physical quantities such as the amplitude
of the modeled wavefield. The standard procedure for eliminating the nonlinear term is to set \( \mathbf{M}(\sigma) = (dQ/d\sigma)Q^{-1} \) and \( \mathbf{P} = dQ/d\sigma \) to obtain a set of eight linear, first-order differential equations

\[
\frac{dQ}{d\sigma} = \mathbf{P} \quad ; \quad \frac{d\mathbf{P}}{d\sigma} = -\frac{1}{v^3} \nabla Q. \quad (10)
\]

This, in short, is the derivation of the dynamic ray-tracing system. Probably the most detailed reference can be found in Červený (1985a). Physical understanding is gained by identifying \( Q(\sigma) = [\partial q_I/\partial \gamma_J]_{q_k=0} \) and \( P(\sigma) = [\partial p_I/\partial \gamma_J]_{q_k=0} \) with \( \gamma_J \) being the ray coordinates (a popular choice, such as sketched in Figure 3, is the takeoff angle or the azimuthal angle) and \( p_I \) represent the slowness component in ray-centered coordinates. Mathematically, the solution of equation (10) describes the transformation from ray- to ray-centered coordinates (Figure 3); physically, the dynamic ray-tracing system can be used to analyze the behavior of paraxial rays; e.g., rays with slightly perturbed ray parameters can be observed solely by computing \( \mathbf{Q} \) and \( \mathbf{P} \) along the central ray. This, on the other hand, allows ready computation of paraxial approximations of Fresnel volumes and the ray Jacobian.

In summary, the equations of dynamic ray tracing can be derived by solving the eikonal equation close to a central ray. The result is accurate for small distances from the central ray, provided the medium has a length scale much larger than the wavelength of the seismic signal and the velocity field and its derivatives vary smoothly and slowly in the ray vicinity. A strict mathematical discussion on the validity conditions of dynamic ray tracing and the GBM can be found in Ben-Menahem and Beydoun, 1985.

**GAUSSIAN BEAMS**

A significant simplification can be achieved by considering a two-dimensional model. One ray coordinate and the initial point are now sufficient to define a ray, and
any point $S$ close to a ray can be described by two ray-centered coordinates (arc length $s$ and normal distance to the central ray $n$, for example). In this 2-D case, $Q$ and $P$ are reduced to simple scalars $q$ and $p$ because all derivatives with respect to $q_2$ and $\gamma_2$ vanish. The dynamic ray tracing system (10) reduces to only two linear, ordinary differential equations

$$\frac{dq}{d\sigma} = p, \quad \frac{dp}{d\sigma} = -\frac{1}{v^3} \frac{\partial^2 v}{\partial q_1^2} q.$$  \hspace{1cm} (11)

Several comments are appropriate:

- Whenever the gradient of $v^{-2}$ in the medium is constant, this system has analytic solutions, which, for example, can be found in Hale (1991).

- Solving system (11) for two linear independent intrinsic, initial choices $Y_1(\sigma_0)$ and $Y_2(\sigma_0)$, for example

  $$\begin{pmatrix} Y_1(\sigma_0) \\ Y_2(\sigma_0) \end{pmatrix} = \begin{pmatrix} q_1(\sigma_0) \\ p_1(\sigma_0) \end{pmatrix}, \begin{pmatrix} q_2(\sigma_0) \\ p_2(\sigma_0) \end{pmatrix} = \begin{pmatrix} 1 [m] \\ 0 [m] \end{pmatrix}, \begin{pmatrix} 0 [s/m] \\ 1 [s/m] \end{pmatrix}$$

  determines the entire solution space of (11). Thus, as soon as $Y_1(\sigma)$ and $Y_2(\sigma)$ are known, we can find solutions of the dynamic ray tracing system analytically for any initial condition specified at $\sigma_0$. The notation $q_1, q_2$ and $p_1, p_2$ is commonly used in the literature and should not be confused with the ray-centered coordinates $q_1, q_2$ introduced in the last chapter. It is important to realize, that in the above discussion, $q_1$ and $q_2$ have units of length, and $p_1$ and $p_2$ units of inverse velocity (see terms in square brackets on the right hand side of the above equation). This fact is often neglected in the literature.

- In two-and-one-half dimensional media (Bleistein, 1984), equation (10) reduces to system (11) plus the restriction that $Q_{22}$ is proportional to $(\sigma - \sigma_0)$. This additional equation leads to the familiar out-of-plane spreading correction.
For acoustic media of constant density, and to leading order in frequency, $u$ in equation (1) can be represented in the form

$$u(\sigma) = \sqrt{\frac{v(\sigma)}{J(\sigma)}} A \exp\left[-i\omega(t - \tau(\sigma))\right],$$

where $A$ is a source-dependent amplitude term, independent of $\sigma$, and $J$ is the ray Jacobian. Recognizing that $J$ can be described by the normal distance of the paraxial rays from the central ray $q$ gives

$$u(\sigma) = \sqrt{\frac{v(\sigma)}{q(\sigma)}} A \exp\left[-i\omega(t - \tau(\sigma))\right]. \quad (12)$$

The key step in introducing the concept of Gaussian beams is to expand the eikonal $\tau$ as in equation (7), but allowing $M$ to be complex valued: $M = \text{Re}(M) + i \text{Im}(M)$. The exponential in equation (12) then becomes

$$\exp\left[-i\omega \left(t - \tau(O_S) - \frac{1}{2} \text{Re}(M) n^2\right)\right] \exp\left[-\frac{\omega}{2} \text{Im}(M) n^2\right].$$

$O_S$ again denotes the orthogonal projection of a point $S$ in the vicinity of the ray, with $n$ being the perpendicular distance (i.e., $n \equiv q_t$). Considering just the real part of $M$ would correspond to the paraxial traveltime extrapolation; the imaginary part additionally introduces a frequency-dependent Gaussian amplitude decay away from the central ray. The distance from the central ray at which the amplitude of the Gaussian beam is 1/e times the amplitude on the central ray is $L(\omega, O_S) = \sqrt{2\omega^{-1} \cdot [\text{Im}(M(O_S))]^{-1}}$, provided that $\text{Im}(M) > 0$. This inequality can be shown to be satisfied once it is true for any arbitrary point along the central ray. A sketch of the amplitude profile perpendicular to the central ray is shown in Figure 4. In two dimensions, $M(\sigma) = p(\sigma)/q(\sigma)$ can be evaluated as a linear combination of the intrinsic dynamic ray tracing results

$$M = \frac{p}{q} = \frac{\epsilon p_1 + p_2}{\epsilon q_1 + q_2}.$$ 

(13)

The beam width $L$ can now be expressed in terms of the complex parameter $\epsilon = \epsilon_1 + i\epsilon_2$. 

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\[ L = \sqrt{\frac{2}{\omega} \left( \frac{(\epsilon_1 q_1 + q_2)^2 + (\epsilon_2 q_1)^2}{-\epsilon_2 [s]} \right)}. \]  

(14)

Several interesting features should be summarized:

- For each beam, the dynamic ray-tracing data \( q_1, q_2, p_1 \) and \( p_2 \) are weighted by a complex parameter \( \epsilon \). The choice of \( \epsilon \) determines the frequency-dependent width of the beam as well as the phase-front curvature along the central ray.

- \( \epsilon_2 \) must be negative. As a result, \( q(\sigma) \) will never be zero, even in situations where the paraxial rays cross the central ray and the real part of \( q \) is zero. Thus, Gaussian beams yield finite amplitudes everywhere, including at caustics of the wavefield.

- Each Gaussian beam carries an \( \epsilon \)-dependent complex quantity in the amplitude function. Consequently, it does not make physical sense to consider individual Gaussian beams. In other words, each beam must be used in concert with other beams.

- A second popular way to define the Gaussian beams can be found in the literature. The so-called parabolic wave-equation method uses the reduction of the wave equation to a parabolic equation for each individual wave type. This procedure, described in detail in Červený and Pšenčík (1983), has the advantage of explicitly deriving the Gaussian beams as high-frequency asymptotic solutions of the wave equation close to rays.

**GAUSSIAN BEAM SUMMATION METHOD**

In the GBM, the source field is expanded into Gaussian beams. Several parameters influence the accuracy of the expansion:

- the number of beams used in the expansion
the width of each beam at some specified position along the central ray

- the phase-front curvature of each beam at some specified position along the central ray

Phase-front curvature, i.e., the second derivative of travelt ime evaluated at the ray, and beam width may vary from beam to beam and must be specified at one point, wherever desired, along each ray. The width and the number of individual beams determine the beam coverage in the area of interest. Since the wave equation is linear, synthetic seismograms may be computed at any point P in the medium as a weighted sum over all beams.

\[ u(P) = \int_{\phi} \Phi(\phi) \ u_\phi(P) \ d\phi, \]  

(15)

where \( u(P) \) is the solution at the receiver, and \( u_\phi(P) \) is the solution of the beam with takeoff angle \( \phi \), evaluated at \( P \). The common procedure for estimating the complex-valued weighting function \( \Phi(\phi) \) is to compare integral (15) with the exact solution of the wave equation in homogeneous media for frequency \( \omega \to \infty \) (Červený, et al., 1982). One major requirement on \( \Phi(\phi) \) is to make the solution for \( u(P) \) independent of the choice of \( \epsilon \) introduced above.

The idea of Gaussian-beam summation is sketched in Figure 5. A source wavefield is expanded into Gaussian beams, which are individually computed from the source to their endpoints. The seismic wavefield at each receiver is then evaluated by integrating over all beams in the vicinity of the receiver. The GBM therefore considers not only the information provided by the nearest beam, but the information of all beams in the neighborhood. This procedure has several advantages. First, rays (which represent the support of the beams) are no longer required to stop at the exact position of each receiver; thus, time-consuming two-point ray tracing can be avoided. Second, the GBM yields stable (but not necessarily accurate) results in regions of the wavefield where the standard ray theory fails (e.g., caustics, shadow zones and critical distance.
for reflections). Third, as compared with seismograms computed by conventional ray tracing techniques, the GBM synthetic data are less influenced by minor details in the model representation. These advantages are discussed and illustrated, for example, in Červený (1985b), where more references can be found.

**BEAMS IN SIMPLE MEDIA**

The behavior of the beams is not intuitive, and some simple examples are appropriate to improve our understanding of their propagation and spreading. First note that, written as a function of arclength, equations (11) yield the following solution in media with constant velocity

\[
q_1(s) = 1 \quad p_1(s) = 0
\]

\[
q_2(s) = v s \quad p_2(s) = 1,
\]

again recalling that \(q_1\) and \(q_2\) have units of distance, and \(p_1\) and \(p_2\) have units of slowness. Using these equations, we can illustrate the spreading of the beams as a function of arclength along the central ray (see Figure 6). The medium has a constant speed of 5000 m/s, and the frequency is 25 Hz. In Figure 6a, \(\epsilon_1\) is set to zero, and \(\epsilon_2\) varies. In Figure 6b, \(\epsilon_2\) is held constant, and \(\epsilon_1\) changes. The shading of the graphs indicates an increase in the magnitude of \(\epsilon_2\) (in Figure 6a) and \(\epsilon_1\) (in Figure 6b). Light gray denotes small magnitudes, dark gray large magnitudes.

Obviously, \(\epsilon_2\) influences the rate of spreading of the beam. Specifically, small \(\epsilon_2\) creates a rapid increase of beamwidth for increasing arclength. On the other hand, \(\epsilon_1\) determines the position along the ray at which the beam width is minimal. From this example, we learn that the intuitive idea that the beams spread with increasing arclength in order to adequately cover the medium with traveltime and amplitude information, is true only for \(\epsilon_1 = 0\). Additionally, for this reason, a physical interpretation of Gaussian beams in terms of Fresnel-volumes (Kim et al., 1990) will be difficult. Fresnel-volumes have minimum radius at the source and receiver, whereas
beams might have their minimum width at any position along the ray (for $\epsilon_1 \neq 0$) and, beyond this point, tend to spread with distance along the central ray. Let us follow the commonly-discussed approach in the literature of setting the curvature of the phase-front of the beam to zero at the source; i.e., phase-fronts are planar at the source, and $\epsilon_1 = 0$. According to equation (14), $\epsilon_2$ specifies the beam width at the source (see Figure 5). In the following, I refer to the source beam width, evaluated at dominant frequency, as the beam parameter.

To illustrate difficulties inherent to the GBM, a simple experiment is sketched in Figure 7. Rays are traced from a single shotpoint and reflected from an interface at a depth of 2.5 km. Resulting synthetic seismograms for various choices of beam parameter are shown in Figure 8. Note that the choice of beam parameter has influence on the generated time sections. Even in this simple model, a poor choice of beam parameter yields spurious arrivals and an abnormal amplitude behavior. This phenomenon may be understood better by considering Figure 9. The diagram shows the dominant-frequency beam widths at the ray ends, as a function of source-to-receiver offset. A poor choice of beam parameter yields beam widths that are too broad at the ray ends. For a beam parameter of 0.1 km, the width of the beams at the ray ends even exceeds the horizontal dimensions of the model. This leads to spurious arrivals because the influence of the beam is not limited acceptably to the vicinity of its emerging point. The seismic section shown in Figure 8b was generated for a choice of beam parameter of 1 km. As one can see in Figure 9, the widths of the beams at the ray ends are much smaller than for a beam parameter of 0.1 km, hardly exceeding 1.8 km even for large offsets. Therefore, as expected, the corresponding seismic section looks acceptable. Do these results imply that a larger beam parameter (i.e., a broader beam width at the source) produces more stable results? Analysis of Figures 8c and 8d proves that this assumption is incorrect. The choice of a 2 km and, more obviously, the choice of 5 km beam parameter again generate spurious arrivals due to very large beam widths at the ray ends. These simple experiments indicate
that a proper choice of beam parameter is crucial in generating accurate seismograms. Specifically, if the beam parameter is too small (i.e., the beam is too narrow at the source), a strong spreading of the beam along the ray can be expected.

One might argue the following:

- The observed spurious arrivals yield only small amplitudes. However, we should remember that this value can be significant compared with amplitudes of rays from some different ray family contributing to the seismograms (e.g., rays with a different history of reflections and transmissions and different arclength).

- One can always find an appropriate beam parameter that guarantees small beam widths along a central ray. Again, this is the case for this simple example, but in general, each ray experiences a different medium along its trajectory and will require an individual specification of optimal beam parameters. These problems will make it difficult to obtain an acceptable weighting function $\Phi(\phi)$ in equation (15).

It is important to note that small beam parameters invariably produce strong beam spreading. Broad beams, however, violate intrinsic assumptions of the beam method. The next section is devoted to this issue.

**BEAMS ACROSS INTERFACES**

There exists a potential danger associated with the transmission of dynamic ray-tracing data. Consider the experiment in Figure 10, where rays from a source at 3.2 km horizontal position are traced through a salt-dome model and near-critical refractions are indicated with the white circles. The analysis of the beam width at the ray ends (i.e., at the positions where the central rays intersect the interface) is shown in Figure 11. Rays corresponding to the near-critical refractions emerge between about 0.2 and 1.2 km horizontal distance as indicated with the black horizontal bar.
As expected, the beam behaves normally for the reflections off the salt dome whereas the beam widths of near-critically-refracted rays are huge. While the reflected beams have beam width on the order of 1 to 2 km, the beam of the near-critical refracted rays spread up to 14 km, and this for a model of just 4 km lateral dimension.

Artifacts due to near-critical transmission can be reduced once we better understand the reason for this beam spreading. If a ray impinges on a boundary dividing two media with different velocities, one must transform the dynamic ray tracing quantity \( q \) across interfaces. To continue the wavefield properly, this is done by applying a phase matching method (Červený and Pšencík, 1984) using the curvature of the splined interface; specifically,

\[
q_{\text{trans}} = q_{\text{inc}} \frac{\cos \alpha_{\text{trans}}}{\cos \alpha_{\text{inc}}}
\]

where \( \alpha_{\text{inc}} \) and \( \alpha_{\text{trans}} \) are the incidence and transmission angles, respectively. Both angles are measured with respect to the interface-normal at the incidence point. For near-critical transmitted rays, \( \alpha_{\text{trans}} \) is close to \( \pi/2 \) and the transformed value of \( q \) (i.e., \( q_{\text{trans}} \)) will be very small. \( q_{\text{trans}} \), in essence, specifies the new initial beam width on the other side of the discontinuity. According to equation (14), a small initial value of \( q \) in effect creates a small initial beam width and, as in the half-space experiment sketched in Figure 7, this small initial beam width produces strong beam spreading!

To avoid spurious arrivals associated with near-critical transmitted rays, these rays have to be treated with caution. Specifically, the GBM program should have an option to reduce amplitudes artificially once the transmission angle approaches values close to \( \pi/2 \). Another situation yielding strong beam spreading is impact of a beam on a strongly curved interface.

**GAUSSIAN BEAM SEISMOGRAMS FOR COMPLEX STRUCTURES**

Conventional ray theory yields inaccurate amplitude and traveltime data in singular regions of the wavefield, such as caustics. The GBM method is well-known to
overcome these difficulties. In particular, the ray Jacobian, which in conventional
ray methods diminishes to zero and produces infinite amplitudes at cusps, is guar-
anteed to stay finite in the GBM. Moreover, using the GBM, the phase shift in the
cusped region can be automatically accounted for and the region outside the cusp
yields nonzero amplitudes. To illustrate these desirable features of the GBM, I com-
pute shot data for the syncline model shown in Figure 12. The syncline causes a ray
triplication and creates the familiar bow-tie reflection pattern. Rays on the second
branch of the syncline reflection have passed through the caustic and are 90-degrees
phase shifted. This is clearly visible on the second leg of the bow tie in Figure 13.

Another similar model (Figure 14) is used to generate the zero-offset sections in
Figure 15a and Figure 15b via the GBM. Figure 15a was computed by assigning a
beam parameter comparable to the dominant wavelength of the seismic signal whereas
in Figure 15b, an alternative choice of beam parameter was applied. The beam widths
are chosen for each beam individually as a function of the dynamic ray tracing data
$q_1, q_2, p_1, p_2$ evaluated at the ray ends of each central ray. This approach has been
reported in Červený (1985b) to produce stable synthetic data free of artifacts. The
main problem with this approach is the determination of an appropriate weighting
function $\Phi(\phi)$ in equation (15), if the evaluated source-beam widths vary significantly
from one beam to the next. Generated zero-offset data for the same model, but using
a Kirchhoff-summation program and a two-point ray tracing program are displayed
in Figure 16a and Figure 16b. Figure 16a is almost identical to the Gaussian beam
results; in particular, the behavior at the caustics of the wavefield is identical to
Figure 15a. On the other hand, the conventional ray tracer (16b) does not handle the
cautics accurately and the amplitudes drop to zero at the caustics. The models above
are exemplary for Gaussian-beam-data tests published in the literature. Tracing
beams in models of increased complexity is rarely discussed. Below, examples are
provided to illustrate and discuss artifacts specific to the Gaussian-beam method
when the model becomes more complex.
A structurally more complex model is shown in Figure 17. The velocity of the second layer is changing laterally while the first layer has a constant velocity. In the following, zero-offset sections for reflections from the lower syncline are investigated, the first interface is simply transmitting the rays. Figure 18a shows a zero-offset section computed after tracing 131 rays from each shotpoint and performing the Gaussian beam summation. What went wrong? Several artifacts are visible:

- Spurious arrivals close to the main reflection curve.
- Spurious arrivals at very small traveltime for several midpoint positions.
- Amplitudes of the second branch of the syncline reflection are very erratic.

After studying the basic properties of the GBM, it is straightforward to understand and diminish the observed artifacts. Most of the spurious arrivals are due to overly large beam widths. The spurious arrivals at small traveltime, for example, are due to refracted energy similar to the case of the refracted rays in the salt dome model (Figure 10). To better understand why refracted energy can contribute to a zero offset trace, recall that unlike a standard ray-tracer, which includes only rays connecting source and receivers in the seismogram generation, the Gaussian beam program adds the contributions of all beams in the vicinity of the receiver. As a result, it is not guaranteed that all these beams are actually reflected from the target, which in this case is the second interface. Attaching additional information describing their individual raypaths to the central rays allows only the beams reflected from the defined target interface to be included in the summation. To further avoid artifacts due to large beams, the program should check if the validity conditions of kinematic and dynamic ray tracing are satisfied. Information from central rays located more than 3 wavelengths from a receiver should not be included into the beam summation; moreover, it is very useful to check if the second term in equation (7) is significantly smaller than the leading order term. Another indicator of a Gaussian-beam-specific
problem is seen in the second branch of the syncline reflection in Figure 18a. Amplitudes of these arrivals are erratic, indicating that the ray coverage is not dense enough to produce accurate data. To obtaining smooth results, each receiver should receive contributions from several beams. As suggested by Beydoun and Keho (1987), the Gaussian beam method generally requires at least 10 beams in the vicinity of the receiver in order to reconstruct the high-frequency part of the wavefield.

Figure 18b shows the result after tracing 260 rays for each zero-offset point, twice as many as used to generate Figure 18a. Additionally, only energy satisfying the validity conditions mentioned above is included in the beam summation. Indeed, the second branch of the triplication is much more continuous with the larger number of rays and virtually all spurious energy disappeared. More theoretical insight into the discretization error of the GBM and its validity conditions is provided in Klimeš (1986) and Ben-Menahem and Beydoun (1985).

**ACCURACY OF THE GAUSSIAN BEAM METHOD**

As discussed above, relatively simple steps can be taken to avoid disturbing artifacts of the GBM once the basic features of beam tracing are understood. Unfortunately, clean-looking seismic sections do not guarantee that the data are necessarily accurate. Data sets created by any method based on approximations or assumptions should always be checked for their accuracy.

This is especially true for rays traced through media with the complexity of the overthrust model shown in Figure 19 (top). This model is composed of geologic blocks with different seismic velocities. Within each layer, the velocity increases with depth. Note that interfaces with strong curvature and sharp edges are present in the model. Complexities in ray tracing are seen in the display of rays and wavefronts for a shot at 1.15 km horizontal position (see Figure 19, bottom). White lines show rays that are reflected from the bottom of the model and emerge at the surface. Energy
that propagates out of the model is simply represented by black wavefronts, with the corresponding rays deleted. Of the 600 rays traced through this model, less than one-sixth end up at the surface; the others either stop at the boundary of the model or they are overcritically incident at velocity discontinuities. In the large shadow zones, such as the one between 1.2 km and 1.7 km horizontal position, data generated by a classical ray-theoretical approach would not show any seismic energy.

To test synthetic data computed for this model, I generate synthetic midpoint gathers for a source-receiver midpoint at 1.6 km. Negative and positive offsets represent the interchange of source and receiver position. The trace labeled by "−1", for example, represents the data recorded at a receiver at 1.1 km horizontal position for a shotpoint at 2.1 km horizontal position. On the other hand, "+1" represents synthetic data recorded at 2.1 km for a shotpoint at 1.1 km. While not being a strict proof for accuracy, the symmetry of the midpoint gather for models of constant density and identical isotropic radiation pattern at all source positions indicates that the data are likely to be accurate, at least within the framework of the GBM. A first midpoint gather, similar to the one shown in Hale (1991) (but including the partitioning of energy at interfaces), is shown in Figure 20a. In this example, the beam width at the source was identical for all rays and the extrapolation range away from the central rays was not restricted. The asymmetry in the figure indicate, that the reciprocity in the created dataset is violated, so that the data are not accurate.

In a second test (Figure 20b), I chose the beam parameter for each beam individually, as a function of the dynamic ray tracing data $q_1, q_2, p_1, p_2$. Moreover, only beams with their central rays emerging within three wavelengths of the receiver contribute to the generated seismic trace. Although the result is much improved over first one, the asymmetry in the figure indicates that again, the data is not sufficiently accurate.

Increasing the complexity of the velocity model will eventually violate the basic assumptions of the Gaussian-beam method. This is easy to understand if one considers a central ray that passes close to a velocity discontinuity or a pinch-out. Any
traveltime or amplitude expansion away from the central ray will inevitably produce inaccurate results. This shortcoming is serious, considering the fact that the Gaussian beams tend to spread significantly in complex media. For example, analysis of the dominant-frequency beam width at the ray ends of central rays traced through the overthrust model revealed, that several beam widths exceed the horizontal dimensions of the model by about one order in magnitude!

DISCUSSION AND CONCLUSION

The combination of triangulated media and dynamic ray tracing is especially promising if analytic velocity profiles are assumed in each triangle. For example, while the overall structure can be rather complex, within each triangular cell with constant gradient of sloth, closed-form solutions of kinematic and dynamic ray tracing are obtained. Moreover, additional information can be added to the model representation to define that rays either transmit, reflect or stop at specified triangle edges.

Unfortunately, aside from its well known advantages over conventional ray theory, the Gaussian Beam approach shows artifacts for modeling in complex media. Theoretically, the summation over all beams in the vicinity of each receiver removes the dependence on the specified beam parameter $\epsilon$. This has been found to be true within a reasonable range of the critical parameter $\epsilon$ and for simple models. In practice, however, I find that for more complex models, the choice of $\epsilon$ is a difficult decision. In complex media, each beam eventually encounters a different velocity field and thus requires an individual optimal beam parameter.

An inappropriate choice of $\epsilon$ is one reason for increased beam spreading. Other situations, such as near-critically transmission cause significant beam spreading, as well, but can be detected in the ray tracing. To comply with intrinsic assumptions of dynamic ray theory, in cases where beam spreading is unacceptable, the complex traveltime and amplitude information cannot be extrapolated too far away from the
central ray. More specifically, the range of extrapolation should not exceed the natural length scale of the medium. Receivers should add up contributions from only nearby emerging beams.

Triangulated models allow the treatment of highly complex structure. It is important to always recall that increasing the complexity of the model will eventually lead to violation of assumptions intrinsic to high-frequency modeling and the GBM. In this case, the wavefield cannot be accurately represented by a summation of the beams and alternative methods, such as finite differencing, need to be applied.

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FIG. 1. Triangulated model of a salt-dome structure. Layer boundaries are shown as black lines. The white lines are edges of auxiliary triangles used in the model building. The seismic propagation speed is indicated by the shading of the triangles.
FIG. 2. The ray-centered coordinate system in 3-D uses as basis vectors the unit tangent $\hat{t}$ to the ray at given $s$ and the unit vectors $\hat{e}_1$ and $\hat{e}_2$ perpendicular to the ray.
FIG. 3. Sketch of different coordinate systems. $q_1$ and $q_2$ correspond to the ray-centered coordinate system, $\gamma_1$ and $\gamma_2$ are the ray coordinates. $x_1$, $x_2$ and $x_3$ represent the cartesian coordinate frame.
FIG. 4. Amplitude profile of a Gaussian beam
FIG. 5. Sketch illustrating the meaning of *beam parameter* and *beam width at ray end*. Note that the rays are the support of the beams and that the amplitude of a beam decreases exponentially away from the ray. The response at the receivers is obtained as a weighted sum over the beams.
FIG. 6. Beam width $L$ as a function of travel distance. In a, $\epsilon_1$ is zero, and $\epsilon_2$ varies.

In b, $\epsilon_2$ is held constant and $\epsilon_1$ varies. The shading of the graph [light to dark] indicates a uniform increase in magnitude of $\epsilon_2$ and $\epsilon_1$. 
FIG. 7. Geometry of an experiment to show the dependence of the GBM results on the choice of beam parameter. Seismic sections for various beam parameters are displayed in Figure 8.
FIG. 8. Synthetic shot records computed for the experiment sketched in Figure 7. The choice of beam parameter is the following: (a) 0.1 km (b) 1.0 km (c) 2.0 km (d) 5.0 km. The arrows point to spurious arrivals. Close examination of the encircled areas shows artifacts due to the fact that beams have become too broad.
FIG. 9. A detailed analysis of the results obtained in the experiment sketched in Figure 7 aids in understanding of some difficulties of Gaussian beam modeling. The diagram displays the dominant-frequency beam width at the receiver as a function of source-to-receiver offset. Each symbol denotes a different choice of the beam parameter at the source.
FIG. 10. Rays traced in a salt-dome model. The source is situated at 3.2 km horizontal position. Near-critical rays emerge in the region of 0 to 1.5 km and yield extremely broad beam widths (see Figure 11). These beams would produce spurious arrivals computed seismic section.
FIG. 11. Analysis of the dominant-frequency beam width at the ray ends in Figure 10.

Data acquired in the region to the right of horizontal position of 1.5 km will yield accurate results.
FIG. 12. Common-shot experiment above a syncline.
FIG. 13. Common-shot data for a shot position at 1.5-km horizontal position acquired above the syncline.
FIG. 14. Model used to compare results from Kirchhoff summation, classical two-point ray tracing, and the Gaussian beam method.
FIG. 15. Gaussian beam synthetic zero-offset seismograms for the model shown in Figure 14. In a, a 0.2-km initial beam width is used, in b the beam width is chosen individually for each beam.
FIG. 16. Synthetic zero-offset seismograms computed for the model shown in Figure 14.

a Kirchhoff summation, b conventional two-point ray tracing.
FIG. 16. Synthetic zero-offset seismograms computed for the model shown in Figure 14.

a Kirchhoff summation, b conventional two-point ray tracing.
FIG. 17. Two-syncline model. The velocity of the second layer is changing laterally while the first layer has a constant velocity.
FIG. 18. Zero-offset data for the model shown in Figure 17. (a) 131 rays are traced at each shotpoint. Reflections and diving rays contribute to the section. (b) 260 rays traced from each shotpoint. This time, only reflecting rays contribute to the time section and validity conditions of dynamic ray theory are checked.
FIG. 19. Rays and wavefronts for a shotpoint situated at 1.15 km horizontal position in the overthrust model above. White lines show rays that are reflected from the bottom of the model and emerge at the surface. Energy that propagates out of the model is represented by black wavefronts only, with the corresponding rays deleted.
FIG. 20. Common midpoint gathers for the model shown in Figure 19. The midpoint is situated at 1.6 km horizontal position. (a) Same beam-width chosen for every ray and no restriction of the extrapolation range. (b) Only beams with their central ray emerging within three wavelengths of the receiver contribute to the generated seismic data. The beam widths are chosen for each beam individually, as a function of the dynamic ray tracing data $q_1, q_2, p_1, p_2$. 