On the Berkhout approach to modeling and inversion of seismic inversion

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ABSTRACT

A major feature of the Berkhout approach to migration/inversion is early discretization of the forward modeling problem for seismic data surveys. Thus, Green’s functions are replaced by propagator matrices and ordinary reflection coefficients are replaced by a poorly understood reflectivity matrix. One of the advantages of this method is that many special features of the underlying model can be characterized by further matrix multiplications. Migration/inversion becomes a matter of determining the reflectivity matrix. In this approach, this determination is reduced to a cascade of matrix inversions. On the other hand, the reflectivity matrix remains elusive to the exploration geophysics community. In my own research, I am developing a continuum analog for the Berkhout formalism. This allows an interpretation of the reflectivity matrix in terms of the more well understood geometrical optics reflection coefficient. It also reveals the inaccessibility of an exact inverse of the propagator matrices, hence the need for using the adjoint as an approximate “well-conditioned” inverse. The analysis also reveals that the “WRW” characterization of the underlying model is conceptual, rather than exact, unless one makes certain compromises on the meaning of W. For anything but the horizontal reflector, W is a dyadic operator. While imaging has been confirmed for the planar reflector by this approach, it is at best only indicated by the current state of analysis for the curved reflector case. Further, extraction of the reflection coefficient from the continuum analog of the reflectivity matrix remains undone for the curved reflector case. Berkhout inversion is a fixed frequency process, with stacking over frequency effecting image enhancement. This leads to the possibility of accounting for dispersion by using different background velocities for different frequencies. This aspect of Berkhout inversion has yet to be exploited.

INTRODUCTION

This is a report on in-progress research. I presented some preliminary results for the previous project review, put the problem aside for a while and then picked it up
again. This research actually began with a discussion with Samuel Gray at the time of Svenfest, but is also based on an ongoing desire by each of us to gain insight into the Berkhout approach to migration/inversion. (Berkhout and van Wulffen Palthe, 1979; Berkhout, 1984, 1985, 1992; Berkhout and Wapenaar, 1989; Fokkema et al., 1993; Wapenaar and Berkhout, 1985a, 1985b 1989, 1993; Wapenaar and James, 1990; Wapenaar, 1993a, 1993b, 1994)

A fundamental problem with gaining an understanding of this method is that the Berkhout school discretizes the forward scattering problem. In this approach, the Green’s function is replaced by a propagator matrix and the geometrical optics reflection coefficient is replaced by a reflectivity matrix in the spatial-frequency domain. The spatial transform of this result provides information about the angularly dependent reflection coefficient at all incidence angles consistent with the aperture of the data. Another subtlety of the method is the introduction of the dipole response in the forward model, rather than the more traditional impulse response; hence, the use of $W$ for a “propagator” rather than $G$. Combining these changes into a succinct descriptor for the Berkhout approach, I use the terminology, “$WRW$ model” for a shorthand descriptor of this form of the forward problem. Here the right-most $W$ is a matrix propagator representing downward propagation, $R$ is a matrix characterizing scattering, so that $RW$ is the initial state of upward scattering at depth. Finally, the left-most $W$ is the upward propagator, making $WRW$ the upward scattered field. Thus, determination of the reflectivity matrix, $R$, amounts to finding (approximate) inverses to the $W$-matrices.

The matrix approach allows for relatively easy incorporation of features of the seismic experiment that are more cumbersome to model in a continuum (Kirchhoff, exploding reflector) integral approach. Of course, in the final analysis all approaches must have an equivalence, but that does not mean that it has to be equally difficult or easy to achieve the same results from different approaches.

On the other hand, there are features of the continuum theory that are not easily accessible to the discrete method. Indeed, any analysis based on applying the method of stationary phase to integral operators are not available in the discrete case. (Although a recent paper by Keller and Knessl, 1993.) suggests an approach to asymptotic evaluation of oscillatory sums.)

The discrete approach has no analog of the 2.5D theory, which allows for processing of a line of data while accounting for out-of-plane spreading. The reason is that 2.5D processing results from applying the method of stationary phase to the integral operator representing 3D migration/inversion under the assumption that there is no out-of-plane variation in the medium (and, hence, in the data).

In the discrete approach, one would have to estimate the out-of-plane operator sum under the same conditions—possible with the Keller/Knessl theory, but as yet undone.

The Berkhout formalism assumes infinite aperture in sources and receivers, as do all other methods. However, again, the availability of the method of stationary
phase provides a mechanism for analyzing finite aperture effects with integral operators, not possible for discrete inversion. For example, one can show in Kirchhoff migration/inversion that diffraction smears are characterized by the passage of a stationary point through the endpoint of integration; the visible smear occurs when the stationary point of the integral over sources and receivers is outside of the interval of integration and the dominant term of the asymptotic expansion of the integral arises from the endpoint of integration.

Using the stationary phase principle again, one can estimate all sorts of geometrical attributes—parameters associated with the specularity of the rays that actually dominate the data that images a reflector. Among these attributes are the incidence angle, source or receiver coordinate for the specular rays, and travel time along the specular ray path, among others. See for example, Geoltrain (1991).

There is another subtle difference between Berkhout inversion and other approaches presently in use. I will explain with the aid of three figures. In Figure 1, we depict a data cube with coordinate axes being midpoint, offset and time/frequency. In Figure 2, we show a typical slice used for a common offset migration/inversion.

![Data Cube Diagram](image)

**Fig. 1.** The elementary data cube.

That is, we take a fixed offset and process over midpoints and time or frequency to obtain an image and parameter estimates. In this approach, we would use data in the orthogonal (horizontal) direction for velocity analysis and, later, to stack for image enhancement. Common shot migration/inversion could be represented by a different vertical slice through the data, again with velocity analysis and stacking being carried out with the use of parallel vertical slices through the data.
Fig. 2. Data slice for a common offset migration/inversion.

Fig. 3. Data slice for Berkhout migration/inversion.
In contrast, in Figure 3 we show the type of data slice used in Berkhout migration/inversion. Note that all sources and receivers are used in the inversion process, leaving the orthogonal direction—time or frequency—for velocity analysis and image enhancement.

I think of alternative approaches as constituting a tradeoff of time for depth to obtain a preliminary reflector map and parameter estimates; the Berkhout approach would seem to be a tradeoff of offset for depth. It seems to me that it would take a "lot" of offset to provide significant depth information. Where that is not available, this method would seem to be much more dependent on the redundancy of data to overcome the artifacts of limited aperture.

On the other hand, an inversion at fixed frequency has an important feature that alternative methods lack, namely, the ability to use different velocities for different frequencies. That is, it would seem that this method can accommodate dispersion more easily than other methods. I do not believe that this aspect of the Berkhout approach has been exploited to date. It also suggests the possibility of developing a velocity analysis scheme in which one examines residual moveout as a function of frequency, thereby developing a frequency dependent background velocity—essentially, deriving a dispersion relationship from residual moveout in frequency.

Analytically, I determine the properties of an inversion operator by applying the operator to Kirchhoff data for a curved reflector. It is this approach that led to an interpretation of our original inversion in terms of the geometrical optics reflection coefficient. This followed from the analysis of Kirchhoff data, despite the fact that the basic inversion formalism was based on the Born approximation—seemingly precluding large changes in medium parameters across an interface and precluding wide offset, near critical reflection and beyond. See, for example, Bleistein (1987).

It is this same type of analysis that I am trying to carry out for the Berkhout formalism.

To date, I have been able to derive the results described below.

1. I have produced a straightforward continuum analog of the \(WRW\) model for the case of a horizontal reflector in a constant background medium. As part of this result, I find a continuum analog for the reflectivity matrix for this model.

2. This derivation provides a motivation for the idea that inversion amounts to finding inverse operators for the two \(W\)'s of the theory. However, I argue that an exact inverse cannot exist because the forward operator has evanescent modes whose rate of exponential decay approaches infinity with \(k_1\), the transverse wave number. Of course, for finite values of \(k_1 > \omega/c\), an approximate inverse leads to an ill-conditioned operator, still not satisfactory. (Here, \(\omega\) is frequency and \(c\) is propagation speed.)

I then show that the adjoint operator, \(W^*\), is a reasonable inverse, in the sense that it inverts the propagating modes and attenuates evanescent modes. (Note
this means in an amplitude-consistent sense in addition to a kinematic sense.)

3. I confirm that adjoint processing provides a means of imaging and a means of estimating the angularly dependent geometrical optics reflection coefficient. This result is achieved by applying an operator based on $W^*$ to an exact representation of the upward scattered wave and calculating the resulting integrals exactly.

4. The simple $WRW$ model does not work for the dipping planar reflector or the curved reflector in a constant background medium. However, I am able to carry through the asymptotic analysis of applying the same adjoint operators to a model of data for the dipping plane. Here, it is necessary to introduce the first approximations in the analysis. The dipping reflector is assumed to emerge at the upper surface “far” from the processing region and I neglect these endpoint effects. I consider this to be a not-to-serious problem in the sense that we are not really interested, in practice, in reflectors that emerge at the upper surface.

I show again how one obtains an image of the reflector and an estimate of the angularly dependent reflection coefficient. However, the latter now requires an accurate estimate of the angle of dip of the reflector. Of course, having an image of the reflector allows one to estimate the dip angle—easy for a planar reflector, harder for a curved reflector. Here, I expect that the stationary phase principal applied to integral versions of the Berkhout inversion will provide a means of obtaining a numerical estimate of the dip of the reflector. This extension has not yet been carried out.

5. Using a recent thesis by a von Vrohnoven, a student of Fokkema’s, (Fokkema et al, 1993) as a point of departure, I am able to obtain the form $WRW$ for the upward propagating field from a single reflector. The difference between my work and van Vrohnoven’s is that her $W$-operators require knowledge of the normal direction all along the reflector. In that case, the $W$’s are no longer propagators; propagators should only depend on the source mechanism and the background medium. In order to overcome this shortcoming, I find it necessary to introduce $1 \times 4$ dyadic propagator operators operating on a $4 \times 4$ dyadic of reflectivity matrices. Furthermore, the von Vrohnoven result applies to the impulse response and not to the dipole response. I believe that to obtain a result for the dipole response will require one to fall back to the asymptotic approximation of the upward scattered field and not try to work directly on an exact representation.

6. For the curved reflector, applying the same adjoint operators produces a more complicated result from which imaging is apparent, but estimation of the reflection coefficient is not yet completed.

This is the present status of this research project. In the following sections, the results are presented in the same order as in the enumerated list.
THE HORIZONTAL REFLECTOR

In this section, I develop some elementary ideas in the context of the simplest reflection problem, namely a point source in two dimensions over a single horizontal reflector. We introduce the exact representation of the upward scattered solution and confirm an assertion in de Bruin [1992] that for the source and receiver on the reflector, this response is just the Fourier transform of the angularly dependent reflection coefficient. We then derive a continuum analog of the solution representation form, \( W_g R W_s \), and use this as motivation for derivation of an approximate inversion operator, \( W^* \), for each of the forward modeling operators, \( W_g \) and \( W_s \). Finally, I check the application of this approximate inverse operator on our exact solution representation and show that it really provides information about the angularly dependent reflection coefficient and also provides a means of imaging the reflector.

The Forward Model

Let us consider the simple example of a point source over a horizontal interface across which only the propagation speed changes. See Figure 4. We are interested in

\[
\begin{align*}
\mathbf{c}_1 & \quad \mathbf{x}_s, \mathbf{z}_s \\
\mathbf{c}_2 & \quad \mathbf{x}_g, \mathbf{z}_g \\
& \quad \mathbf{x} \\
& \quad \mathbf{H} \\
& \quad \mathbf{z}
\end{align*}
\]

FIG. 4. Source and receiver over a horizontal reflector.

the upward scattered wave. The solution to this problem in two dimensions can be found in many books including Bleistein [1984] eq. (8.1.1);

\[
\begin{align*}
u(x_g, z_g, x_s, z_s, \omega) &= -\frac{1}{4\pi i} \int_C R(k_1, \omega) \exp\{i\Phi\} \frac{dk_1}{k_3}, \\
z_s > H, \quad z_g > H.
\end{align*}
\]  

(1)

Here,

\[
R(k_1, \omega) = \frac{k_3(k_1, \omega) - k_4(k_1, \omega)}{k_3(k_1, \omega) + k_4(k_1, \omega)},
\]  

(2)

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and

\[ \Phi = k_1[x_s - x_g] + k_3[2H - (z_g + z_s)] \]

\[ = k_1s + k_3n. \tag{3} \]

In these equations, as seen in Figure 4, \((x_s, z_s)\) are the source coordinates, \((x_g, z_g)\) are the receiver coordinates, \(H\) is the normal distance from the source point to the reflector, and \(c_1\) and \(c_2\) are the propagation speeds above and below the reflector, respectively.

The variables, \(s\) and \(n\) are shown in Figure 5. They are the projected distances—parallel and normal to the reflector—from the image of the source point. Equivalently, they are the parallel and normal distances from the source point to the receiver point along the specular ray path. As defined, \(n\) is a signed quantity that is positive for the source and the receiver above the reflector. In modeling, this will always be the case; in downward continuation of the sources and receivers, this need not be the case, since the downward continuation is a mathematical process that is not limited by the physical constraints of the problem.

The functions, \(k_3\) and \(k_4\) are given by

\[ k_3 = \sqrt{\omega^2/c_1^2 - k_1^2}, \quad k_4 = \sqrt{\omega^2/c_2^2 - k_1^2}, \tag{4} \]

with both square roots defined to be positive at \(k_1 = 0\). Also, the contour \(C\) on the right side of (1) extends from \(-\infty\) to \(\infty\), passing over the branch points of \(k_3\) and \(k_4\) in the left half \(k_1\)-plane and under the branch points in the right half \(k_1\)-plane. This is a classic technique in the method of complex variables for defining the square roots so that they are real and positive for \(\omega^2/c_1^2 > k_1^2\), and \(\omega^2/c_2^2 > k_1^2\), respectively, and they have a positive imaginary part for \(\omega^2/c_1^2 < k_1^2\), and \(\omega^2/c_2^2 < k_1^2\), respectively.
This solution is the response to an impulsive point source at \((x_s, z_s)\). In the absence of the reflector, the solution is the free space Green’s function, which, in the wave number domain, is given by

\[
G(k_1, z, x_s, z_s, \omega) = -\frac{1}{2ik_3} \exp\{-ik_1 x_s + i k_3 |z - z_s|\}.
\]  

We are, instead, interested in the response to a dipole source. In particular, to make the constants come out nicely, introduce the solution \(W\) to the equation,

\[
\nabla^2 W + \frac{\omega^2}{c_1^2} W = 2\delta'(z - z_s)\delta(x - x_s).
\]  

In the \(k_1\)-plane, the solution to this equation is

\[
W(k_1, z, x_s, z_s, \omega) = \text{sign}(z - z_s) \exp\{-ik_1 x_s + i k_3 |z - z_s|\},
\]  

obtained from the previous result by taking the derivative with respect to \(z\) and multiplying by \(-2\). The representation of the dipole response in the space-frequency domain is obtained by making a corresponding adjustment in (1), namely, multiplication by \(-2ik_3\). This function will still be called \(u\), because there will be no further need of the former result in the discussion to follow. Thus,

\[
u(x_g, z_g, x_s, z_s, \omega) = \frac{1}{2\pi} \int_C R(k_1, \omega) dk_1 \exp\{i\Phi\}
\]  

In this application, the dipole response is evaluated at \(z > z_s\) so that the sign function in \(W\) is positive and can be omitted.

**Reflectivity**

Our point of departure for comparison with the Berkhout approach is a discussion in Cees de Bruin’s thesis [102]. He points out that the dipole response with source and receiver on the reflecting surface is just the Fourier transform \(^1\) of the angularly dependent reflection coefficient. Here, we will verify that observation for the exact field representation, (8). All we need to do is to take the limit in that equation as \(z_s\) and \(z_g\) both approach \(H\). See Figure 6. It is necessary to think of this as a limiting process from above rather than a direct evaluation because there is a different limit of the “scattered field” from below, namely, the transmitted wave. It is the limit from above that we define as the field on the reflector for this discussion. The result is

\[
u(x_g, H, x_s, H, \omega) = \frac{1}{2\pi} \int_C R(k_1, \omega) dk_1 \exp\{ik_1 s\}. \tag{9}
\]

\(^1\) The Fourier transform here is multiplication by \(\exp\{-i\kappa_1(z_g - z_s)\}\) and integration over \(x_g\), or, equivalently, multiplication by \(\exp\{-i\kappa_1 s\}\) and integration over \(s\).
Indeed, this result is just the Fourier transform of the reflection coefficient (2) expressed as a function of \( k_1 \) and \( \omega \). For \( k_1^2 \leq \omega^2/c^2 \), we can introduce the incidence angle \( \gamma \) with respect to the normal by

\[
k_1 = \frac{\omega}{c_1} \sin \gamma
\]  

(10)

and rewrite the reflection coefficient in the form

\[
R((\omega/c_1) \sin \gamma, \omega) = \frac{\cos \gamma/c_1 - 1/c_2^2 - \sin^2 \gamma/c_1^2}{\cos \gamma/c_1 + 1/c_2^2 - \sin^2 \gamma/c_1^2}.
\]

(11)

or, in terms of the ray parameter \( q \) with \( k_1 = (\omega/c_1)q \),

\[
R(\omega q/c_1, \omega) = \frac{\sqrt{1-q^2/c_1} - 1/c_2^2 - q^2/c_1^2}{\sqrt{1-q^2/c_1} + 1/c_2^2 - q^2/c_1^2}.
\]

(12)

These latter two forms prove useful for integration (summation) over all \( \omega \) for fixed incidence direction.

Returning to the form (9), we define the function,

\[
R(x_g, x_s, \omega) = u(x_g, x_s, \omega)
\]

(13)

\[
= \frac{1}{2\pi} \int_C dk_1 \, R(k_1, \omega) \exp \{ik_1 s\}.
\]

In this particular case, \( R(x_g, x_s, \omega) \) is actually a function of the difference, \( x_g - x_s \), and is just the dipole response, as predicted. However, in more general examples, it should be anticipated that the result will be somewhat more complicated than this, but still related to the reflection coefficient. If the coordinates \( x_g \) and \( x_s \) are discretized through \( x_g = j \Delta \xi \), and \( x_s = i \Delta \xi \), then \( R \) becomes a matrix at the depth, \( H \),

\[
R_{ij} = u(i \Delta \xi, j \Delta \xi, \omega).
\]

(14)
This is very close to Berkhout's *reflectivity matrix*, except that this matrix is defined by an integral in (13), whereas in Berkhout's derivation, it would be the corresponding finite discrete sum for discrete values of $\omega$. With this caveat in mind, below, the function, $R(x_g, x_s, \omega)$, in which both depth variables are evaluated at the same depth, will be called the reflectivity matrix, even in its continuous form. The term, *reflectivity function*, will be reserved for the case in which the two depth-arguments are independent. This will be discussed below.

![Graph showing magnitude of $|R(q\omega/c_1, \omega)|$ as a function of $q$.](image)

**Fig. 7.** Reflection coefficient as a function of $q$, equation (12).

Figure 7 is a *Mathematica* rendition of the absolute value of the reflection coefficient as a function of $q$ in the range $(-2, 2)$, which includes the range of real incident angles and a portion of the range of evanescent modes. Note that this is a complex valued even function of $q$ and that its Fourier transform, the reflectivity matrix, even at spatial position $x_g - x_s = 0$, is complex valued. For the purpose of imaging, then, it makes sense to examine the absolute value of the Fourier transform of $R$. Figure 8 is the absolute value of a 128 point Fourier transform of $R$ in the previous figure, as a function of $x_g - x_s$ with the origin at the center position. The 128X128 discretized version (13) has this plot as its 65th row. This is a Zoepritz matrix, with each row just a shift of this one. In this case, a single row carries all of the information about $R$. Equivalently, in a laterally homogeneous medium, a single shot and double spread of receivers contains all information of all translates of this configuration, equivalent to the translates of the central row of the reflectivity matrix. These figures should be compared with Figure 3.4 in de Bruin [102].

These observations suggest an approach to inversion, namely, that one create a dipole response at depth from the ensemble of observations of dipole response experiments. This is a form of downward continuation of the data. Before we proceed to do this, we will examine the structure of the solution (8) with an eye towards the matrix form of the Berkhout approach and with an eye towards writing an operator.
Fig. 8. Fourier transform of the reflection coefficient with zero position as the center point of the plot. In a 128x128 matrix representation of the reflectivity matrix for this example, this is the 65th row (or column).

form for the observed field that lends itself to the process of creating an operator inversion that achieves the intended goal.

The Form WRW

It will be shown here that the integral representation (8) has the form of the inverse Fourier transform of three functions, namely, two dipole Green’s functions and the reflectivity in the $k_1$-domain. To be more specific, let us define the spatial-frequency domain dipole response in the absence of the reflector as the (distributional) inverse Fourier transform of the result (7), namely,

$$W(x, z, \xi, \zeta, \omega) = \frac{\text{sign}(z - \zeta)}{2\pi} \int_C \exp \{ik_1[x - \xi] + ik_3[z - \zeta]\} \, dk_1. \quad (15)$$

Furthermore, let us define a reflectivity function $R$ of four spatial variables and frequency,

$$R(\xi_g, \zeta_g, \xi_s, \zeta_s, \omega) = R(\xi_g, \xi_s, \omega) \delta(\zeta_s - H) \delta(\zeta_g - H). \quad (16)$$

In this particular example, the reflectivity function of four spatial variables is defined in such a manner as to restrict it to the only depth where reflection data is nonzero, namely, on the reflector at depth $H$. The reflectivity matrix—the Fourier transform of the reflection coefficient—becomes the weighting function of the distributions that restrict this particular reflector to the depth $H$. This is the continuum mechanism for creating the reflection data with nonzero values only on the given reflector.

Now, we consider the integral,
\[ I = \int W(x_g, z_g, \xi_g, \zeta_g, \omega) R(\xi_g, \zeta_g, \xi_s, \zeta_s, \omega) \cdot W(\xi_s, \zeta_s, x_s, z_s, \omega) d\xi_s d\xi_g d\zeta_s d\zeta_g, \]

\[ = \int W(x_g, z_g, \xi_g, H, \omega) R(\xi_g, H, \xi_s, H, \omega) W(\xi_s, H, x_s, z_s, \omega) d\xi_s d\xi_g. \]

(17)

By using the Fourier representations for \( W \) (15) and for \( R \) (13), \( I \) can be recast as

\[ I = \frac{1}{(2\pi)^3} \int d\xi_s d\xi_g dk'_1 dk''_1 dk_1 \cdot \exp \{ ik'_1[x_g - \xi_g] + i k_3(k'_1)[H - z_g]\} \]

\[ \cdot R(k_1, \omega) \exp \{ ik_1[\xi_g - \xi_s]\} \]

\[ \cdot \exp \{ ik''_1[\xi_s - x_s] + i k_3(k''_1)[H - z_s]\}. \]

(18)

The integrals over \( \xi_s \) and \( \xi_g \) yield delta functions in the wavenumber variables, which allow us to carry out those integrals, as well, leaving only the integral in \( k_1 \). In particular, we use the result

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi_g \exp \{ i[k_1 - k'_1] \xi_g \} = \delta(k_1 - k'_1), \]

with a similar result in \( \xi_s \) and \( k''_1 \). Consequently, the function \( I \) in (17) becomes just the wavefield \( u \) as given by (8):

\[ I = u(x_g, z_g, x_s, z_s, \omega). \]

(19)

If we think of the operations of multiplication by the functions \( W(x_g, z_g, \xi_g, H, \omega) \) and \( W(\xi_s, H, x_s, z_s, \omega) \) and integration over \( \xi_g \) and \( \xi_s \) as operators, \( W_g \) and \( W_s \), then (17) and (19) can be rewritten as the operator equation

\[ u(x_g, z_g, x_s, z_s, \omega) = W_g RW_s. \]

(20)

Of course, in operator form, this result can be thought of as the integrations that were used here, or, in discrete form, as matrix multiplications, which is the Berkhout, et al representation.

This representation describes the response only from the reflector at depth \( H \). In the discrete approach, one must then sum over all depths. Berkhout, et al would
tie the depth variable $\zeta_s$ to the depth variable $\zeta_g$ and then carry out one summation. Here, I prefer to leave those two variables separate as in (16) and think of a double sum or double integral representing separate propagation in sources and receivers. Using the notation $W_s$ and $W_g$ to include the integrations over depth in (17), I introduce the integral operator notation,

$$u(x_g, z_g, x_s, z_s, \omega) = W_g R W_s,$$

(21)
as an operator form of this result in terms of all of the integrations that appear in (17).

...of Inverses and Adjointst

For the purpose of migration or inversion, we seek a method of propagating the data from the source/receiver surface to the reflecting surface, in such a manner that we obtain (or reasonably approximate) the reflectivity matrix on that surface. Thus, it is our objective to transform the data as described by (8) to data at points $\xi_s, \zeta_s, (\xi_g, \zeta_g)$, with $\zeta_s > z_s$ and $\zeta_g > z_g$.

In terms of our operators, $W_s$ and $W_g$, (or $W_g$ and $W_s$), we seek inverse operators, say, $[W_s]^{-1}$ and $[W_g]^{-1}$. That is, we seek something that will operate on these operators and produce a delta function at depth.

First, let me describe something that does not achieve this end. One might think to multiply the observed data by functions $W_s^-$ and $W_g^-$—given by (15) with the exponent replaced by its negative—and then integrate over $x_s$ and $x_g$, essentially carrying out a convolution over sources and receivers. Unfortunately, the integral representations for these two new functions do not converge because the integrand in each is exponentially growing for imaginary $k_3$. Disregarding this for the moment, the formal integrations over all sources and receivers produce delta functions in the $k_1$—variables, just as in the analysis of the integral (18), which, in turn allow for “easy” calculation of the $k$—domain integrals to obtain the spatial domain reflectivity, $R(\xi_g, \zeta_g, \xi_s, \zeta_s, \omega)$. Indeed, these functions $W_s^-$ and $W_g^-$ are an attempt to downward propagate the propagating modes and to simultaneously compensate for the attenuation of the evanescent modes. However, they are not valid.

What is happening here is that the attenuation rate approaches infinity as $|k_1|$ does. Thus, an inversion operator that compensates for attenuation would have to have an exponential growth rate that becomes infinite with $k_1$. One cannot create such an inverse operator.

What does seem to work, is to base the operator inversion on the incoming Green’s function. Indeed, if one tried to represent $u$ at some point between the reflecting surface and the source receiver surface in terms of Green’s functions, one would find that the Green’s function of choice is the incoming delta function response. The reason is that we have to form the integral of $u^* G^* - G^* L u$ over a domain bounded by the observation surface and some lower surface below the output point where
$u$ is to be evaluated and above the reflector. Here $\mathcal{L}$ is the wave operator. Using Green's theorem transforms this integral into a surface integral over the bounding surfaces with integrand, $u \partial G^*/\partial n - G^* \partial u/\partial n$. On the upper surface, the value of $u$ is the observed field and its normal derivative is easily determined in the $k_1$-domain under the assumption that it is an upward propagating wave. On the lower surface, we do not know the fields. Thus, we need to choose $G^*$ so that this integral is zero. Since $u$ is upward propagating on this lower surface, this can be guaranteed by choosing a Green's function with that property. The Green's function of choice is the inward propagating Green's function. Here, "inward" means towards the point at depth where $u$ is to be evaluated. On the upper surface, that Green's function is also inward propagating or now downward propagating, while $u$, itself, is still upward propagating. Hence the integral over the upper surface is nonzero, in general.

The inward propagating Green's function is given by

$$G^*(x, z, \xi, \zeta) = \frac{1}{4\pi i} \int_C \frac{dk_1}{k_3^*(k_1)} \exp \left\{ -ik_1(x - \xi - i k_3^*(k_1)[z - \zeta]) \right\}. \tag{22}$$

In this equation, $k_3^*(k_1)$ is the complex conjugate of the function $k_3(k_1)$. Thus, on the path of integration, this function agrees with $k_3$ when they are both real but is the negative of $k_3$ when they are imaginary. This assures that in the latter domain, $-i k_3^*(k_1)$ has a negative real part and the integral converges. It is an exercise in asymptotics to show that $G^* = O(\exp \{-i \omega r/c_1\}/\sqrt{\omega r/c_1})$, with $r$ being radial distance between source and receiver. For the temporal Fourier transform that I use, this is the incoming Green's function.

The incoming dipole response, $W^*$, corresponding to this Green's function is

$$W^*(x, z, \xi, \zeta) = \frac{\text{sign}(z - \zeta)}{2\pi} \int dk_1 \exp \left\{ -ik_1(x - \xi - i k_3^*(k_1)[z - \zeta]) \right\}. \tag{23}$$

Now consider the integral,

$$I = \int dx_s W(x, z, x_s, z_s, \omega) W^*(\xi, \zeta, x_s, z_s, \omega), \tag{24}$$

with both $z$ and $\zeta$ greater than $z_s$. We use the Fourier representations (15) and (23) to rewrite this integral as

$$I = \frac{1}{(2\pi)^2} \int dx_s dk_1 dk_1' \exp \left\{ ik_1[x - x_s] + ik_3(k_1)[z - z_s] \right\} \cdot \exp \left\{ -ik_1'[\xi - x_s] - ik_3^*(k_1')[\zeta - z_s] \right\}. \tag{25}$$

As above, the integral over $x_s$ is a delta function, which allow us to calculate the integral over $k_1'$ as well. The result is

$$I = \frac{1}{(2\pi)} \int dk_1 \exp \left\{ ik_1[x - \xi] + ik_3(k_1)(z - z_s) - ik_3^*(k_1)(\zeta - z_s) \right\}$$

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\[
\frac{1}{2\pi} \int_{\omega^2/c_1^2 \geq k_1^2} dk_1 \exp \{ik_1[x - \xi] + ik_3[z - \zeta]\} \\
+ \frac{1}{2\pi} \int_{\omega^2/c_1^2 < k_1^2} dk_1 \exp \{ik_1[x - \xi] + ik_3(k_1)[z + \zeta - 2z_s]\}.
\]

We are more interested in the Fourier transform here than in the function, \(I\), itself.

\[
\tilde{I} = \begin{cases} 
\exp \{ik_3(k_1)[z - \zeta]\}, & \omega^2/c_1^2 \geq k_1^2 \\
\exp \{ik_3(k_1)[z + \zeta - 2z_s]\}, & \omega^2/c_1^2 < k_1^2.
\end{cases}
\]

(27)

In particular, when the two depths agree, \(z = \zeta\),

\[
\tilde{I} = \begin{cases} 
1, & \omega^2/c_1^2 \geq k_1^2 \\
\exp \{ik_3(k_1)[2z - (z_s + z_g)]\}, & \omega^2/c_1^2 < k_1^2.
\end{cases}
\]

(28)

That is, for \(z = \zeta\), \(\tilde{I}\) is equal to unity in the propagating range and attenuates to zero outside that range, with the rate of attenuation increasing with depth and increasing with \(|k_1|\). If \(\tilde{I}\) were identically equal to unity at \(z = \zeta\), then \(I\) would be a delta function, \(\delta(x_g - x_s)\). Thus, we can think of \(I\) as an approximate delta function that at least behaves "right" in the range of propagating wave numbers.

Note that if we were to discretize the operator \(I\) as defined by (24) then the indicated product would become a matrix multiplication of \(W\) with the transpose of \(W^*\). Since \(W^*\) is the complex conjugate of \(W\), this is a multiplication of \(W\) with its adjoint. In that sense, \(I\) view the operation of multiplication by \(W^*\) and integration as being the continuous analog of matrix multiplication with the adjoint of the matrix \(W\). Thus, I view the operator that I propose here as being the continuous analog of the Berkhout school's multiplication with the adjoint matrices associated with the downward propagator matrices.

Of course, this is just the simplest constant background case and we have yet to see what the effect of this operator is on more complicated reflectors, such as dipping planes and arbitrary curved reflectors. However, I believe this is a good start.
Downward Continuation

We are now prepared to test our adjoint operators as approximate inverse operators by applying one in source variables and one in receiver variables to our representation (9) of the observed field. Thus, we consider the integral

$$I = \int dx_1 dx_g W^*(x_g, z_g, \xi_g, \zeta_g, \omega) u(x_g, z_g, x_s, z_s, \omega) W^*(\xi_s, \zeta_s, x_s, z_s, \omega).$$  \hspace{1cm} (29)

Physically, the integral over $x_g$ downward propagates all the responses for a fixed source to the position $\xi_g$. Correspondingly, by reciprocity, the role of sources and receivers can be interchanged. Viewing all source points as receiver locations for a fixed geophone position now viewed as the source, we can perform exactly the same downward propagation of sources. This is the integral over $x_s$.

As previously, the integrations here will be carried out by using the Fourier representations of the three functions under the integral sign, with $W^*$ given by (23) and the observed field given by (9). Thus, the previous equation is replaced by

$$I = \frac{1}{8\pi^3} \int dx_1 dx_g dk''_1 dk''_1 \cdot \exp \left\{ -ik'_1[x_g - \xi_g] - ik'_3(k'_1)[\xi_g - z_g] \right\} \cdot R(k_1, \omega) \exp \left\{ ik_1[x_g - x_s] + ik_3(k_1)[2H - (z_g + z_s)] \right\} \cdot \exp \left\{ -ik''_1[\xi_s - x_s] - ik''_3(k''_1)[\xi_s - z_s] \right\}.$$  \hspace{1cm} (30)

We calculate these integrals as above. Namely, the integrals in $x_s$ and $x_g$ are delta functions that allows us to determine the integrals in $k'_1$ and $k''_1$, yielding the result,

$$I = \frac{1}{2\pi} \int dk_1 R(k_1, \omega) \cdot \exp \left\{ ik_1[\xi_g - \xi_s] + ik_3(k_1)[2H - (z_g + z_s)] - ik''_3(k''_1)[\zeta_g + \zeta_s - (z_g + z_s)] \right\}$$  \hspace{1cm} (31)

or

$$I = \frac{1}{2\pi} \int_{\omega^2/c_s^2 \geq k_1^2} dk_1 R(k_1, \omega) \exp \left\{ ik_1[\xi_g - \xi_s] + ik_3(k_1)[2H - (\xi_g + \zeta_s)] \right\} + \frac{1}{2\pi} \int_{\omega^2/c_s^2 < k_1^2} dk_1 R(k_1, \omega) \cdot \exp \left\{ ik_1[\xi_g - \xi_s] + ik_3(k_1)[2H + \xi_g + \zeta_s - 2(z_g + z_s)] \right\}.$$  \hspace{1cm} (32)
This equation can be equivalently written as

\[ I = \frac{1}{2\pi} \int_{\omega^2 / c_1^2 \geq k_1^2} dk_1 R(k_1, \omega) \exp \{ ik_1 s + ik_3(k_1)n \} \]

\[ + \frac{1}{2\pi} \int_{\omega^2 / c_1^2 < k_1^2} dk_1 R(k_1, \omega) \exp \{ ik_1 s + ik_3(k_1)[n + 2d] \}. \]  

Here, \( s \) and \( n \) are as in Figure 5, except that now they are measured from the downward continued source and receiver positions, \( (\xi_s, \zeta_s) \) and \( (\xi_g, \zeta_g) \). The distance, \( d \) is the sum of the normal distances from the original source and receiver positions to the propagated source and receiver positions, respectively; that is, \( d = \zeta_s - z_s + \zeta_g - z_g \).

We evaluate the result (32) at \( \zeta_s = \zeta_g = H \) to obtain

\[ I = \frac{1}{2\pi} \int_{\omega^2 / c_1^2 \geq k_1^2} dk_1 R(k_1, \omega) \exp \{ ik_1 [\xi_g - \xi_s] \} \]

\[ + \frac{1}{2\pi} \int_{\omega^2 / c_1^2 < k_1^2} dk_1 R(k_1, \omega) \exp \{ ik_1 [\xi_g - \xi_s] + 2ik_3(k_1)[2H - (z_g + z_s)] \} . \]

The Fourier transform of this result is

\[ \tilde{I} = \begin{bmatrix} R(k_1, \omega), & \omega^2 / c_1^2 \geq k_1^2 \\ R(k_1, \omega) \exp \{ 2ik_3(k_1)[2H - (z_g + z_s)] \}, & \omega^2 / c_1^2 < k_1^2 \end{bmatrix}. \]

We see that this result produces the reflection coefficient for \( \omega^2 / c_1^2 \geq k_1^2 \). Furthermore, for \( k_1^2 > \omega^2 / c_1^2 \), it produces the reflection coefficient multiplied by a factor that decays exponentially with increasing \( k_1^2 \) and with increasing depth \( H \). In fact, as noted above, the multiplier of \( ik_3 \) in the second case is twice the sum of the normal distances from the source and the receiver to the reflector. It will be seen below that this is the form that the decay rate takes in the case of the dipping reflector, as well.

Figure 9 shows the function \( \tilde{I} \) in (35) for the case \( 2\omega(2H - z_g - z_s)/c_1 = 8\pi \).

This latter evaluation effects the decay rate in the second line in (35). The choice of constant made here occurs for \( f[H - (z_g + z_s)/2]/c_1 = 1 \), with \( f \) being frequency in Hz. For example, this corresponds to a reflector at 500m depth, with source and receiver at depth zero, at a frequency of 10Hz and a propagation speed of 5Km/s. If we were to decrease the propagation speed or increase the frequency to values more realistic for this depth, then the decay rate would be even greater. Similarly, if we were to increase the depth to something more consistent with this frequency and propagation speed, the same thing would happen. The point here is that for relatively conservative choices of parameters the exponential decay in the evanescent range of \( k_1 \) acts effectively like a bandpass filter eliminating all evanescent information from the output.
\[ \tilde{I}(q\omega/c_1, \omega) \]

**Fig. 9.** Plot of the function in equation (26).

\[ I(n\Delta x) \]

**Fig. 10.** Fourier transform of the function of Figure 3.
Figure 10 shows the transform of this result, as a function of a counting index $n$. This result is scaled by $\omega/c_1$. As with the exact reflection coefficient, this result peaks at zero argument, which is the center position of the 128 point transform, with peak value that is actually higher, $\approx 2.6$, than for the exact reflectivity function in Figure 8, $\approx 2.2$.

Imaging

In order for the integral $I$ in (33) to be an effective integral for imaging the reflector, we would want this integral to peak in the $z$–direction, as well. It is clear that the integral will peak if both $s$ and $n$ are zero. If we are interested in peaking only in the $z$–direction, then we must set $s = 0$. The way to do this is to make the source and receiver coincident. We propose this as the method of imaging for this approach to migration and inversion. In the next section, we show that this provides an image for the case of the dipping reflector, as well. Indeed, for $s = 0$, the representation (33) assures us that the output will peak only for $n = 0$.

Thus, we consider the evaluation of $I$ for $\xi_g = \xi_s$, but for $\zeta_s = \zeta_g \neq H$. The result is predominantly given by the first line in (33) with $\xi_g = \xi_s$. In Figure 11,

![Graph](image)

**Fig. 11.** $I$ in (24) for coincident source receiver but variable depth.

we show this output. The units represent approximately 32m. Again we see a sharp peak. That is, the operations represented by (29), which I can symbolically write as $W_g^* u W_s^*$, provides an output that peaks at the reflector depth for coincident source and receiver. This is what we would want an imaging operator to do. However, in addition to that, the Fourier transform of the downward continued data is the reflection coefficient as a function of $k_1$ and $\omega$. Of course, the lateral invariance of this problem assures us that all vertical lines will look alike, as long as we process the data
for coincident source and receiver. Note that this also makes the output a function of two spatial variables which is necessary for an image of a reflector in two dimensions.

In three dimensions, the three coordinates of the source and receiver point would be equal in pairs, making the output a function of three spatial variables as it should be for that case.

Below, when we consider the dipping reflector, we will display a two-dimensional imaging output for this operator inversion. Here, there is no point in showing more than Figure 11 because all vertical lines will be identical.

Asymptotic Analysis

It is worthwhile to examine $I$, as defined by (33), asymptotically in order to gain some idea about whether or not processing for $I$ will reveal the presence of the reflector, that is, whether this function peaks when $\zeta_g = \zeta_s = H$. In carrying out this analysis, a change of variable of integration from $k_1$ to $\gamma$ through $k_1 = (\omega/c_1) \sin \gamma$, or to $q$ through $k_1 = (\omega/c_1)q$. In either case, an overall multiplier of $\omega/c_1$ will appear in the representation of $I$ and will appear in our discussion of the asymptotic results, below. Indeed, this factor also carries the dimension of $I$, namely, inverse length or wave number.

The two summands of $I$ as represented in (33) behave quite differently. In the second term, $k_3$ is purely imaginary and has a length scale multiplier in depth, $2H + \zeta_g + \zeta_s - 2(z_g + z_s)$, that is never zero. Furthermore, $k_3$ is singular at the endpoints of integration with an infinite derivative there. These are the dominant critical points in the asymptotic analysis of these integrals. In fact, this second line can be shown to be

$$\frac{\omega}{2\pi c_1} O \left[ (\omega[2H + \zeta_g + \zeta_s - 2(z_g + z_s)]/c_1)^{-2} \right],$$

which is less than the order of the first term in (33), to be derived below.

As might be expected, the analysis of this first term is sensitive to the value of the relative depth, $2H - (\zeta_g + \zeta_s)$. It should be noted that when this variable is zero, as in (34), the phase is linear in $k_1$; that is, there are no stationary points in this integral. When this variable is nonzero, the integrand has a stationary point. In this case, one can show that this first line and, hence, $I$ can be estimated by

$$I = \frac{\omega}{2\pi c_1} O \left[ (\omega r/c_1)^{-1/2} [2H - (\zeta_g + \zeta_s)]/r \right].$$

Here,

$$r = \sqrt{[\xi_g - \xi_s]^2 + [2H - (\zeta_g + \zeta_s)]^2}.$$  \hspace{1cm} (37)

This result is not valid when $2H - (\zeta_g + \zeta_s) = 0$ because the second derivative of the phase becomes infinite there. That is the case where the phase is linear, as noted above. Thus, this should be viewed as a qualitative estimate only for this quantity.
bounded away from zero. When \(2H - (\xi_g + \xi_s) = 0\), \(I\) is given by (33). In particular, when, in addition, \(\xi_g - \xi_s = 0\),

\[
I = \frac{\omega}{2\pi c_1} O(1),
\]

clearly larger than the result at other depths and for non-conincident source and receiver at depth \(H\).

It seems to me that the operations indicated in (29) provide a reasonable approximation to the reflectivity function we seek, since they at least provide an estimate of the reflection coefficient in the \(k_1\)-domain in the propagating range, \(-\omega/c_1 < k_1 < \omega/c_1\). Therefore, with acknowledgement of the error of the approximation demonstrated here, I define

\[
R(\xi_g, \xi_g, \xi_s, \xi_s, \omega) = \int dx_d dx_g W^*(x_g, z_g, \xi_g, \xi_g, \omega) u(\xi_g, \xi_g, x_s, z_s, \omega) W^*(\xi_s, \xi_s, x_s, z_s, \omega).
\]  

(38)

Recapitulation

In summary, we have started with the model of a dipole source over a horizontal reflector and verified a basic premise for this example that the response is the Fourier transform of the reflection coefficient when the source and receiver are moved down to the reflector. We have also verified the structure \(WRW\) for the field representation and used that as a motivation for a mechanism of inverting data to produce an image of the reflecting interface and an estimate of the reflection coefficient. We showed that an exact inverse of the forward propagators is not available because the attenuation rate of the forward propagators approaches infinity with \(k_1\), requiring that the exponential growth rate of the inverter would have to do the same.

We then introduced the adjoint operator as an approximate inverse operator and showed that it acted as an inverse operator for the propagating range of \(k_1, \omega^2/c_1^2 > k_1^2\), and \(\omega^2/c_2^2 > k_1^2\). When these operators in source and geophone coordinates were applied to the data, the result was, indeed, an approximate inverse of the data. Imaging was achieved by plotting the output of this inversion for coincident source and receiver coordinates. Examination of the output at the depth where peaking occurred, yielded the spatial Fourier transform of the reflection coefficient restricted to the propagating range of \(k_1\) with some error in the evanescent range. However, Fourier inversion of that spatial output yields the reflection coefficient for the propagating modes or, equivalently, for all real angles of incidence on the reflector.

Contrast with Conventional Migration/Inversion

I do not use the word “conventional” for alternative approaches to imply something perjorative about the method being discussed here, but only as a reference for discussion and comparison.
Note that in this method, the frequency has been held fixed throughout the discussion; imaging and parameter estimation were achieved for fixed frequencies by using data from all sources and receivers simultaneously. Subsequently, one could use summation over frequency for image enhancement through stacking, or summing over fixed ratio, \( ck_1/\omega \) for stacking at varying incidence angle. Crudely speaking, one uses all sources and receivers for imaging, then frequency for stacking.

Conventional migration approaches the problem by migrating data from one fixed offset or one fixed source and all frequencies and then sums over offsets (first case) or source locations (second case) for stacking purposes.

It is my view that the method considered here reverses the role of summing over frequency and stacking over sources or midpoints when compared to other migration/inversion methods such as wave equation migration, Fourier migration/inversion, or Kirchhoff migration/inversion.

It is this reversal of roles that allows for the possibility of processing with a frequency dependent velocity, essentially by changing the velocity for each separate inversion. Thus, dispersion could be built into this approach to inversion more easily than in the conventional approaches.

**THE DIPPING REFLECTOR**

Here, we extend the discussion to the case of a dipole source over a dipping reflector.

![Diagram of Point Source over a Dipping Reflector](image)

**FIG. 12.** Point source over a dipping reflector
reflector as shown in Figure 12. The objective of this discussion is to extend the ideas of the previous section to this case in order to examine the effect of reflector dip on the results of the previous discussion. In summary, the results are not so clean. The modeling formula does not lead to as clean a result as (8), nor is the function $R(\xi_s, \zeta_s, \xi_g, \zeta_g, \omega)$ as clean. On the other hand the inversion operations $W_g^* u W_s^*$ does produce as simple a result as in the previous case. We can achieve imaging and we can achieve parameter estimation. It is all just a little more difficult than in the case of the horizontal reflector.

The Forward Model

We begin be presenting the solution for the reflection from the dipping reflector in Figure 12. This solution can be derived from the previous one by first solving for the impulse response in a rotated coordinate system in which the $x'$-axis is parallel to the reflector. In this new system, the impulse response is given by (1). Then, a change of variables back to the original coordinates yields the solution to the present problem. By taking $-2\partial/\partial z$ of this solution, we obtain the dipole response solution.

The rotation of coordinates is given by

$$x' = (x - x_s) \cos \phi + (z - z_s) \sin \phi,$$

$$z' = -(x - x_s) \sin \phi + (z - z_s) \cos \phi,$$

$$H' = (x_s - x_0) \sin \phi - z_s \cos \phi.$$ \hfill (39)

In the last equation, $x_0$ is the point where the dipping reflector emerges at the upper surface, $z = 0$ and $H'$ is the normal distance from the source point to the reflector. In fact, by setting $H' = 0$ in this last equation, we obtain the equation of the reflector,

$$(x - x_0) \sin \phi - z \cos \phi = 0.$$ \hfill (40)

The dipole response solution for the dipping reflector is given by

$$u(x_g, z_g, x_s, z_s, \omega) = \frac{1}{2\pi} \int_C R(k_1, \omega) \frac{k(-k_1) \cdot \hat{n}}{k_3(k_1)} dk_1 \exp\{i\Phi\},$$

$$z_s < (x_s - x_0) \tan \phi, \quad z_g < (x_g - x_0) \tan \phi.$$ \hfill (41)

In this equation, $R(k_1, \omega)$ is again given by (2); $\hat{n} = (-\sin \phi, \cos \phi)$ is the downward pointing unit normal to the reflector and

$$k(k_1) = (k_1, k_3(k_1)), \quad k(-k_1) = (-k_1, k_3(k_1)),$$ \hfill (42)

with the second equation arising from the symmetry of $k_3$. 

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Finally,
\[
\Phi = k_1 [(x_g - x_s) \cos \phi + (z_g - z_s) \sin \phi]
\]
\[
+k_3(k_1) [(x_s + x_g - 2x_0) \sin \phi - (z_g + z_s) \cos \phi]
\]
\[
= k_1 s(x_g, z_g, x_s, z_s) + k_3(k_1)[n(x_s, z_s) + n(x_g, z_g)],
\] (43)
\[
s(x_g, z_g, x_s, z_s) = (x_g - x_s) \cos \phi + (z_g - z_s) \sin \phi,
\]
\[
n(x, z) = (x - x_0) \sin \phi - z \cos \phi.
\]

The functions, \(k_3(k_1)\) and \(k_4(k_1)\) are given by (4). We will have need of functions \(k_3\) and \(k_4\) of other arguments here; hence, the introduction of their explicit argument in this discussion.

Although the phase is much more complicated here, its representation in terms of \(s\) and \(n\) is just as it was in (3) for the flat reflector case. \(^2\) A new feature here is the ratio \(k(-k_1) \cdot \hat{n}/k_3(k_1)\). This factor is equal to unity when the reflector is flat; hence, it was not present in the discussion of the previous section.

We take the point of view that \(x_0\) is far to the left of the domain of interest, so that the reflector is actually more than a “few wavelengths” deep. In fact, one can obtain the previous solution from the present one by allowing \(\phi \to 0\) while \(x_0 \to \infty\) in such a manner that \(-x_0 \sin \phi \to H\). This assumption that \(x_0\) is “far away” is essentially a high frequency assumption. Below, it will be necessary to make approximations based on this.

One can see from the solution representation, (41), (43), that the dipole response can no longer be the Fourier transform of the reflection coefficient, itself. What we can say here is that for the source and the receiver both on the reflector, the dipole response is the Fourier transform of the reflection coefficient multiplied by this new factor, \(k(-k_1) \cdot \hat{n}/k_3(k_1)\); that is,
\[
u(x_g, (x_g - x_0) \tan \phi, x_s, (x_s - x_0) \tan \phi, \omega) = \frac{1}{2\pi} \int_C R(k_1, \omega) \frac{k(-k_1) \cdot \hat{n}}{k_3(k_1)} dk_1 \exp\{ik_1 s\};
\]
\[
s = (x_g - x_s) \sec \phi.
\] (44)

On the other hand, we should expect that the reflectivity matrix will be just (41) evaluated at equal depth values, \(z_g = z_s\), as a function of the two variables, \(x_g\) and

\(^2\)Note, however, that we have redefined the variable \(n\) here to write the total normal distance as the sum of the two distances from source to reflector and receiver to reflector.
\[ R_{ij}(z, \omega) = \frac{1}{2\pi} \int_C R(k_1, \omega) \frac{k(-k_1) \cdot \hat{n}}{k_3(k_1)} dk_1 \exp\{i\Phi\}; \]

\[ \Phi = k_1(x_g - x_s) \cos \phi + k_3(k_1) [(x_s + x_g - 2x_0) \sin \phi - 2z \cos \phi], \]

continuous; \hspace{1cm} (45)

\[ = k_1 [(i - j)\Delta x \cos \phi] + k_3(k_1) [((i + j)\Delta x - 2x_0) \sin \phi - 2z \cos \phi], \]

discrete.

As a check on this result, let us consider its Fourier transform with respect to \( x_g \), defined by

\[ \tilde{R}_{ij}(z, \omega) = \int R_{ij}(z, \omega) dx_g \exp \{ -i\kappa_1 [x_g - x_s] \}. \hspace{2cm} (46) \]

Integration with respect to \( x_g \) yields a Dirac delta function, namely, \( \delta[f(k_1, \kappa_1)] \), with

\[ f(k_1, \kappa_1) = k_1 \cos \phi + k_3(k_1) \sin \phi - \kappa_1. \hspace{2cm} (47) \]

Thus, the integral in \( k_1 \) can now be carried out, by determining the zero of this equation. We need the results,

\[ k_1(\kappa_1) = \kappa_1 \cos \phi - k_3(\kappa_1) \sin \phi, \]

\[ k_3(\kappa_1) = \kappa_1 \sin \phi + k_3(\kappa_1) \cos \phi, \hspace{2cm} (48) \]

\[ \frac{\partial f(k_1, \kappa_1)}{\partial k_1} = \frac{k(k_1) \cdot \hat{n}}{k_3(k_1)}. \]

By using these results in (46), we find that

\[ \tilde{R}_{ij}(z, \omega) = \frac{k(-k_1) \cdot \hat{n}}{k(k_1) \cdot \hat{n}} R(k_1, \omega) \exp \{ 2ik_3(k_1) n(x_s, z) \} \]

\[ = \frac{\kappa_1 \sin 2\phi + k_3(\kappa_1) \cos 2\phi}{k_3(\kappa_1)} R(k_1, \omega) \exp \{ 2ik_3(k_1)n(x_s, z) \}. \hspace{2cm} (49) \]

Thus, we see that after Fourier transform, we obtain the reflection coefficient at a wave number that would seem to be “rotated” through the negative of the dip angle, multiplied by a factor which also is tied to the dip. However, note that when \( n = 0 \), that is, when the source point is moved to the reflector at the depth \( z \), \( R(k_1(\kappa_1), \omega) \) is the Fourier transform of the reflection coefficient, but at a wavenumber that is “corrected” for dip.
Downward Continuation

Let us go directly now to applying the operators $W^*_g$ and $W^*_s$ to the solution representation (41). That is, we repeat the operator analysis, (29), with $u$ replaced by the solution (41). Our objective is to see how close the result is to claimed reflectivity matrix, (45).

In place of (30), we now obtain

\[
I = \frac{1}{8\pi^3} \int_{x_s, x_s > x_0} dx_s dx_g dk'_1 dk'_2 dk''_1 dk_1 \cdot \exp \{-ik'_1[x_s - \xi_s] - ik''_1(k'_1)[\zeta_s - z_s]\}
\]

\[
\cdot R(k_1, \omega) \frac{k(-k_1) \cdot \hat{n}}{k_3(k_1)} \exp \{ik_1s + ik_3(k_1)[n(x_s, z_s) + n(x_g, z_g)]\}
\]

\[
\cdot \exp \{-ik''_1[\zeta_s - z_s] - ik''_3(k'_1)[\zeta_s - z_s]\}. \tag{50}
\]

Here, $s$ and $n$ are given in (43).

If the integrations over $x_s$ and $x_g$ ranged from $-\infty$ to $\infty$, then the results of those integrations would be $2\pi \delta(k''_1 - k_1 \cos \phi + k_3(k_1) \sin \phi)$ and $2\pi \delta(-k'_1 + k_1 \cos \phi + k_3(k_1) \sin \phi)$, respectively. Here is where we make our approximation based on the assumption that $x_0$ is "very far" from the domain of interest. As a practical matter, for dipping and curved reflectors in the earth's subsurface, we rarely have to be concerned with their emergence at the upper surface. This is a technicality of this particularly simple problem. Thus, we proceed on the assumption that we can neglect the effects of the endpoint, $x_0$, in these two integrals.

We now carry out the integrals in $k'_1$ and $k''_1$ by in these variables at the zeroes of the arguments of the delta functions; that is for

\[
k''_1 = k_1 \cos \phi - k_3(k_1) \sin \phi \quad \text{and} \quad k'_1 = k_1 \cos \phi + k_3(k_1) \sin \phi. \tag{51}
\]

Associated with these values, we find that

\[
k_3(k''_1) = k_1 \sin \phi + k_3(k_1) \cos \phi \quad \text{and} \quad k_3(k'_1) = -k_1 \sin \phi + k_3(k_1) \cos \phi, \tag{52}
\]

The result of carrying out these integrations and evaluations is

\[
I = \frac{1}{2\pi} \int_C R(k_1, \omega) \frac{k(-k_1) \cdot \hat{n}}{k_3(k_1)} dk_1 \exp \{i\Phi\},
\]

\[
\Phi = k_1[(\xi_g - \xi_s) \cos \phi + (\zeta_g - \zeta_s) \sin \phi]
\]

\[
+ k_3(k_1)[(\xi_g + \xi_s - 2x_0) \sin \phi - (z_g + z_s) \cos \phi]
\]

\[
- k_3^*(k_1)[\zeta_g - z_g + \zeta_s + z_s] \cos \phi.
\]

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The evaluation of $\Phi$ becomes more meaningful if we write the result in separate forms for the propagating values of $k_1$ and the evanescent values of $k_1$. We find that

$$\Phi = \begin{cases} 
  k_1 \cdot s(\xi_g, \zeta_g, \xi_s, \zeta_s) + k_3(k_1) \cdot [n(\xi_g, \zeta_g) + n(\xi_s, \zeta_s)], & k_1^2 \leq \omega^2/c_1^2, \\
  k_1 \cdot s(\xi_g, \zeta_g, \xi_s, \zeta_s) + k_3(k_1) \cdot [n(\xi_g, \zeta_g) + n(\xi_s, \zeta_s) + 2d], & k_1^2 > \omega^2/c_1^2.
\end{cases} \quad (54)$$

In this equation, $d$ plays the same role as it did in the horizontal reflector case; it is the sum of the normal distances from the actual source and receiver positions to the downward continued source and receiver positions:

$$d = [\zeta_g - z_g + \zeta_s - z_s] \cos \phi.$$ 

Since $n(\xi_g, \zeta_g) + n(\xi_s, \zeta_s) + 2d$ is always positive, and $\Im k_3$ is positive in the evanescent region, $ik_3[n + 2d]$ provides exponential decay for the evanescent values of $k_1$ and

$$I = \frac{1}{2\pi} \int_{\omega^2/c_1^2 \geq k_1^2} dk_1 R(k_1, \omega) \frac{k(-k_1) \cdot \hat{n}}{k_3(k_1)}$$

$$\cdot \exp \{ik_1 s + ik_3(k_1)[n(\xi_g, \zeta_g) + n(\xi_s, \zeta_s)]\}$$

$$+ \frac{1}{2\pi} \int_{\omega^2/c_1^2 < k_1^2} dk_1 R(k_1, \omega) \frac{k(-k_1) \cdot \hat{n}}{k_3(k_1)}$$

$$\cdot \exp \{ik_1 s + 2ik_3(k_1)[n(\xi_g, \zeta_g) + n(\xi_s, \zeta_s) + 2d]\}.$$ \quad (55)

When we compare this result with the predicted reflectivity matrix, (45), we see that they agree for the propagating range of values of $k_1$ and disagree for the evanescent range of values of $k_1$. This is exactly as it was in the case of the horizontal reflector.

We have shown here that the reflectivity matrix is approximated by our adjoint operators in exactly the same was as it was for the case of the horizontal reflector. Now we must turn the questions of imaging and inversion using this reflectivity that we have created.

**Imaging**

Previously, we proposed that imaging is achieved with this method by evaluating the inversion output, (55), for coincident source and receiver. As in the previous section, in this limit, $s = 0$, and $I$ is a function of $n$ and $d$. That is, for the purpose of imaging, we should evaluate

$$I = \frac{1}{2\pi} \int_{\omega^2/c_1^2 \geq k_1^2} dk_1 R(k_1, \omega) \frac{k(-k_1) \cdot \hat{n}}{k_3(k_1)} \exp \{2ik_3(k_1)n(\xi_s, \zeta_s)\}$$

$$64$$
\[ + \frac{1}{2\pi} \int_{\omega^2/c_1^2 < k_1^2} dk_1 R(k_1, \omega) \frac{k(\mathbf{\mathbf{k}}_1) \cdot \hat{n}}{k_3(k_1)} \exp \left\{ 4ik_3(k_1)[n(\xi, \zeta_s) + d] \right\}. \]

Figure 13 is a Mathematica rendition of the output of (56) for the case of a plane at

\[ W_g^* u W_s^* \]

**COINCIDENT SOURCE & RECEIVER**

**FIG. 13.** Imaging for a dipping reflector

30° dip. The reflector is clearly visible here. Again, I used a frequency of 20Hz and propagation speed of 5000m/sec. The integration was truncated at the evanescent boundary, justified by our experience with the horizontal reflector.

**Inversion**

Here we will show how to extract the reflection coefficient from the result (55). First, note that \( I \) is initially a function of four spatial variables, \( \xi_s, \zeta_s, \xi_g, \zeta_g \) as well as frequency. In the Berkhout approach, the downward continuation is carried out with the two depth variables being the same; that is, \( \zeta_g = \zeta_s \), thus, making \( I \) a function of three spatial variables and frequency. Now let us consider the Fourier transform of this function with respect to \( \xi_g \) with the phase shift suggested when we first introduced this type of Fourier transform above equation (9):

\[ \tilde{I} = \int I \exp \{-i\kappa_1(\xi_g - \xi_s)\} d\xi_g. \] (57)

Actually, we have already carried out a similar integration in determining \( \tilde{R}_{ij} \) in (46). The main difference here is that the phase in (55) is different for the propagating
and evanescent ranges of the variable $k_1$. However, this does not change the support of the delta function or its evaluation. Consequently, we find that

$$\tilde{I} = \frac{\kappa_1 \sin 2\phi + k_3(\kappa_1) \cos 2\phi}{k_3(\kappa_1)} R(k_1(\kappa_1), \omega) \exp i\Psi,$$

(58)

where, just as in the analysis of (46),

$$\Psi = 2 [ - \kappa_1 \sin \phi + k_3(\kappa_1) \cos \phi ] [ (\xi_s - x_0) \sin \phi - \zeta_s \cos \phi ]$$

$$= 2k_3(k_1(\kappa_1)) \cdot n(\xi_s, \zeta_s).$$

(59)

Here, $k_1(\kappa_1)$ and $k_3(k_1(\kappa_1))$ are defined by (48).

As a result of our previous imaging, we know how to choose $\xi_s$ and $\zeta_s$ to make $n = 0$; it is a matter of placing the output point on the reflector. In that case, we find

$$\tilde{I} = \frac{\kappa_1 \sin 2\phi + k_3(\kappa_1) \cos 2\phi}{k_3(\kappa_1)} R(k_1(\kappa_1), \omega), \quad n = 0.$$

(60)

That is, by evaluating the Fourier transform of the downward continued field, $W_g^* u W_s^*$, on the reflector, we obtain the exactly the result predicted in (49). If we introduce the incidence angle $\gamma$ as in (10), but now for $\kappa_1$,

$$\kappa_1 = \frac{\omega}{c_1} \sin \gamma,$$

(61)

then from (48),

$$k_1(\kappa_1) = \frac{\omega}{c_1} \sin(\gamma - \phi),$$

(62)

and

$$R((\omega/c_1) \sin(\gamma - \phi), \omega) = \frac{\cos(\gamma - \phi)/c_1 - \sqrt{1/c_2^2 - \sin^2(\gamma - \phi)/c_1^2}}{\cos(\gamma - \phi)/c_1 + \sqrt{1/c_2^2 - \sin^2(\gamma - \phi)/c_1^2}}$$

(63)

At the moment, we have no way of determining $\phi$ except from the graphical output, such as Figure 13. However, given an analytical formula, we should anticipate a modified asymptotic downward continuation operator whose output will differ from the one here by cos $\phi$, similar to the Kirchhoff inversion results in the Bleistein, (1987), et al, approach. Also, the range of the angle $\gamma$ for which we actually have reliable output will be a function of the completeness of the angular aperture of the original experiment; here, we have assumed that we have sources and receivers from $x_0$ to $\infty$, which is clearly not the case, in practice. Here, again, asymptotic analysis, most likely by the method of stationary phase, will reveal the extent of the aperture. (I expect results similar to the ones obtained in the analysis of Kirchhoff inversion.)
Summary

We have examined an exact forward model of the upward propagating wave from a dipping planar reflector. We have applied the inversion formalism that was motivated by the previous study of the horizontal reflector. Imaging was achieved for a fixed frequency, suggesting that dispersion could be accommodated by processing data at different frequencies with different background velocities. Estimation of the reflection coefficient is a little more obscure, here, because it is masked by a factor that depends on the dip angle. Furthermore, the transverse wave number at which the reflection coefficient is to be evaluated is also a function of the dip angle. However, we believe that the basic objective of exhibiting a continuum analog of Berkhout inversion for the case of a dipping planar reflector has been accomplished.

EXTENSION OF WRW BEYOND THE PLANAR REFLECTOR

This section is based on a thesis by von Vroonhoven (1993), in which a derivation is given of the extension of the form WRW to curved reflectors. The result here differs from von Vroonhoven's: her W-functions depend on the normal to the reflector and therefore are not pure propagators, while the ones given here do not depend on properties of the reflector. However, the price we pay for this is a slightly more complicated, still fairly simple, representation.

The derivation starts from a Kirchhoff integral representation of the upward scattered field from a single reflector. Von Vroonhoven gives a detailed derivation, but it can also be found elsewhere, including Bleistein (1984). The most interesting derivation is probably in Baker and Copson (1939).

The upward scattered field can be represented in terms of its values on a reflecting surface by

$$ u(x, x_s) = \int_{x' \in S} \left[ u(x', x_s) \frac{\partial G(x', x)}{\partial n'} - G(x', x) \frac{\partial u(x', x_s)}{\partial n'} \right] dA(x'). $$

In this equation, $G$ is the free space Green's function; $u$ is the upward scattered field and $S$ is the reflecting surface.

By reciprocity, \(^3\) we can interchange $x$ and $x_s$. In fact, we could interchange these variables under the integral sign, only, leaving them unchanged on the left side of the equation. That is,

$$ u(x, x_s) = \int_{x' \in S} \left[ u(x', x) \frac{\partial G(x', x_s)}{\partial n'} - G(x', x_s) \frac{\partial u(x', x)}{\partial n'} \right] dA(x'). \quad (65) $$

\(^3\)When the acoustic wave equation is written in non-self-adjoint form, minor adjustments must be made to account for non-symmetric reciprocity. In fact, von Vroonhoven starts from self-adjoint coupled equations for pressure and particle velocity, so that this derivation applies to her equations. This is only appropriate, since this derivation follows hers.
The point of doing this is that the scattered field under the integral sign no longer depends on the upper surface source point, \( x_s \), while the Green’s function (propagator) now does. The function, \( u(x', x) \) is the field at \( x' \) due to a fictitious source at \( x \) of the same type as the true source at \( x_s \). The field at \( x' \) is to be interpreted as the limit of the upward propagating field for an observation point above the reflector moved down onto the reflector.

The point, \( x \) is not to be our ultimate observation point. That will be \( x_g \). The representation (64), used again, allows to write

\[
u(x_g, x_s) = \int_{x \in \mathcal{S}} \left[ u(x, x_s) \frac{\partial G(x_g, x)}{\partial n} - G(x_g, x) \frac{\partial u(x, x_s)}{\partial n} \right] dA(x) \tag{66}\.
\]

Now, for \( u \) and \( \partial u/\partial n \), use (65). That means we move the source point in \( u(x', x) \) onto the reflecting surface, as well. The result is

\[
u(x_g, x_s) = \int_{x \in \mathcal{S}} \int_{x' \in \mathcal{S}} \left\{ \left[ u(x', x) \frac{\partial G(x, x_s)}{\partial n'} - G(x', x_s) \frac{\partial u(x', x)}{\partial n'} \right] \frac{\partial G(x_g, x)}{\partial n} \right.
\]

\[
\left. - G(x_g, x) \frac{\partial}{\partial n} \left[ u(x', x) \frac{\partial G(x', x_s)}{\partial n'} - G(x', x_s) \frac{\partial u(x', x)}{\partial n'} \right] \right\} dA(x) dA(x'). \tag{67}\]

We expand the second line and rewrite this result as

\[
u(x_g, x_s) = \int_{x \in \mathcal{S}} \int_{x' \in \mathcal{S}} dA(x) dA(x') \left\{ u(x', x) \frac{\partial G(x', x_s)}{\partial n'} \frac{\partial G(x_g, x)}{\partial n}
\right.
\]

\[
\left. + G(x', x_s) G(x_g, x) \frac{\partial^2 u(x', x)}{\partial n' \partial n} \right.
\]

\[
\left. - \frac{\partial G(x_g, x)}{\partial n} G(x', x_s) \frac{\partial u(x', x)}{\partial n'} \right.
\]

\[
\left. - G(x_g, x) \frac{\partial G(x', x_s)}{\partial n'} \frac{\partial u(x', x)}{\partial n} \right\} \tag{68}\.
\]

It only remains to write this result in terms of propagators and a reflection operator. To do so, it is necessary to separate the normal derivatives (surface effects) from the Greens’ functions (propagators). Thus, we introduce the horizontal 4-tuple,

\[
\mathcal{G}(x_g, x) = (G(x_g, x), \nabla G(x_g, x)) \tag{69}\.
\]

and the \( 4 \times 4 \) reflectivity dyad

\[
\mathcal{R}(x', x) = \begin{bmatrix}
\frac{\partial^2 u(x', x)}{\partial n' \partial n} & -\frac{\partial u(x', x)}{\partial n'} \hat{n}(x) \\
-\frac{\partial u(x', x)}{\partial n} \hat{n}^T(x') & u(x', x) \hat{n}^T(x') \hat{n}(x)
\end{bmatrix} \tag{70}.
\]
In this equation, $T$ denotes transpose, so that the first line here connotes a scalar followed by a three component horizontal vector and the second line connotes a three component vertical vector followed by a $3 \times 3$ dyadic.

By using these representations in (68) we find that we can rewrite that integral as

$$ u(x_g, x_s) = \int_{x \in S} \int_{x' \in S} dA(x) dA(x') G(x_g, x') R(x', x) \mathcal{G}^T(x, x_s). \tag{71} $$

This result clearly has the form

$$ u(x_g, x_s) = \mathcal{W}_g R \mathcal{W}_s, \tag{72} $$

where the operators $\mathcal{W}_g$ and $\mathcal{W}_s$ are propagators from the reflector to the geophone and from the source to the reflector, respectively, and $R$ is a reflectivity operator which carries the surface information through the reflectivity dyad in (70). The operations performed are integrations over the reflector. $R \mathcal{W}_s$ creates the upward scattered field at the reflector through integration over the variable $x$ and $\mathcal{W}_g$ operating on this result propagates this surface field back to the geophone $x_g$ through integration over the variable $x'$.

For the case of the dipping planar reflector in two dimensions, through calculations much like the ones that were carried out earlier with the Fourier representations of all of these functions, one finds that each of the terms in (68) is identical and equal to one fourth of the exact solution presented earlier for this problem. The calculation of one of these terms will be carried out here. The others follow in a quite similar matter. (For completeness of this short note, I will include equations that were stated in the earlier notes on the Berkhout approach to inversion.)

The previously derived result for the upward propagating wave from a dipping planar reflector is (eqs 41, 42)

$$ u(x, x', \omega) = \frac{1}{2\pi} \int_C R(k_1, \omega) \frac{k \cdot \hat{n}}{k_3(k_1)} dk_1 \exp\{i\Phi\}, \tag{73} $$

$$ z' < (x' - x_0) \tan \phi, \quad z < (x - x_0) \tan \phi. $$

In this equation, $\hat{n} = (-\sin \phi, \cos \phi)$ is the downward pointing unit normal to the reflector; the vector, $k$ is given by $k = (k_1, k_3(k_1))$; and

$$ R(k_1, \omega) = \frac{k_3(k_1, \omega) - k_4(k_1, \omega)}{k_3(k_1, \omega) + k_4(k_1, \omega)}, \tag{74} $$

Finally,

$$ \Phi = k_1 [(x' - x) \cos \phi + (z' - z) \sin \phi] $$

$$ + k_3(k_1) [(x' + x - 2x_0) \sin \phi - (z' + z) \cos \phi] $$

69
\[ s(x', x) = (x' - x) \cos \phi + (z' - z) \sin \phi, \]
\[ n(x) = (x - x_0) \sin \phi - z \cos \phi. \]

For the Green's functions we have the results,
\[ G(x_g, x) = -\frac{1}{4\pi i} \int \frac{dk'_1}{k_3(k'_1)} \exp i \left\{ k'_1(x_g - x) + k_3(k'_1)(z - z_g) \right\}, z > z_g, \]
\[ G(x', x_s) = -\frac{1}{4\pi i} \int \frac{dk''_1}{k_3(k''_1)} \exp i \left\{ k''_1(x' - x_s) + k_3(k''_1)(z - z_s) \right\}, z > z_s. \]

In (68), let us consider the first integral,
\[ I_1 = \int_{x \in S} \int_{x' \in S} dsds'u(x', x) \frac{\partial G(x_s, x')}{\partial n'} \frac{\partial G(x_g, x)}{\partial n} \]

Here, \( ds \) and \( ds' \) are differential arclengths along the reflector and
\[ x = x_0 + s \cos \phi, \quad z = s \sin \phi, \]
\[ x' = x_0 + s' \cos \phi, \quad z = s' \sin \phi. \]

Substitution of the above three representations for the functions appearing in this equation yields the equation
\[ I_1 = \frac{1}{32\pi^3} \int \frac{k \cdot \hat{n}}{k_3(k_1)} \frac{\tilde{k}' \cdot \hat{n}}{k_3(k'_1)} \frac{\tilde{k}'' \cdot \hat{n}}{k_3(k''_1)} dk_1 \cdot \hat{n}dk_1' \cdot \hat{n}dk_1'' \cdot \hat{n}R(k_1, \omega) \exp \{ i\Psi \}. \]

In this equation,
\[ \tilde{k} = (-k_1, k_3(k_1)), \]
and
\[ \Psi = k_1(s' - s) + k_1'(x_g - x_0 - s' \cos \phi) + k_3(k'_1)(s' \sin \phi - z_g) \]
\[ + k''_1(x_s - x_0 - s'' \cos \phi) + k_3(k''_1)(s \sin \phi - z_s) \]

The integrations in \( s \) and \( s' \) now yield a pair of delta functions \( \delta(k_1 - k'_1 \cos \phi + k_3(k'_1) \sin \phi) \) and \( \delta(k_1 - k''_1 \cos \phi + k_3(k''_1) \sin \phi) \) which allow us to carry out those
integrations. The derivatives of these delta function arguments with respect \(k'_1\) and \(k''_1\), respectively, are exactly the functions, \(\hat{k}' \cdot \hat{n}/k_3(k'_1)\) and \(\hat{k}'' \cdot \hat{n}/k_3(k''_1)\). In evaluating the delta functions, it is necessary to divide by these derivatives. The solutions for \(k'_1\) and \(k''_1\) are given in equations (51) and (52) of the earlier notes, namely,

\[
\begin{align*}
k'_1 &= k_1 \cos \phi + k_3(k_1) \sin \phi, \\
k_3(k'_1) &= -k_1 \sin \phi + k_3(k_1) \cos \phi, \\
k''_1 &= k_1 \cos \phi - k_3(k_1) \sin \phi, \\
k_3(k''_1) &= k_1 \sin \phi + k_3(k_1) \cos \phi.
\end{align*}
\]  

(83)

When these results are used in (80) we obtain

\[
\begin{align*}
u(x_g, x_s, \omega) &= \frac{1}{8\pi} \int_C R(k_1, \omega) \frac{\mathbf{k} \cdot \mathbf{n}}{k_3(k_1)} dk_1 \exp\{i\Phi\},
\end{align*}
\]  

(84)

where, now, in the definition of \(\Phi\) in (43) above, \(\mathbf{x}\) is replaced by \(x_g\) and \(x'\) is replaced by \(x_s\). This is one fourth of the previously derived upward scattered field for this problem, as stated earlier. In each of the other integrals in (68), the derivatives of the delta functions produce exactly the right factor to “cancel” the corresponding multiplier in \(\hat{k}\), just as occurred here.

Having the form (71) is not apparently of any particular use, except that it makes literal the conceptual structure \(WRW\) for downward propagation, reflection, and upward propagation. As was demonstrated earlier, the application of the scalar adjoints, \(W^*uW^*\), produces an approximate inverse for both the horizontal and the dipping planar reflector. It remains to analyze the curved reflector.

**INVERSION OF DATA FOR A CURVED REFLECTOR**

We now consider processing of reflecton data for an arbitrary curved reflector. Clearly, in this case, we cannot write down an exact solution, so we must content ourselves with an asymptotic solution, namely, a Kirchhoff-approximate solution. We will apply the same inversion operator, \(W^*\), as given by equation (23) and show that we can image the reflector by calculating \(W^*uW^*_g\) for coincident downward propagated source and receiver. This analysis will be carried out in two dimensions, as were the earlier discussions of inversion. However, the asymptotic analysis of the amplitude for extraction of the reflectivity by this method has not been accomplished yet.

Application of the Kirchhoff approximation requires certain constraints on the curvature of the reflector, namely, that it satisfy the inequality,
\[ \Lambda = 2\pi f L/c \gg 1. \]

Here, \( f \) is the frequency in Hz, \( L \) is the radius of curvature of the reflector at the point(s) where the Kirchhoff approximation is to be applied and \( c \) is the propagation speed, as in previous sections.

Thus, we are abandoning analysis of the exact solution for this discussion. However, it should be pointed out that the very concept of a reflection coefficient is an asymptotic—geometrical optics—attribute of the scattering process for curved reflectors. Hence, we see no inconsistency here in using an asymptotic approximation. Quite the contrary, if we seek reflectivity, it would be questionable to use anything more than asymptotic analysis for a curved reflector.

For this discussion, we introduce the notation, \( u_I(x, x_s, \omega) \) for the downward propagating dipole response, (15). We assume that this field gives rise to an upward propagating wave \( u_R(x, x_s, \omega) \) at a reflecting curve, \( C \). Starting from (65) and using the Sommerfeld radiation condition, one can derive the following representation for \( u_R \) in terms of \( u_I \):

\[
\begin{align*}
    u_R(x_g, x_s, \omega) &= -\int_{x \in C} \left[ u_R(x(\sigma), x_s, \omega) \frac{\partial G(x_g, x(\sigma))}{\partial n} \\
    &+ G(x_g, x(\sigma)) \frac{\partial u_R(x(\sigma), x_s, \omega)}{\partial n} \right] d\sigma.
\end{align*}
\]

See Baker and Copson (1939), Bleistein (1984) In this equation, \( \sigma \) is arclength on the reflector, \( C \), \( G \) is the free-space Green's function, and \( \partial / \partial n \) is the downward directed normal derivative on the reflector.

The Kirchhoff approximation amounts to replacing the upward reflected field, \( u_R \), and its normal derivative on \( C \) by their geometrical optics approximations, namely,

\[
\begin{align*}
    u_R(x(\sigma), x_s, \omega) &= R(x(\sigma), x_s) u_I(x(\sigma), x_s, \omega), \\
    \frac{\partial u_R(x(\sigma), x_s, \omega)}{\partial n} &= -R(x(\sigma), x_s) \frac{\partial u_I(x(\sigma), x_s, \omega)}{\partial n}.
\end{align*}
\]

Here, \( R \) is the geometrical optics reflection coefficient, calculated for the incidence angle between the geometrical optics ray from \( x \) to \( x(\sigma) \) and the upward normal \( \hat{n}(\sigma) \) on the reflector. We prefer not to write down \( R \) in this spatial form, because we will immediately modify this result by using a wavenumber representation for both \( u_I \) and \( R \), below.

To this end, we return to the representation of the dipole response, (15), which we use for \( u_I \) and write down a corresponding representation for its normal derivative,
as well:

$$u_I(x, x_s, \omega) = \frac{1}{2\pi} \int_C \exp \{i\Phi(k_1, x, x_s)\} \, dk_1,$$

$$\frac{\partial u_I(x, x_s), \omega}{\partial n} = \frac{1}{2\pi} \int_C i\hat{k}(k_1) \cdot \hat{n} \exp \{i\Phi(k_1, x, x_s)\} \, dk_1.$$  \hspace{1cm} (87)

Here, $k_3(k_1)$ is given by the first expression in (4), and

$$x = (x, z), \quad x_s = (x_s, z_s), \quad k(k_1) = (k_1, k_3(k_1)),$$

$$\Phi(k_1, x, x_s) = k_1[x(\sigma) - x_s] + k_3(k_1)(z(\sigma) - z_s), \quad z > z_s.$$  \hspace{1cm} (88)

We introduce corresponding representations for $G$ and its normal derivative:

$$G(x_g, x, \omega) = -\frac{1}{4\pi i} \int_C \frac{1}{k_3(k'_1)} \exp \{i\Phi(-k'_1, x, x_g)\} \, dk'_1,$$

$$\frac{\partial G(x_g, x, \omega)}{\partial n} = -\frac{1}{4\pi} \int_C \frac{k(-k'_1) \cdot \hat{n}}{k_3(k'_1)} \exp \{i\Phi(k'_1, x, x_g)\} \, dk'_1.$$  \hspace{1cm} (89)

$$k(-k'_1) = (-k'_1, k_3(k'_1)) \quad x_g = (x_g, z_g), \quad z > z_g.$$

As a new feature, here, we will introduce a representation for the reflection coefficient under the integral sign in (87). Since the representation in the Fourier domain is just a plane wave decomposition, we need only use the plane wave representation of the reflection coefficient for a wave incident on a reflector with normal direction, $\hat{n}$, namely,

$$R(k_1, \omega, \sigma) = \frac{\hat{n} \cdot \hat{k} - \text{sign}[\omega \hat{n} \cdot \hat{k}] \sqrt{\omega^2/c^2 + \omega^2/c^2 + (\hat{n} \cdot \hat{k})^2}}{\hat{n} \cdot \hat{k} + \text{sign}[\omega \hat{n} \cdot \hat{k}] \sqrt{\omega^2/c^2 + \omega^2/c^2 + (\hat{n} \cdot \hat{k})^2}}, \quad \hat{k} = k(k_1).$$  \hspace{1cm} (90)

**Remark:** $R$ is really a function of $k_1/\omega$ and $\sigma$. To see this, one needs only to divide the numerator and the denominator here by $\omega$ and to realize that

$$k_3(k_1) = \omega \sqrt{1 - (k_1/\omega)^2},$$

because for real values of $k_3$, $\text{sign}(k_3) = \text{sign}(\omega)$; see (4).

We substitute the results (86) - (89) in (85) to obtain the result,

$$u_R(x_g, x_s, \omega) = \frac{1}{2(2\pi)^2} \int_{x \in C} \int_{k_1} dk_1 \int_{k'_1} dk'_1 \frac{k(k_1) \cdot \hat{n} + k(-k'_1) \cdot \hat{n}}{k_3(k'_1)} R(k_1, \omega, \sigma) \cdot \exp \{i\Phi(k, x, x_s) + i\Phi(k', x_g, x)\}.$$  \hspace{1cm} (91)
This equation should be compared to (41), which is the result for the dipping plane. We see here two additional integrations for this solution. However, if we specialize to the case of a dipping plane, this leading order asymptotic solution actually yields the exact result, (41). This is a somewhat trick calculation to carry out and I will only outline it here.

First, note that for a linear reflector, the phase is linear in $\sigma$ and $R$ is actually independent of $\sigma$. Thus, the $\sigma$-integration yields a Dirac delta function with argument,

$$(k_1 - k'_1) \cos \phi + (k_3(k_1) + k_3(k'_1) \sin \phi,$$

with $\phi$ being the dip angle as in the earlier sections.

We use this delta function to evaluate the $k'_1$-integral when this argument is zero, or when

$$k_1 \cos \phi + (k_3(k_1) \sin \phi = \lambda = k'_1 \cos \phi - k_3(k'_1) \sin \phi.$$

Here, we show the “auxiliary” variable, $\lambda$, because the representation (91) reduces to the result, (41) when the remaining $k_1$-integral is rewritten as an integral in $\lambda$.

We propose to apply the inversion operators, $W^*$, in source and receiver, fashioned from the basic definition (23) as follows. Set

$$W^*(x_s, \xi_s) = \frac{\text{sign}(\zeta_s - z_s)}{2\pi} \int d\kappa_1 \exp \left\{ -i \Phi(k_1, \xi_s, x_s) \right\}.$$  \hspace{1cm} (92)

and

$$W^*(x_g, \xi_g) = \frac{\text{sign}(z_g - \zeta_g)}{2\pi} \int d\kappa'_1 \exp \left\{ -i \Phi(-k'_1, \xi_g, x_g) \right\}.$$  \hspace{1cm} (93)

We do not need to be concerned with complex conjugates in this discussion. In considering only the leading order asymptotic solution, we are neglecting evanescent waves—imaginary $k_3$ values. It is our point of view that the evidence of the analysis of the previous sections is overwhelming that the evanescent energy contributes little to the inversion process in this method. Thus, we do not consider this approximation as a serious loss of accuracy in our analysis.

We now multiply the upward propagating wave representation (91) by these two functions and integrate over $x_s$ and $x_g$ in order to carry out the approximate inversion, $W^*uW^*$, as in (29). This equation takes the form,

$$W^*uW^* = \frac{1}{4(2\pi)^4} \int dx_s dx_g dk_1 dk'_1 d\kappa_1 d\kappa'_1 d\sigma$$

$$k(k_1) \cdot \hat{n} + k(-k'_1) \cdot \hat{n}$$

$$k_3(k'_1)$$

$$\cdot R(k_1, \omega, \sigma)$$

$$\cdot \exp \left\{ i \Psi(k_1, k'_1, \kappa_1, \kappa'_1, x, x_s, x_g) \right\},$$  \hspace{1cm} (94)

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where,
\[
\Psi(k_1, k'_1, \kappa_1, \kappa'_1, x, x_g) = \Phi(k_1, x, x_g) + \Phi(-k'_1, x, x_g)
- \Phi(\kappa_1, \xi, x_g) - \Phi(-\kappa'_1, \xi, x_g)
\]
or,
\[
\Psi(k_1, k'_1, \kappa_1, \kappa'_1, x, x_g) = k_1(x - x_g) + k_3(k_1)(z - z_g)
-k'_1(x - x_g) + k_3(k'_1)(z - z_g)
-\kappa_1(\xi - x_g) - k_3(\kappa_1)(\zeta - z_g)
+\kappa'_1(\xi_g - x_g) - k_3(\kappa'_1)(\zeta_g - z_g).
\]  
(95)

As in previous discussions of this process, the integrals in \(x\) and \(x_g\) produce Dirac delta functions with arguments, \(k_1 - \kappa_1\) and \(k'_1 - \kappa'_1\), respectively. Hence, we can then carry out those two integrations, as well, yielding the result for (94),
\[
W^*uW^* = -\frac{1}{4(2\pi)^2} \int dx dx_g dk_1 dk'_1 d\sigma \\
\cdot \frac{k(k_1) \cdot \hat{n} + k(-k'_1) \cdot \hat{n}}{k_3(k'_1)} R(k_1, \omega, \sigma) \\
\cdot \exp \left\{ i\Phi(k_1, x, \xi) + i\Phi(k'_1, x, \xi_g) \right\},
\]  
(96)

\[\xi = (\xi, \zeta), \quad \xi_g = (\xi_g, \zeta_g).\]

At first glance, it is not clear that such an integral could provide an image of the reflector and information about the reflection coefficient. In fact, this integral will peak on the reflector. To make this more plausible, we propose to introduce alternative integration variables in place of \(k_1\) and \(k'_1\). For each point on the reflector—that is, for each \(\sigma\)—we use the dip angle of the tangent to define this new pair of wave number variables, as follows:
\[
\lambda_1 = k_1 \cos \phi + k_3(k_1) \sin \phi, \quad \lambda'_1 = -k'_1 \cos \phi + k_3(k'_1) \sin \phi.
\]  
(97)

This makes the new wave numbers functions of \(\sigma\) as well as functions of the old wave number variables. For these variables, one can verify the auxiliary relationships,
\[
k_3(\lambda_1) = -k_1 \sin \phi + k_3(k_1) \sin \phi, \quad \text{and} \quad k_3(\lambda'_1) = k_1 \sin \phi + k_3(k_1) \sin \phi.
\]  
(98)
In terms of these new variables, the representation (96) becomes

\[
W^* u W^* = -\frac{1}{4(2\pi)^2} \int_{\mathcal{C}} d\sigma \int d\lambda_1 d\lambda_1' \frac{k_3(\lambda_1) + \lambda_1' \sin 2\phi + k_3(\lambda_1') \cos 2\phi}{k_3(\lambda_1')} \frac{\lambda_1 \sin \phi + k_3(\lambda_1) \cos \phi}{k_3(\lambda_1)} \tag{99}
\]

\[R(k_1, \omega, \sigma) \exp \{i\Theta(\lambda_1, s_s, n_s) + i\Theta(\lambda_1', s_g, n_g)\}.
\]

In this equation,

\[
\Theta(\lambda_1, s_s, n_s) = \lambda_1 s_s + k_3(\lambda_1) n_s, \\
\Theta(\lambda_1', s_g, n_g) = \lambda_1' s_g + k_3(\lambda_1') n_g, 
\tag{100}
\]

with the signed distances as shown in Figure 14 and given by

\[
s_s = (x - \xi_s) \cos \phi + (z - \zeta_s) \sin \phi, \quad n_s = -(x - \xi_s) \sin \phi + (z - \zeta_s) \cos \phi, \\
s_g = (x - \xi_g) \cos \phi + (z - \zeta_g) \sin \phi, \quad n_g = -(x - \xi_g) \sin \phi + (z - \zeta_g) \cos \phi. \tag{101}
\]

**Remark:** For the planar reflector, integration in \(\sigma\) now yields the delta function, \(\delta(\lambda_1 + \lambda_1')\). Evaluation of the \(\lambda_1'\) integral again leads to the result, (41).

FIG. 14. Curved reflector coordinates.
Imaging

This is as far as the analysis has progressed on this problem at this time. However, there is enough information here to predict imaging. To see why this is so, consider the final formulas for inversion, (96) or (99). Note that to obtain an image of the reflector in earlier sections, we proposed that the data be processed for coincident source and receiver, that is for

\[ \xi_s = \xi_g = \xi. \]

Let us consider that case now and, further, let us suppose that \( x \) is on the reflector, \( C \). In this case, there is one choice of \( \sigma \), say \( \sigma = \sigma_0 \), for which \( s_s = s_g = n_s = n_g = 0 \) and the oscillatory exponential no longer appears in the integrand. In this case, we expect that the result of the \( \lambda_1 \) and \( \lambda'_1 \) integrals would be much larger than for any other choice of \( \sigma \). That is, the pair of integrals in \( \lambda_1 \) and \( \lambda'_1 \) behave something like a delta function, \( \text{delta}(\sigma - \sigma_0) \). This is mathematical imaging!

To go further with this type of argument, suppose that \( \xi \) is “near” \( C \) and we choose for \( \sigma_0 \) the value that identifies the point on \( C \) closest to \( \xi \). We expect this value to be a critical value of the asymptotic analysis of the \( \sigma \)-integral. In this case, let us consider linearizing the exponent around \( \sigma = \sigma_0 \) and evaluating the amplitude at \( \sigma = \sigma_0 \) as a first order approximation of the integral.

We already know the result of this linearized analysis: \( C \) is replaced by a dipping plane with dip angle, \( \phi = \phi(\sigma_0) \). In this case, we showed in this section, that the double integral in \( \sigma \) and \( \lambda'/\lambda \) effectively reduce to the processing for a dipping linear reflector of an earlier section, just as the forward Kirchhoff model reduces exactly to the model of a plane when \( C \) was specialized to this case. Analysis of the processing formalism for the dipping plane was carried out earlier and, indeed, we saw that the output of the formalism was an image of the reflector.

Thus, out intuition tells us that the result we seek is contained in the formulas (96) or (99); it only remains to be carried out in detail.

In summary, what we have done here is model propagating part of the energy from a reflector as by the Kirchhoff approximation and apply the \( W^*uW^* \) formalism to that representation. We have proceeded far enough with the analysis to see that it is at least very likely that this output will produce an image of the curved reflector. The details of this latter analysis are a subject for further study.

CONCLUSIONS

The objective of this paper was to expose research in progress on the development of a continuum analog of the Berkhout inversion formalism. Beyond the desire for a basic understanding, it is hoped that this study will lead to methods of adapting results based on stationary phase for integral inversion operators to the discrete Berkhout inversion. We have in mind here the development a 2.5D Berkhout formalism as well as methods of processing for geometrical attributes, such as incidence
angle, travel time, etc.—with this discrete formalism. Furthermore, it is hoped that by asymptotic analysis of the continuum analog, the artefacts of limited aperture for this method can be better understood.

An important new idea that has been exposed here is that for full aperture data from a reflector, each fixed frequency leads to an image of the reflector. This suggests the possibility of processing with different velocities for different frequencies, thereby accounting for dispersion. It also suggests the possibility of developing a velocity analysis scheme in which one examines residual moveout as a function of frequency, thereby developing a frequency dependent background velocity—essentially, deriving a dispersion relationship from residual moveout in frequency.

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