Inversion of Reflection Traveltimes for Transverse Isotropy

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ABSTRACT

In conventional velocity analysis of reflection data, it is often assumed that the short-spread stacking velocity equals the root-mean-square vertical velocity. This assumption is not valid in the presence of anisotropy, where it is impossible to recover the vertical velocity (or the reflector depth) using hyperbolic moveout analysis on short-spread common-midpoint (CMP) gathers, even if both P- and SV-waves are recorded.

Hence, we examine the feasibility of inverting long-spread (nonhyperbolic) reflection moveouts for parameters of transversely isotropic media with a vertical symmetry axis. One possible solution is to also recover the quartic term of the Taylor series expansion for $t^2 - x^2$ curves for P- and SV-waves, and to use it to determine the anisotropy. However, this procedure turns out to be unstable, due to ambiguity in the joint inversion of intermediate-spread (i.e., spreads of about 1.5 times the reflector depth) P- and SV-moveouts. The nonuniqueness cannot be overcome by using long spreads (twice as large as the reflector depth), if only P-wave data are included. A general analysis of the P-wave inverse problem proves the existence of a broad set of models with different vertical velocities, all of which provide a satisfactory fit to the exact traveltimes. This strong ambiguity is explained by trade-off between vertical velocity and the parameters of anisotropy on gathers with a limited angle coverage.

The accuracy of the inversion procedure may be significantly increased by combining both long-spread P- and SV-moveouts. The high sensitivity of the SV-moveout near the SV-wave velocity maximum to the reflector depth permits a less ambiguous inversion. In some cases, the SV-moveout alone may be used to recover the vertical S-wave velocity and, hence, the depth. Success of this inversion depends on the spreadlength and degree of SV-wave velocity anisotropy, as well as on the constraints on the P-wave vertical velocity.

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INTRODUCTION

One of the common assumptions in conventional velocity analysis of reflection seismic data is the identification of the moveout velocity (determined by semblance analysis on CMP gathers) with the vertical root-mean-square (RMS) velocity (c.f., Taner and Koehler, 1969). If the RMS velocity in a horizontally layered isotropic medium is found, recovery of the interval velocities, and time-to-depth conversion, can be easily performed using variations of the Dix (1955) formula. This simple approach is invalid for anisotropic media, since the short-spread moveout moveout velocity is not equal to the RMS vertical velocity (c.f., Tsvankin and Thomsen 1994, cited below as TT1, and many prior works cited therein). In the presence of anisotropy, inversion of moveout velocities by means of the Dix formula results in errors in interval velocities and, therefore, in inaccurate estimates of the reflector depth. A good example of mis-ties in time-to-depth conversion due to anisotropy was given by Banik (1984).

Not only velocity analysis, but practically all other conventional seismic processing and interpretation techniques become inaccurate if the medium is anisotropic. Recent papers have discussed errors caused by anisotropy in dip moveout (DMO) removal (Larner, 1993; Tsvankin, 1994), migration (Lynn et al., 1991; Alkhalifah and Larner, 1994), and amplitude-versus-offset (AVO) analysis (Wright, 1987). However, distortions in velocity analysis are especially dangerous, because they propagate into all subsequent processing steps.

Inversion of reflection data in the presence of anisotropy has two principal aspects. On the one hand, it is important to be able to look "past" anisotropy (i.e., correct for anisotropy) when recovering vertical velocity and performing such processing steps as time-to-depth conversion, migration, and dip moveout. For instance, Alkhalifah and Larner (1994) showed that accurate 2-D imaging in transversely isotropic media requires good estimates of the anisotropy parameters of Thomsen (1986) — δ and ϵ. On the other hand, it may be also important to look "at" the anisotropy, for example by using the anisotropic coefficients in lithology inversion.

Here, we consider a common anisotropic model: horizontally-layered, transversely isotropic media with a vertical symmetry axis (VTI media). Seismic velocities in such media vary in the vertical plane, but not azimuthally. VTI formations have been documented in a number of publications (e.g., White et al., 1983; Robertson and Corrigan, 1983; Banik, 1984; Sams et al., 1993). The present conclusions may be extended to azimuthally anisotropic media if the surveys are performed along the principal directions of such anisotropy, i.e., if the incidence plane represents a plane of symmetry.

Most previous work on the inversion of reflection data in anisotropic media has been focused on recovering the anisotropic coefficients in the case when the vertical velocity (or the layer-thicknesses) is known (e.g., Banik, 1984; Winterstein, 1986; Sena, 1991). For instance, Byun and Corrigan (1990) suggested a technique to obtain all five elastic constants for layered transversely isotropic media from *P*- and *SH*-data (we omit the qualifiers in "quasi-P-wave" and "quasi-SV-wave"). They developed a "skewed" hyperbolic formula to recover the long-spread P-wave moveout curve and employed a numerical algorithm to find the elastic parameters, in a layer-stripping mode. Sena (1991) derived an analytic version of the "skewed" hyperbolic formula using the weak anisotropy approximation and applied it to obtain the interval elastic parameters without time-consuming numerical search. However, as shown by TT1, the domain of validity of that formalism is rather limited. In principle, if the vertical velocities (or layer thicknesses) are known, the anisotropic coefficients may be determined from short-spread moveouts (P, SV, and SH) alone. Byun and Corrigan (1990) and Sena (1991) had to use long-spread P-wave moveout (along with vertical velocity) because they did not include SV data.

Here we treat the more general problem, important in the exploration context, where all model parameters (except for the type of symmetry and orientation of the symmetry axis) are unknown. In this case, the inverse problem cannot be solved by means of the conventional hyperbolic moveout analysis on short-spread gathers, even if all waves (P, SV, and SH) are recorded. The goal of this paper is to examine the feasibility of including long-spread (nonhyperbolic) reflection moveouts in the inversion procedure. First, using analytical results of TT1, we examine an inversion technique based on the quartic Taylor series for $t^2 - x^2$ curves. This algorithm turns out to be unstable, due to trade-off between quadratic and quartic moveout coefficients. Then, we carry out direct numerical analysis of the objective function for the kinematic inverse problem and establish the conditions necessary to avoid ambiguous solutions, given realistic uncertainty in traveltimes.

TRAVELTIME INVERSION USING THE QUARTIC TAYLOR SERIES

Squared arrival times of reflected waves may be approximated by the Taylor series expansion near vertical (Taner and Koehler, 1969; Hake, et al., 1984):

$$t_T^2 = A_0 + A_2 x^2 + A_4 x^4 + \dots, \tag{1}$$

with the coefficients

$$A_0 = t_0^2, \quad A_2 = \left.\frac{dt^2}{dx^2}\right|_{x=0}, \quad A_4 = \frac{1}{2}\frac{d}{dx^2}\left(\frac{dt^2}{dx^2}\right)\Bigg|_{x=0}, \tag{2}$$

t_0 is the vertical arrival time. The short-spread moveout velocity is given by $V_2^2 = 1/A_2$. In the conventional hyperbolic approximation, expansion (1) is truncated after the second (quadratic) term and the measured moveout velocity is identified with the analytic short-spread value V_2. In this section, we briefly explore the natural idea of including the next (quartic) Taylor series coefficient in the inversion procedure.

The transversely isotropic model may be characterized by the elastic moduli C_{ij}, or alternatively by the P- and S-wave vertical velocities (V_{P0} and V_{S0}) plus three dimensionless parameters of anisotropy, introduced by Thomsen (1986):

$$\epsilon = \frac{C_{11} - C_{33}}{2\,C_{33}}\,, \tag{3}$$

$$\gamma = \frac{C_{66} - C_{44}}{2\,C_{44}}\,, \tag{4}$$

$$\delta = \frac{(C_{13} + C_{44})^2 - (C_{33} - C_{44})^2}{2\,C_{33}(C_{33} - C_{44})}\,. \tag{5}$$

ϵ and γ are the conventional measures of *P*- and *SH*-wave velocity anisotropy (respectively), and are close to the fractional differences between the horizontal and vertical velocities. The parameter δ influences P-SV propagation, especially the *P*-wave velocity at near-vertical incidence. All three parameters equal zero in isotropic media.

The short-spread moveout velocities (also called normal-moveout velocities) in a horizontally layered transversely isotropic model are given by Hake et al. (1984):

$$V_2^2 = \lim_{x \to 0} \frac{dx^2}{dt^2} = \frac{1}{t_0} \sum_{i=1}^{N} V_{2i}^2\,\Delta t_i\,, \tag{6}$$

where V_{2i} and Δt_i are the short-spread moveout velocity and two-way vertical traveltime in layer i. The values of V_{2i}^2 for different wave types can be expressed through the anisotropic coefficients as (Thomsen, 1986):

$$V_{2i}^2(P) = V_{P0i}^2(1 + 2\delta_i), \tag{7}$$
$$V_{2i}^2(SV) = V_{S0i}^2(1 + 2\sigma_i), \tag{8}$$
$$V_{2i}^2(SH) = V_{S0i}^2(1 + 2\gamma_i), \tag{9}$$

with

$$\sigma = \left(\frac{V_{p0}}{V_{s0}}\right)^2 (\epsilon - \delta). \tag{10}$$

The coefficient σ was introduced in TT1 as the most influential parameter in the SV-wave moveout and velocity equations. Coefficient σ reduces to zero both in isotropic and elliptically anisotropic media; for elliptical anisotropy, $\epsilon = \delta$.

It is clear from (7-9) that the short-spread moveout velocity given by (6) equals the RMS vertical velocity ($V_{RMS} = \frac{1}{t_0}\sum_i V_{0i}^2 \Delta t_i$) only if the anisotropic coefficients (4,5,10) are zero. Hence, if we equate the measured stacking (moveout) velocity to V_2 and try to derive interval vertical velocities V_{0i}^2 from V_2^2 by applying the Dix formula (as is usually done in conventional processing), we get instead the values V_{2i}^2, which contain contributions of the anisotropic parameters δ_i, σ_i, or γ_i (depending on wave type). Thus, application of the Dix formula in anisotropic formations results in erroneous interval velocities, hence inaccurate estimations of reflector depths, i.e., in mis-ties in time-to-depth conversion (c.f., e.g., Banik, 1984).

If the reflector depth (or at least one of the vertical velocities) is known, the short-spread moveout velocities are sufficient to recover the anisotropic coefficients. For instance, if the P-wave vertical velocity V_{P0} in a certain layer is determined, we can find the vertical S-wave velocity using the vertical P and S traveltimes t_{P0} and t_{S0} for this layer: $V_{S0} = V_{P0}t_{P0}/t_{S0}$. Having thus obtained both vertical velocities, we can then recover the anisotropies from the moveout velocities (7-9). If only P data are available, the short-spread moveout and vertical velocity enable us to determine a single anisotropic coefficient δ from equation (7).

The question we will address is: how can we invert reflection moveouts for the true vertical velocities and parameters of anisotropy, given the simple VTI model, without such prior information. From equations (6-9) it is clear that conventional hyperbolic moveout analysis does not provide enough data to solve this problem. Even if all three short-spread moveout velocities in a VTI layer are measured (plus the ratio $V_{P0}/V_{S0} = t_{S0}/t_{P0}$, independent of anisotropy), these four measurements are insufficient to determine the five parameters. In particular, neither vertical velocity may be determined. The combination of the short-spread velocities [equations (7-9)] and the vertical arrival times is sufficient to solve the inverse problem only with an artificial assumption, e.g., elliptical anisotropy or no anisotropy. It is difficult even to *detect* the presence of transverse isotropy in short-spread CMP gathers, especially if only P data are available. The only diagnostic of anisotropy on short spreads is the difference between the moveout velocities of SV and SH-waves.

The inadequacy of short-spread moveout represents a fundamental problem in velocity analysis for anisotropic media. In isotropic media, nonhyperbolic (long-spread) moveout is necessary only in certain applications (e.g., in AVO analysis, suppression of multiples, processing of shallow reflections), while velocity inversion (for horizontally layered media) can be performed using short spreads alone. However, in the presence of anisotropy, recovery of the true vertical velocity from reflection traveltimes requires (at a minimum) analysis of nonhyperbolic moveout on long spreads.

Thus, while in conventional processing nonhyperbolic moveout is usually considered as a hindrance that distorts velocity estimation and deteriorates the quality of stacked sections, such information is necessary for solution of the inverse problem in anisotropic media. In fact, we prefer to work with maximum deviations from hyperbolic moveout, in order to separate the vertical velocities and the parameters of anisotropy.

This strategy is obviously hopeless for elliptically anisotropic media, where P-, SV-, and SH-moveouts in a single layer are purely hyperbolic (in multilayered media, moveout is nonhyperbolic due to ray bending). In elliptical media ($\sigma = 0$), however, the SV-wave short-spread moveout velocity alone can provide us with the true vertical velocity and reflector depth [c.f., equation (8)]. In any case, elliptical anisotropy is an idealization based on mathematical convenience, whose occurrence in nature is vanishingly rare (Thomsen, 1986).

One possible way to use nonhyperbolic moveout in the inversion procedure is to recover the fourth-order Taylor series coefficients A_4 from long-spread reflection moveouts. Analytical expressions for the coefficient A_4 are derived in TT1 for P- and SV-wave reflections; in a single transversely isotropic layer we have

$$A_4(P) = -\frac{2(\epsilon-\delta)}{t_{P0}^2\, V_{P0}^4} \frac{1+\frac{2\delta}{1-V_{S0}^2/V_{P0}^2}}{(1+2\delta)^4}, \tag{11}$$

$$A_4(SV) = \frac{2\sigma}{t_{S0}^2\, V_{S0}^4} \frac{1+\frac{2\delta}{1-V_{S0}^2/V_{P0}^2}}{(1+2\sigma)^4}, \tag{12}$$

$$A_4(SH) = 0\,. \tag{13}$$

For multilayered media, the coefficient A_4 is given by

$$A_4(P \text{ or } SV) = \frac{\left(\Sigma_i\, V_{2i}^2 \Delta t_i\right)^2 - t_0\, \Sigma_i V_{2i}^4 \Delta t_i}{4\left(\Sigma_i V_{2i}^2 \Delta t_i\right)^4} + \frac{t_0 \Sigma_i A_{4i} V_{2i}^8 \Delta t_i^3}{\left(\Sigma_i V_{2i}^2 \Delta t_i\right)^4}, \tag{14}$$

which includes (in the first term) ray-bending due to the layered structure. Here, A_{4i} is the quartic coefficient A_4 [equations (11-13)] for layer i. Formulas (11-14) are valid for arbitrary (not just weak) transverse isotropy; the values of anisotropic parameters govern only the maximum offset $x_{\max}$ to which the Taylor series (1) may be accurately applied.

In principle, expressions (11-14) make it possible to obtain the vertical velocities and anisotropic parameters from the second- and fourth-order Taylor series coefficients. The main steps of such an algorithm are:

1. Find the three Taylor series coefficients for the $t^2 - x^2$ curves corresponding to the reflections from the top and from the bottom of any particular layer, preferably using all three (P, SV, and SH) modes.

2. Apply the Dix-type formulas derived in TT1 to recover the Taylor series coefficients A_{2i}, A_{4i} for the layer. If the SV-wave is not recorded, as shown in TT1, the SV-wave coefficients may be obtained from the coefficients of the P and converted P-SV waves.

3. Invert the coefficients A_{2i} (V_{2i}) [equations (7,8,9)] and A_{4i} [equations (11-13)], in combination with the vertical arrival times, for the vertical velocities and anisotropies.

For P or SV propagation, we are searching for four unknown parameters for each layer: V_{P0}, V_{S0}, δ, and ϵ (or σ); the thickness of the layer can be obtained from the vertical velocities and arrival times. It is important to mention that P-wave traveltimes are determined almost entirely by three parameters: V_{P0}, δ, and ϵ. Although the ratio V_{P0}/V_{S0} can slightly change the quartic coefficient $A_4(P)$ [equation (11)], the influence of the shear-wave vertical velocity V_{S0} on P-wave traveltimes is practically negligible, even for long spreads and strong anisotropy (see TT1 and Tsvankin, 1994). However, the quadratic and quartic P-wave Taylor series coefficients alone (7,11) are not sufficient to recover the three unknowns: V_{P0}, δ, and ϵ.

If both P and SV data are used, the ratio V_{P0}/V_{S0} (independent of the unknown reflector depth z) can be determined from the ratio of vertical arrival times, and so the number of unknowns is still three. In principle, the problem can be solved using the P and SV second-order Taylor series coefficients [short-spread moveout velocities in equations (7,8)], plus one of the fourth-order coefficients (11,12). The other fourth-order coefficient provides redundancy.

The above algorithm seems to be quite straightforward. However, the crucial point in this inversion is in step 1, i.e., in the possibility of recovering the fourth-order coefficient A_4 from reflection data. The analytic three-term (fourth-order) Taylor series (1) diverges from the exact traveltimes even for $x_{\max}/z \approx 1.5$ (TT1). For these spreads and plausible values of anisotropy, the quartic coefficient of the three-term Taylor series determined by the least-squares method from the exact traveltimes is substantially different from the analytic values [equations (11,12)]. Hence, the quartic Taylor series (1) may not be used in the inversion.

TT1, however, introduced a better nonhyperbolic moveout approximation:

$$t_A^2 = t_0^2 + A_2 x^2 + \frac{A_4 x^4}{1 + A\, x^2}. \tag{15}$$

with

$$A = V_h^2 A_4/(1 - A_2 V_h^2),$$

where V_h is the horizontal velocity. It has a form similar to that for weak anisotropy, but remains numerically accurate in the description of /it P-wave moveout for strong anisotropy and long spreads ($x_{\max}/z = 2$ and larger). TT1 shows that approximation (15) may be successfully used for nonhyperbolic moveout correction, even for pronounced deviations from the hyperbola, which cannot be handled by the quartic Taylor series (1).

In order to obtain the coefficients A_2, A_4, and A, we have performed the least-squares fitting of equation (15) to calculated arrival times in models discussed by TT1. When the exact P- wave traveltimes are used, the quartic coefficient A_4 can be recovered with relatively good accuracy for intermediate spreads up to about $1.5z$ (Figure 1).

However, if plausible errors in traveltimes are admitted, the second-order coefficient A_2 remains relatively well-determined, while the fourth-order coefficient A_4 does not. Small variations in traveltimes cause significant deviations of A_4 from the exact value, for both P- and SV-waves. This means that models with markedly different quartic coefficients and slightly different quadratic coefficients may have almost identical moveout curves; examples of this kind will be discussed in the next section.

The failure of the "direct" inversion technique, based on the quartic Taylor series, stems from the ambiguity in the joint inversion of P and SV intermediate-spread moveouts ($x_{\max} \approx 1.5z$). The results discussed in this section have prompted us to address the general issue of ambiguity in the inversion of reflection traveltimes for transverse isotropy.

NUMERICAL ANALYSIS OF THE NONUNIQUENESS OF THE INVERSE PROBLEM

The results of the previous section show that the major problem in the inversion of reflection traveltimes for transverse isotropy is not how to carry out the inversion, but, rather, what types of data are necessary for unambiguous inversion. Ambiguity is a typical feature of most geophysical problems; usually the interpreter is satisfied with a solution that fits the experimental data and seems reasonable from the geological standpoint. This approach is difficult to follow in anisotropic media, because our understanding of what anisotropy is reasonable in real rocks is still rather poor. Therefore, here we examine directly the objective function for the problem at hand, to find out what kind of ambiguity exists and what data are necessary for unambiguous inversion, given realistic uncertainty in traveltimes.

We consider the inversion of P- and SV-reflection moveouts for the simple model of a single transversely isotropic layer. In the following analysis, it is convenient to replace the parameters δ and ϵ (or δ and σ) as independent variables by the short-spread moveout velocities of the P- and SV-waves [hereafter denoted as V_{P2} and V_{S2} and determined through equations (7,8)]. Therefore, the layer will be described by four velocities: V_{P0}, V_{S0}, V_{P2}, V_{S2}, and the unknown thickness z. The parameters A_4 and A of equation (15) may be calculated directly from these.

Application of any formalized inversion algorithm would enable us to recover some "best" set of model parameters, but the degree of ambiguity of the traveltime problem would remain unknown. Instead, we use the following procedure to give a direct estimate of the nonuniqueness of the inverse problem:

1. The moveout curves for P- and/or SV-waves were calculated for two models, Taylor sandstone and Dog Creek shale (Figures 2 and 3), taken from Thomsen (1986), and used extensively in TT1. Both have positive σ, an important characteristic discussed in TT1.

2. For each, the model parameters were systematically varied, within a reasonable range, and a multidimensional objective function (error surface in model-parameter space) was constructed in the neighborhood of the exact solution.

3. The set of equivalent models (given a certain level of accuracy, and for a certain kind of input data) was determined.

In essence, we have performed an extensive search in the model space to determine the behavior of the objective function near the exact solution. As the objective function, we used the root-mean-square value of time residuals calculated with respect to the reference (exact) curve:

$$\Delta t_{RMS} = \sqrt{\frac{1}{M} \sum_{j=1}^{M} \Delta t_j^2}, \tag{16}$$

where M is the number of receivers.

Inversion of P-wave traveltimes

Since P-waves constitute the overwhelming majority of all seismic data being acquired in the oil industry, the most important question is whether long-spread P-wave moveout alone is sufficient for unambiguous inversion. As shown above, intermediate-spread ($x_{\max} \approx 1.5z$) P-wave moveout is not sufficient to resolve the quartic moveout coefficient A_4. In this section, we extend the spreadlength up to $x_{\max} = 2z$ (corresponding to a maximum incidence angle of 45 degrees) to determine what kind of information can be recovered from long-spread P-wave data.

Since the shear vertical velocity V_{S0} has a negligible influence on P-wave traveltimes, in the analysis of the objective function for the P-wave inverse problem we deal with three variables: V_{P0}, V_{P2} and V_{S2}; in calculating V_{S2}, we used the correct value for V_{P0}/V_{S0} ratio. The depth of the reflector z was computed through the vertical velocity, as $z = V_{P0} t_{P0}/2$, and t_{P0} was fixed at the correct value.

Figures 4 and 5 illustrate our numerical procedure. First, we calculated exact traveltimes for the reference model (in this case, Taylor sandstone) on the spread $x_{\max} = 2z$. Then, for each pair (V_{P2}, V_{S2}) of moveout velocities within a certain range around the exact (reference) values, we scanned vertical velocity V_{P0} and calculated the reflection times for each model. Then we computed the RMS time residual (16) with respect to the reference model, and picked the model with the minimum Δt_{RMS}. The values of Δt_{RMS} for these best-fit models are shown in the plane ($\bar{V}_{P2}$, $\bar{V}_{S2}$) in Figure 4, where $\bar{V}_{P2}$ and $\bar{V}_{S2}$ are the parameters normalized by the short-spread moveout velocities for the reference model; the corresponding values of V_{P0} are shown in Figure 5. The centers of the plots in Figures 4 and 5 represent the results for the exact (reference) model. Figure 4 may be considered as a special projection of the objective function containing only local minima of Δt_{RMS} for each pair (V_{P2}, V_{S2}).

Comparison of Figures 4 and 5 makes it possible to estimate the ambiguity of the P-wave traveltime inversion. The figures show only narrow intervals of V_{P2} and V_{S2} (limited within ± 2 percent of the correct values), indicating highly resolved moveout velocities, and Δt_{RMS} for the best-fit models are indeed small (Figure 4). Nonetheless, the corresponding vertical P-wave velocity may be far different from the value for the reference model (Figure 5). This means that there is a broad set of models with different V_{P0}, whose time responses almost coincide with one another, even for the spreadlength $x_{\max} = 2z$.

For brevity, similar figures for Dog Creek shale are omitted. However, the results for both Taylor sandstone and Dog Creek shale are summarized in Figure 6. The error in V_{P0} is calculated as the maximum deviation in the vertical velocity among the models with a given time residual. For instance, some models with $\Delta t_{RMS} \leq 2$ ms have vertical velocities that are different by 20 percent from the correct value. As mentioned above, the depth z of the boundary is changed along with V_{P0}, to keep t_{P0} constant.

This procedure shows that the P-wave traveltime inversion problem is highly ambiguous, even for long spreads ($x_{\max} = 2z$). The actual nonuniqueness is even greater, since we have considered the moveout velocities V_{P2}, V_{S2} to be well-resolved. This ambiguity is caused by the trade-off between the velocities V_{P0}, V_{P2}, and V_{S2} (or between V_{P0} and anisotropies δ and ϵ). The most influential parameter is the short-spread moveout velocity V_{P2}, which must be close to the correct value if time residuals are to be small. Keeping V_{P2} constant, we may change V_{P0} and V_{S2} together so that the average time residual remains almost the same up to at least $x_{\max} = 2z$ (Figure 4).

If V_{P2} coincides with the exact value, we may achieve an almost ideal coincidence of the traveltimes (Figure 6) using V_{P0}, differing by about 3-4 percent from the correct value. When V_{P2} contains an error of about 1-3 percent, it is still possible to get small time residuals by compensating for this change by much more pronounced alterations in V_{P0} and V_{S2} (implying a corresponding change in the quartic coefficient).

The above results demonstrate that the extension of spreadlength to $2z$ has not even eliminated the trade-off between the quadratic and quartic moveout terms found in the previous section on smaller (intermediate) spreads. For instance, a model with $V_{P0} = 3.609$ km/s, $\epsilon = 0.021$, $\delta = -0.087$, $z = 3.215$ km and the reference model of Taylor sandstone (with $z = 3$ km) yield practically indistinguishable P-wave moveout curves up to $x_{\max} = 2z$, although the magnitude of the quartic coefficient A_4 for Taylor sandstone is 24 percent higher than that for the erroneous model. However, the short-spread moveout velocity for Taylor sandstone is 1 percent smaller, and the trade-off between the hyperbolic and nonhyperbolic terms [see equation (15)] almost eliminates the difference in P-wave traveltimes between the two models up to at least $x_{\max} = 2z$.

Thus, the only parameter tightly constrained by P-wave traveltimes on the spread $x_{\max} = 2z$ is still the short-spread moveout velocity. The magnitude of P-wave nonhyperbolic moveout, although not insignificant for $x_{\max} = 2z$, nevertheless is not sufficient to recover the quartic coefficient with acceptable accuracy.

The accuracy in V_{P0} is less for Dog Creek shale than for Taylor sandstone because the P-wave moveout for the former model is more close to a hyperbola, due to the smaller quartic Taylor series term (TT1). Clearly, for purely hyperbolic moveout the vertical velocity cannot be resolved at all (the conventional velocity analysis "succeeds" only because of the artificial assumption of zero anisotropy). The difference between the results for the two models would be even more pronounced if we normalized Δt_{RMS} by the vertical arrival time t_0. From equations (7) and (8), it is clear that percentage errors in δ and $\epsilon(\sigma)$ are much greater than the corresponding errors in V_{P0} and V_{S2}.

In short, the objective function for the P-wave inverse problem has too flat a minimum near the exact solution to ensure a nearly unique inversion result, even for relatively small errors in traveltimes. While some of the kinematically equivalent models can be disregarded on the basis of unrealistic values of the anisotropic coefficients, many other models are equally plausible, unless some additional information is available. These conclusions are valid for transversely isotropic models with typical values of the anisotropic coefficients ϵ and δ; the present procedure can be used to test the ambiguity of any inverse problem.

Inversion of P and SV data

One natural way to reduce ambiguity is to combine P-wave data with SV-wave data. SV traveltimes depend on the same three unknowns used in the P-wave problem (V_{P0}, V_{S2}, and V_{P2} or, alternatively, V_{P0}, ϵ, and δ) and the shear vertical velocity V_{S0}. Further, since V_{S0} can be determined through V_{P0} as $V_{S0} = V_{P0}t_{P0}/t_{S0}$, the number of unknowns remains the same, while the amount of data is increased. As before, the depth of the layer is expressed through the P-wave vertical velocity as $z = V_{P0}t_{P0}/2$; again, we fix t_{P0} and t_{S0} at the correct values. In general, we can expect the vertical times to be better resolved than the short-spread moveout velocities which, in turn, are better resolved than the quartic moveout coefficients.

As mentioned above, here we consider models with positive σ. Media with negative σ ($\epsilon - \delta < 0$) require a special analysis because SV-moveout may become strongly nonhyperbolic even on short spreads (TT1). However, existing measurements at seismic frequencies indicate predominantly positive σ (Thomsen, 1986; TT1).

Despite the addition of SV traveltimes, the inversion remains nonunique for intermediate SV-spreadlength ($x_{\max} = 1.5z$). As an example, one of the equivalent models for Dog Creek shale is shown in Figure 7. Since V_{P2} and V_{S2} in this particular model are different from the correct values ($V_{P2} = 2.054$, $V_{S2} = 1.250$), the minima of the curves $\Delta t(V_{P0})$ for both the P-wave and the SV-wave are shifted from the correct vertical velocity ($V_{P0} = 1.875$ km/s). For $V_{P0} = 1.775$ km/s (which is 5.3 percent less than the correct value), the time residuals for both waves are small: Δt_{RMS} (P-wave)=0.73 ms, Δt_{RMS} (SV-wave)=1.8 ms.

Another example is the equivalent model for Taylor sandstone discussed in the previous section ($V_{P0} = 3.609$ km/s, $\epsilon = 0.021$, $\delta = -0.087$, $z = 3.215$ km). We have shown that P-wave traveltimes for this model and Taylor sandstone are almost identical up to $x_{\max} = 2z$. Moreover, if $V_{S0} = 1.960$ km/s is used, the values of t_{S0} and V_{S2} for this model and Taylor sandstone practically coincide with each other implying that intermediate-spread SV-wave moveout is not sufficient to resolve the trade-off between the vertical velocities and anisotropic parameters.

This general nonuniqueness in the joint inversion of P and SV data for the case of intermediate SV-wave spreads explains the failure of the inversion algorithm based on the quadratic and quartic moveout coefficients. Clearly, it is necessary to use longer spreads to reduce this ambiguity.

As shown in Figure 8, a significant improvement in the accuracy of the inversion procedure can be achieved by extending the SV-wave spread to $x_{\max} = 2z$. The residuals in Figure 8 are calculated as RMS averages for both the P- and SV-moveouts. If the SV-wave spreadlength is limited by 1.5z, the error in V_{P0} is about 10 percent for the models with $\Delta t_{RMS} = 2$ ms. Combination of P and SV data for the spread length $x_{\max} = 2z$ (Figure 8) makes the recovery of the vertical velocity for Dog Creek shale much more accurate.

The success of this inversion results from the high sensitivity of the SV moveout near $x = 2z$ to the depth of the boundary z (and hence to V_{S0}). Due to the influence of the velocity maximum, located at incidence angles 40-45 degrees, the SV-wave moveout curve exhibits a sharp turn caused by the rapid decrease in the "instantaneous" moveout velocity (for discussion and examples see TT1). The strongly nonhyperbolic SV-moveout near the velocity maximum cannot be described either by the three-term Taylor series (1) or by a more elaborate approximation (15). However, as shown in TT1, the accuracy of equation (15) can be significantly increased by determining its coefficients numerically by a least-squares fit.

If we pick the wrong vertical velocity, we get the wrong depth of the boundary, and this anomalous part of the traveltime curve moves into a different range of offsets. Even for relatively small errors in V_{S0} and z (and V_{P0}, since V_{P0}/V_{S0} ratio is fixed by the vertical traveltimes), the departure of the SV-wave moveout from the curve for the reference model is so significant that it cannot be easily compensated for by changes in the parameters of anisotropy. This implies that the inclusion of long-spread SV-moveout leads to a significant reduction in the trade-off between the model parameters.

The more pronounced SV-wave traveltime anomaly near $x = 2z$ for Dog Creek shale than for Taylor sandstone is due to the fact that for the former model the velocity maximum is larger and is located at lower incidence angles (Figures 2 and 3). Consequently, the joint P-SV inversion for Dog Creek Shale is more accurate than for Taylor sandstone (Figure 9). The time residuals for the SV-wave at large offsets (x close to $2z$) are 2 to 4 times higher than the RMS value over all offsets. Therefore, it may be possible to distinguish between different models on this basis, even for relatively low values of the RMS residual.

In the above discussion, the vertical arrival times have been fixed at the correct values. Changes in t_{P0} and t_{S0} may lead to a certain increase in the maximum error in V_{P0} and V_{S0}, but do not materially alter our conclusions.

One important problem to be addressed in the practical implementation of the joint inversion of P and SV data is the possible presence of local minima of the objective function (multimodality). In the above analysis we have seen that the objective function has a well-defined global minimum and no local minima in the vicinity of the exact solution. Nevertheless, local minima might exist elsewhere in the model space. However, we can expect to get good initial estimates of at least two parameters in our problem – the short-spread moveout velocities. Also, since the absolute value of the anisotropic coefficient δ rarely exceeds 10-15 percent, we have a reasonable approximation for the vertical P-wave velocity too. Therefore, our initial model cannot be far from the exact solution.

Inversion of SV-wave traveltimes

Given the strongly nonhyperbolic SV-moveout near the velocity maximum, might long-spread SV-wave data alone be sufficient for unambiguous inversion? In the weak-anisotropy approximation, the SV-wave velocity depends only on V_{S0} and V_{S2} (or V_{S0} and σ) (Thomsen, 1986; TT1). However, numerical results show that for the models we consider here, the influence of V_{P0} and V_{P2} (δ) on the SV-wave traveltimes cannot be neglected. In the following, we switch back to δ as an independent parameter because we find it useful to separate the influence of the P-wave vertical velocity and the parameter of anisotropy (δ) on the SV-moveout. The depth of the boundary is again determined through the correct vertical time as $z = V_{S0}\, t_{S0}/2$.

Figures 10 and 11 clearly show the dependence of the time residuals and of the best-fit V_{S0}, respectively, on V_{P0} (for fixed V_{S2}). The influence of V_{P0} is stronger for small values of the ratio V_{P0}/V_{S0} (Figure 11). If the changes in V_{P0} are within ± 25 percent of the correct value, the vertical S-velocity for Dog Creek shale may be recovered with sufficient accuracy (Figure 12). Variations in δ do not substantially change the maximum error in V_{S0}. Although this inversion can provide us with good estimates of only two parameters, V_{S0} and $V_{S2}(\sigma)$, this is all that is needed to carry out accurate time-to-depth conversion.

For Taylor sandstone, the long-spread SV-wave moveout is not so sensitive to the depth of the boundary as for Dog Creek shale. Hence, the inversion becomes more ambiguous due to the trade-off between the model parameters. However, an extension of the SV-wave spread to $x_{\max} = 2.1z$ brings about a substantial improvement in the accuracy of the moveout procedure (Figure 13).

Thus, long-spread SV-wave data is marginally good for traveltime inversion. The vertical S-velocity can be accurately determined if the spread is at least $2z$ long, and the vertical P-wave velocity is loosely constrained.

DISCUSSION

Ambiguity in the inversion of P-wave reflection moveout in transversely isotropic media may be significantly reduced by combining long-spread P and SV data. In multilayered media, inversion can be performed from the top to the bottom in a layer-stripping mode. Due to the accumulation of errors with depth, however, the accuracy for any internal layer would be lower than that for the present results pertaining to a single-layer model. Feasibility of the joint inversion of the P, SV, and SH-data in VSP geometry was shown by Leary et al. (1987), who used the traveltimes of the direct arrivals to determine the parameters of inhomogeneous transversely isotropic media near a fault zone.

Although we have proved the viability of the joint inversion of P- and SV-wave reflection traveltimes, the practical realization of this approach is a challenging task. Both acquisition and processing of long-spread P- and SV-data is expensive and complicated. One of the potential obstacles is the presence of cusps on SV-wave wavefronts that occur for relatively strong anisotropy (Musgrave, 1970) and may seriously impede the analysis of SV-wave moveout.

Application of the proposed algorithm requires the recovery of nonhyperbolic moveouts from long-spread CMP gathers. While deviations from a hyperbola are an advantage in traveltime inversion, they are difficult to account for in moveout-correction procedures. Previously developed algorithms for nonhyperbolic moveout correction are based on the quartic moveout equation (May and Straley, 1979; Gidlow and Fatti, 1990). Our approximation (15) is more accurate than the quartic polynomial because it contains an additional independent parameter and converges at large x.

However, even equation (15) may fail to describe the long-spread SV-wave moveout in the case of strong anisotropy. The semblance search at high incidence angles is also hindered by phase shifts in post-critical reflections. A possible solution is to try to pick SV-wave traveltimes at large offsets or to use forward modeling (e.g., ray tracing) in order to find the moveout curve that maximizes the stacked trace.

In our modeling, we have assumed a horizontally homogeneous, azimuthally anisotropic medium. It is likely that in many cases the assumption of horizontal homogeneity may be violated at large offsets, and this may lead to significant distortions of the nonhyperbolic partion of moveout curves. Further complications might be caused by the presence of azimuthal anisotropy.

Due to the above difficulties, it is important to find out what other kinds of additional information may be used to supplement P-wave traveltimes in the inversion procedure. In areas with sufficient well control, one may use check shots or sonic logs to recover the true vertical velocity and then obtain the anisotropic coefficients from the short-spread moveout velocities. The elastic parameters, determined at well sites, can then be used to constrain the inversion of surface data between the wells.

One of the ways to overcome the limited angle coverage of reflection moveouts from horizontal interfaces is to use reflections from dipping planes (Alkhalifah and Tsvankin, 1994) or head waves, which propagate along interfaces with the velocity of the faster underlying medium. The head waves formed at shallow boundaries have been successfully used in isotropic media (Lankston, 1989). P-head waves formed at horizontal boundaries in transversely isotropic media can provide us with the horizontal velocity that gives an additional equation for V_{P0} and ϵ, making the inversion more stable.

We have not discussed the dynamic properties (amplitudes, waveforms) of reflected waves in transversely isotropic models. However, the high sensitivity of body-wave amplitudes in anisotropic media to velocity maxima and minima (Tsvankin and Chesnokov, 1990) is a potentially useful feature in the inversion procedure.

In this paper we have considered horizontally-homogeneous models. The above results suggest that the reconstruction of 2-D anisotropic velocity fields from reflection traveltimes is a highly ambiguous problem.

CONCLUSIONS

We have examined the feasibility of inverting reflection traveltimes from horizontal interfaces for the parameters of transversely isotropic model, in the case when vertical velocities are unknown. Conventional hyperbolic moveout analysis on short-spread gathers does not provide enough information to solve this problem, even if all three waves (P,SV, *SH*) are recorded. Correct determination of the vertical velocities and accurate time-to-depth conversion require, at a minimum, analysis of nonhyperbolic moveout on long-spread gathers.

One way to incorporate information from nonhyperbolic moveout into the inversion procedure is to recover the quartic Taylor series terms of moveout ($t^2 - x^2$) curves and use them along with the short-spread moveout velocities for P- and SV-waves. However, this algorithm fails due to the fundamental ambiguity of the inverse problem.

In order to determine the degree of ambiguity, and find out what kind of data is necessary for unambiguous inversion, we have carried out numerical analysis of the objective function (RMS time residuals) for the inversion of P- and SV-wave reflection traveltimes. The results show that P data alone are insufficient for accurate determination of vertical velocity, even if long spreads are used ($x_{\max} = 2z$). The degree of nonuniqueness may be significantly reduced by combining long-spread P and SV-wave data. This improvement is ensured by the high sensitivity of the SV-moveout near the velocity maximum to the depth of the boundary. The accuracy of the inversion is thus higher for the models with stronger nonhyperbolic moveout.

In some cases, the SV-wave moveout alone may be used to recover the vertical S-wave velocity and parameter σ. Success of this inversion depends on the spreadlength and the degree of SV-wave velocity anisotropy, as well as on plausible constraints on the P-wave vertical velocity.

For multilayered media, the joint inversion of P and SV data may be performed in a layer-stripping mode. The accuracy for any internal layer, however, is likely to be lower in comparison with our estimates made for a single-layer case.

Practical realization of the above algorithm, in any case, is not straightforward. Acquisition and processing of multicomponent long-spread reflection data is technically complicated and expensive. Recovery of strongly nonhyperbolic moveouts is time-consuming and requires advanced methods of moveout correction. Moreover, the present analysis may break down in the presence of cusps on SV-wavefronts. The results of the inversion on long spreads may be also impeded by horizontal inhomogeneities and azimuthal anisotropy. Therefore, whenever possible, P-wave data from horizontal reflectors should be supplemented with additional information (e.g., well data, dip moveout, head waves) in order to reduce the ambiguity of the inverse problem.

We considered the simple case of transversely isotropic media with a vertical symmetry axis. There is no doubt that for more complicated azimuthally anisotropic models the degree of nonuniqueness in the traveltime inversion is even higher.

One more general conclusion that may be drawn from this study is that inversion algorithms in anisotropic media should be designed to use the data mostly sensitive to changes in model parameters. Due to the large number of independent variables, "blind" formal inversion in the presence of anisotropy is usually unstable.

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FIG. 1. Error in the P-wave quartic Taylor series coefficient obtained using approximation t_A. The parameters of t_A are found by the least-squares method from the exact reflection times for a layer of Taylor sandstone (Figure 2).

FIG. 2. Phase velocities of the P- and SV-waves for Taylor sandstone. Elastic parameters are taken from Thomsen (1986): $V_{P0} = 3.368$ km/s, $V_{S0} = 1.829$ km/s, $\epsilon = 0.110$, $\delta = -0.035$ ($\sigma = 0.492$).

FIG. 3. Phase velocities of the P- and SV-waves for Dog Creek shale. Elastic parameters are taken from Thomsen (1986): $V_{P0} = 1.875$ km/s, $V_{S0} = 0.826$ km/s, $\epsilon = 0.225$, $\delta = 0.1$ ($\sigma = 0.644$).

FIG. 4. P-wave RMS time residuals (in ms) calculated with respect to the reference model of Taylor sandstone. $\bar{V}_{P2}$ and $\bar{V}_{S2}$ are the parameters V_{P2} and V_{S2} normalized by the exact values (the short-spread moveout velocities for the reference model). The plot shows the smallest residual for each pair of ($\bar{V}_{P2}$, $\bar{V}_{S2}$), obtained by scanning the vertical velocity V_{P0}. The spreadlength $x_{\max} = 2z$, $z = 3$ km, $t_{P0} = 1.781$ s.

FIG. 5. The vertical P-wave velocity (normalized by the exact value of V_{P0}) for the models whose time residuals are shown in Figure 4.

FIG. 6. Accuracy of the inversion of long-spread P-wave data. The error in V_{P0} is calculated as the maximum deviation in the P-wave vertical velocity among the set of models with a given RMS time residual. The upper and lower curves show the maximum over- and underestimation of V_{P0} respectively. The reference models (Figure 2,3) are Taylor sandstone (Model 1, $t_{P0} = 1.781$ s) and Dog Creek shale (Model 2, $t_{P0} = 3.200$ s). The spreadlength $x_{\max} = 2z$. Only a subset of models with the constrained SV-wave short-spread moveout velocity is taken into account (V_{S2} is ± 2 percent of the exact value).

FIG. 7. A family of equivalent models in the joint inversion of P- and SV-traveltimes. Time residuals are calculated as the RMS averages for both P and SV data. The spreadlengths are $2z$ (P-wave) and $1.5z$ (SV-wave). The velocities $V_{P2} = 2.047$ km/s and $V_{S2} = 1.253$ km/s are different from the values for the reference model of Dog Creek shale ($V_{P2} = 2.054$ km/s, $V_{S2} = 1.250$ km/s), and the minimum time residual is shifted towards vertical P-velocities, which are smaller than the correct $V_{P0} = 1.875$ km/s. The vertical times are $t_{P0} = 3.200$ s and $t_{S0} = 7.264$ s.

FIG. 8. Influence of the SV-wave spreadlength $x_{\max}$ on the accuracy of the joint inversion of P and SV reflection times for Dog Creek shale. The error in V_{P0} is calculated as the maximum deviation in the vertical P-velocity among the set of models with a given RMS time residual (which includes both P- and SV-residuals). The spreadlength for the P-wave equals $2z$.

FIG. 9. Accuracy of the joint inversion of P and SV data for Taylor sandstone (Model 1) and Dog Creek shale (Model 2). The spreadlength $x_{\max} = 2z$ for both P- and SV-waves.

FIG. 10. SV-wave RMS time residuals calculated with respect to the reference model of Dog Creek shale. $\bar{V}_{P0}$ and $\bar{V}_{S2}$ are the parameters V_{P0} and V_{S2} normalized by the exact values. The parameter δ is fixed at the correct value ($\delta = 0.1$). The plot shows the smallest residual for each pair of (V_{P0}, V_{S2}), obtained by scanning the vertical S-wave velocity V_{S0}. The spreadlength $x_{\max} = 2z$, $t_{S0} = 7.264$ s.

FIG. 11. The vertical S-wave velocity (normalized by the exact value of V_{S0}) for the models whose time residuals are shown in Figure 10.

FIG. 12. Accuracy of the inversion of long-spread SV data for Dog Creek shale. The error in V_{S0} is calculated as the maximum deviation in the vertical S-wave velocity among the set of models with a given SV-wave RMS time residual. This plot summarizes the results of Figures 10 and 11. The P-wave vertical velocity is constrained by ± 25 percent of the exact value, and δ is fixed at the correct value ($\delta = 0.1$).

FIG. 13. Influence of spreadlength on the accuracy of the inversion of SV data for Taylor sandstone. The vertical P-wave velocity is constrained by $1.5 < V_{P0}/V_{S0} < 2.45$, and δ is fixed at the correct value ($\delta = -0.035$). The vertical time $t_{S0} = 3.280$ s.