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Mathematica^{TM} and the Method of Steepest Descents: Part II

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ABSTRACT

Mathematica is a symbolic manipulator with graphical capabilities. During the fall 1990 semester, I used Mathematica on my NeXT workstation to create graphics for teaching the method of steepest descents. This required level curve plots and surface plots for the real or imaginary part of functions of a complex variable. By the end of this topic in the course, I was doing near-research-level analysis of complex valued functions. The response of my class was also very positive. In Part I of this paper, I described some of the more elementary examples presented in that class. Here, I describe the application of the technique to the problem of scattering by a half plane, including the analysis of the exponent that leads to refracted and evanescent waves in a layered medium.

INTRODUCTION

In Part I of this paper [Bleistein, 1992], I described the application of Mathematica to create contour plots and surface plots of the real and imaginary parts of complex valued functions. More specifically, I did this in the context of the method of steepest descents, about which I have written extensively, including chapters in two books [Bleistein, 1984; Bleistein and Handelsman, 1986]. As discussed in the introduction to Part I, this project arose as part of the teaching of a course entitled "Multi-valued functions and their applications."

Part I of this paper dealt with standard textbook examples of exponents, starting from a simple saddle and monkey saddle and continuing on to discussion of the exponent of the integral representations of Airy functions and Bessel functions. Here, I will describe the application to the exponents that arise in propagation of acoustic waves from a line source over a half space of higher propagation speed. I discuss first the analysis of the exponent for the direct or free-space wave and then the exponent for the refracted/evanescent wave.

I should also point out that some of my research was carried out on a more primitive graphics terminal by using Mathematica in terminal mode and including an appropriate file describing the graphics profile of the terminal. This was done with more sparse pictures at 2400 baud resulting in lesser quality graphics, but adequate for many purposes. There is one distinct advantage to the graphics at 2400 baud. The curves evolve more slowly on the screen and, at times, insight is gained from the evolution.

I also used the Mathematica graphics in teaching the Cagniard-de Hoop method [de Hoop, 1960; Cagniard, 1962; Aki & Richards, 1980] in this course. In this method, straightforward conformal mappings become useful, as well as the analysis of the position of singularities in the mapped plane. Some of these applications will appear in a forthcoming paper [Bleistein and Cohen, 1992].

BACKGROUND: ON THE METHOD OF STEEPEST DESCENTS

This is a briefer review of material already presented in Part I.

The method of steepest descents is concerned with integrals of the form

$$I(\lambda) = \int_C g(z) \exp\{\lambda w(z)\} dz. \tag{1}$$

Here, we use the notations, $z = x + iy$ and $w(z) = u(x, y) + iv(x, y)$, with x, y, u, v, all being real. In this integral, C is a contour in the complex z plane. The method is concerned primarily with asymptotic approximations of this integral for large positive values of λ, although the method is also relevant to numerical approximations for any choice of λ.

The objective of the method is to use Cauchy's theorem to justify replacing the given contour of integration by one or a sum of contours that have been chosen for particularly rapid convergence of the resulting integrals. The rapid convergence is achieved by choosing these contours in such a manner that $u(x, y) = \text{Re } w(z)$ decreases most rapidly from its reference value at a so-called *critical point.* Candidate critical points include points where w and/or g fails to be analytic and points where the first and, possibly, higher derivatives of w vanish. These are the saddle points.

In Part I, we explained how to recognize these paths from contour plots of $v(x, y)$. We refer the reader there for further discussion except that we repeat here the standard terminology. Paths along which $u(x, y)$ decreases (increases) are called paths of descent (ascent). Paths along which this decrease (increase) is most rapid compared to neighboring paths are called paths of steepest descent (ascent). At each point on these paths, the tangents are called directions of steepest descent (ascent).

In Part I, the following examples were discussed:

$$w(z) = z^2; \ \ w(z) = z^3; \ \ w(z) = z - z^3/3, \ \ w(z) = i[\cos z + .5(z - \pi/2)].$$

Clearly, for simple examples, the analysis of the paths of steepest descent can all be done by hand. However, even in those cases, exposition is greatly improved by the computer graphics such as the ones presented in Part I and below. Furthermore, hands-on experience by the student becomes a valuable learning tool. Finally, as an indication of the power of this additional graphical tool, in the last example presented here, computer implementation helped me to see a saddle point that is the solution of a fourth order equation in z^2. This was a case in which I already knew that the saddle point was out there somewhere and the graphical output helped me locate it. I would suggest that a similar analysis is possible where the presence of such saddle points is only suspected.

POINT SOURCE OVER A HALF SPACE

The physical problem is as follows. Consider a point source in two dimensions—or a line source in three dimensions—for an acoustic medium of variable propagation speed. In particular, the source resides in an upper half space with a propagation speed, c_1, and horizonatal boundary. Below is a medium with propagation speed, $c_2 > c_1$.

The solution to this problem has been well-studied, as in my own book [Bleistein, 1984]. A more complete discussion can be found, for example, in Brekhovskikh [1980]. Here we will discuss the exponents of the Fourier representation of the solution on both sides of the interface.

Exponent for the Primary Wave

By primary, we mean the signal emanating from the source point. This part of the signal is unchanged from what it would be with no interface at all, that is, for the propagation speed being c_1 throughout.

The Fourier representation of this wave has as exponent,

$$\Phi(k, \omega, x_1, x_3) = i\left[kx_1 + k_3|x_3|\right], \tag{2}$$

where,

$$k_3 = \begin{cases} \sqrt{\omega^2/c_1^2 - k^2}, & k^2 > \omega^2/c_1^2, \\ i\sqrt{k^2 - \omega^2/c_1^2}, & k^2 < \omega^2/c_1^2. \end{cases} \tag{3}$$

Here, k is the horizontal wave number and ω is the frequency; x_1 and x_3 are the horizontal and vertical coordinates, respectively.

Our objective is to rewrite this exponent in the form of (1) in dimensionless variables. (We proceed under the assumption of positive ω; adjustments to this discussion for negative ω are straightforward but not relevant to the discussion here.) To this end, set

$$x_1 = \rho\cos\theta,\ x_3 = \rho\sin\theta,\ k = z\omega/c_1, \tag{4}$$

and rewrite (2) as

$$\Phi = \lambda w(z),\ \lambda = \rho\omega/c_1,\ w(z) = i\left[z\cos\theta + \sqrt{1 - z^2}\sin\theta\right]. \tag{5}$$

For this example, then, large λ means that the observation point is at a distance ρ that is large compared to the wave number, ω/c_1, or "many" wavelengths away from the source, for the frequencies of interest. We should think, then, of some source signature, not explicitly stated, with some defined passband of frequencies of interest and an observation range for which λ is "large"— typically, three or more wave lengths.

For the purposes of this illustrative example, we must make a choice of θ; we choose $\theta = \pi/3$. Thus, the exponent to be analyzed in this case is

$$w(z) = .5i\left[z + \sqrt{3}\sqrt{1 - z^2}\right]. \tag{6}$$

Note that for this exponent, we are beyond the point where explicit formulas for $u(x, y)$ and $v(x, y)$ are helpful. The multi-valued square root here is chosen to be positive at $z = 0$ and to be positive imaginary for $z = x,\ |x| > 1$, so that Re $w(z) < 0$ in this latter range. This is equivalent to moving to the outer part of the x axis by passing *above* the branch point at $z = -1$ and passing *below* the branch point at $z = +1$. In what follows, we will take this function to have *branch cuts* that are vertical—upward from $z = 1$, downward from $z = -1$.

As a side issue, note that the choice of branch cuts are not crucial to the method. Evaluation of the square root is carried out by knowing its value at one point—$z = 0$—and knowing how one arrived at any other point on a connected path. The choice of values of a multi-valued function is then determined by continuity considerations. The particular choice of cuts assures us that the given path of integration and the paths of steepest descent derived below will not intersect the branch cut. In fact, one need not introduce branch cuts at all to proceed with the analysis, as long as one defines the square root continuously, as noted. However, for computational purposes, one must define a single-valued, albeit discontinuous, function in order to proceed. The square root is single valued in the cut plane, but discontiuous across the cuts.

For the function defined by (6),

$$w'(z) = .5i\left[1 - \frac{z\sqrt{3}}{\sqrt{1-z^2}}\right],\ w''(z) = -\frac{i\sqrt{3}}{2\left[1-z^2\right]^{3/2}}. \tag{7}$$

One can check that

$$w'(.5) = 0,\ w(.5) = i,\ w''(.5) = -4i/3. \tag{8}$$

Thus, for this example, there are a saddle point and two branch points in the function $w(z)$.

In Figure 1, we show the level curves of $v(x, y)$. Perhaps surprisingly, the branch cuts show up as darkened sections of the plot. What is happening here is that the constant value of $v(x, y)$ on one side of the cut does not occur at all on the other side of the cut. Apparently, under these circumstances, Mathematica simply takes the contour out of the figure entirely to the nearest edge over one step in x. This can be seen more clearly in Figure 3 below, where we have limited the plot to only the level curves of $v(x, y)$ that produce the steepest ascent and descent paths through the saddle points. The steepest ascent paths intersect the branch cuts and behave as we have described. The ensemble of such "broken" level curves taken together comprise the darkened section of the figure around the branch cut.

The saddle point is also visible through the now familiar signature of the level curves moving in and away from that point, but otherwise locally remaining in quarter sectors near that point. Although we do not have the steepest descent and ascent paths here – the level curves through $z = .5$—the plot does support the complex function theory that predicts that the paths of steepest descent make angles of

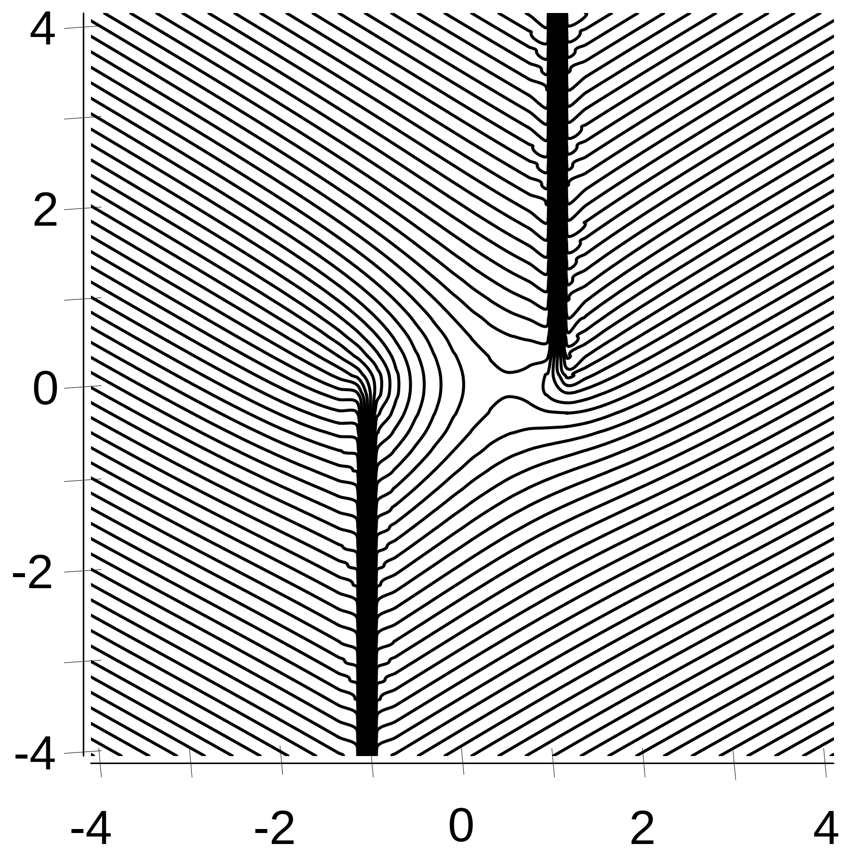

FIG. 1. Level curves of Im $\{i[.5(z+\sqrt{3}\sqrt{1-z^2})]\}$.

$-\pi/4$, $3\pi/4$, with the positive x axis, while the paths of steepest ascent make angles of $\pi/4$, $-3\pi/4$, with this axis. (Recall that these are the pair of level curves of $v(x,y)$ that must cross at the saddle point.)

This figure was produced with the following command line.

```
ContourPlot[Im[I ( (x + I y)/2 +
      Sqrt[1 - x - I y]
      Sign[Arg[1 - x - I y] + Pi/2]
      Sqrt[x + I y + 1]
      Sign[Arg[x + I y + 1] + Pi/2]
      Sqrt[3]/2 )], {x,-4,4},{y,-4,4},
      ContourLevels -> 81 ,
      PlotPoints -> 40]
```

The square roots of $1 \pm z$ with appropriate branch cuts each take two commands to describe. The reason is that Mathematica produces a principal value square root with argument ϕ restricted to $-\pi < \phi \leq \pi$. To get a square root with a different angular range, we must take the alternate branch of the square root in the sector where we are not using the principal value; essentially, change the sign of the square root across the branch cut. We leave it to the reader to verify that the function

$$\sqrt{z}\,\mathrm{sign}\,[\mathrm{Arg}[z] - \alpha]$$

puts the branch cut of $\sqrt{z}$ at $\arg[z] = \alpha$, and that the proper location for the branch cuts for $1 \pm z$ is at $\alpha = -\pi/2$.

The choice of the number of ContourLevels and PlotPoints here was obtained by trial and error to produce an informative and esthetically pleasing plot.

In Figure 2, we show the surface $u(x,y)$. This figure was produced with the following command line.

```
Plot3D[Re[I ( (x + I y)/2 +
      Sqrt[1 - x - I y]
      Sign[Arg[1 - x - I y] + Pi/2]
      Sqrt[x + I y + 1]
      Sign[Arg[x + I y + 1]+ Pi/2]
      Sqrt[3]/2 )], {x,-4,4},{y,-4,4},
      BoxRatios -> {4,4,2},
      ViewPoint -> {5,10,10}]
```

In the figure, we see the branch cuts as "creases" across which—again, in one x sample width—Re $w(z)$ jumps from its value on one side of the branch cut to its value on the other side. Above the x axis, between ± 1, the surface is level, because $w(z)$

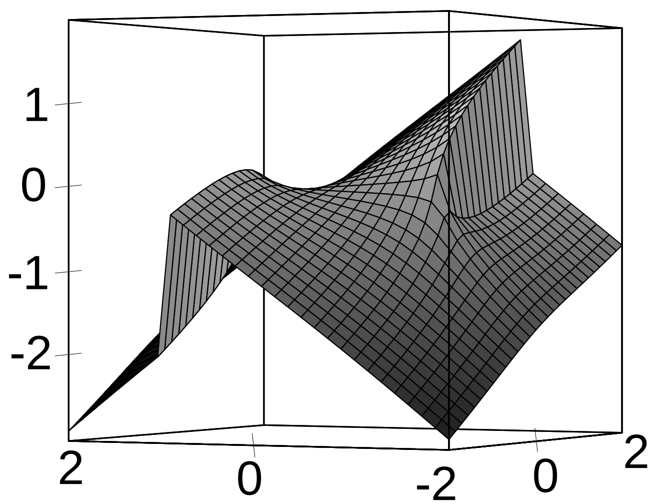

FIG. 2. Surface of Re $\{i[.5(z+\sqrt{3}\sqrt{1-z^2})]\}$.

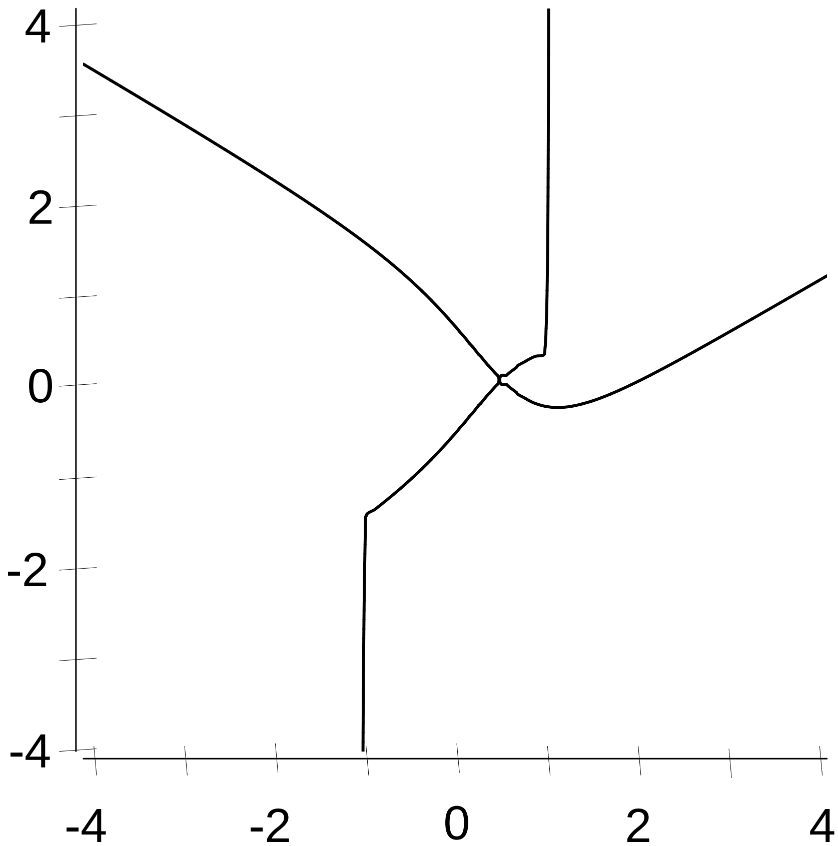

FIG. 3. Steepest descent and ascent directions through the saddle point at $z = .5$.

is purely imaginary there: $u(x, y) = 0$. The steepest descent and ascent directions on this surface make angles of 45° with this line. The position of the saddle point is not obvious on this figure, but the features of $u(x, y)$ “in the large” are quite apparent. In particular, one can see the hills and valleys of the surface much more clearly here than from Figure 2

Figure 3 shows the steepest descent and ascent paths through the saddle point. Though the neighborhood of the saddle point itself is clear, here, the plot breaks down at the branch cuts in the manner describe above in the discussion of Figure 1. Unfortunately, the Mathematica command, Show, does not work with ContourPlot. If it did, we could overlay Figure 1 and Figure 3 to see the fit of the steepest paths here within the level curves of the former figure. However, the reproductions here are to scale and the reader might be able to do that with copies of the figures.

Exponent for the Transmitted Wave

As a second example for this paper and last in this series, we discuss the exponent for the transmitted wave for this same problem. In dimensional variables the exponent of interest in this case is also derived in Bleistein [1984], equation (8.1.11). In slightly different notation,

$$\Phi(k,\omega,x_1,x_3) = i\left[kx_1 + k_4[x_3 - h] + k_3 h\right], \tag{9}$$

with

$$k_4 = \left\{ \begin{array}{ll} \sqrt{\omega^2/c_2^2 - k^2}, & k^2 > \omega^2/c_2^2, \\ i\sqrt{k^2 - \omega^2/c_2^2}, & k^2 < \omega^2/c_2^2. \end{array} \right\}, \tag{10}$$

and k_3 as defined in (3). This exponent represents a wave that propagates vertically a distance h to the interface with vertical wave number k_3 and then propagates a distance $x_3 - h$ with vertical wave number k_4. Of necessity, the distance h must be many wavelengths for high frequency asymptotics to be valid. Thus, we introduce the new dimensionless variables, z, α, β, by

$$k = z\omega/c_1,\ x_1 = \alpha h,\ x_3 - h = \beta h, \tag{11}$$

and choose $c_2 = 2c_1$ for this example. In this case, (9) becomes

$$\Phi = \lambda w(z),\ \lambda = \omega h/c_1,\ w(z) = i\left[\alpha z + \beta\sqrt{.25 - z^2} + \sqrt{1 - z^2}\right]. \tag{12}$$

We must make a specific choice of α and β for making plots. We choose values below the region of critical reflection for the specific relationship between c_1 and c_2, so that we can anticipate seeing some effect of the evanescent waves that we expect to be propagating in this region. The particular choice I made was $\alpha = .7$ and $\beta = .4$. Of course, in an unfamiliar problem, such choices must be arrived at by analysis or trial and error. In this case, $w(z)$ as defined in (12) becomes

$$w(z) = i\left[.7z + .4\sqrt{.25 - z^2} + \sqrt{1 - z^2}\right]. \tag{13}$$

For this function

$$\begin{aligned} w'(z) &= i\left[.7 - .4\frac{z}{\sqrt{1/4 - z^2}} - \frac{z}{\sqrt{1 - z^2}}\right], \\ w''(z) &= -i\left[\frac{.4}{\{1/4 - z^2\}^{3/2}} - \frac{1}{\{1 - z^2\}^{3/2}}\right]. \end{aligned} \tag{14}$$

We can be sure that $w(z)$ has a saddle point on the real axis. To see why this is so, note that $w'(0) = .7i$ and $w'(z) \to -i\infty$ as $z = x \to .5$, while remaining purely imaginary for $z = x$ in this interval. Therefore, $w'(z)$ must pass through zero and the location of that zero is the saddle point. Furthermore, $w''(z) \neq 0$ on this interval; from (14) it can be seen to be negative-imaginary here.

Replacing the multiplier .4 by zero is equivalent to taking an observation point in the physical domain right on the interface. We can check, in this case, that there is a

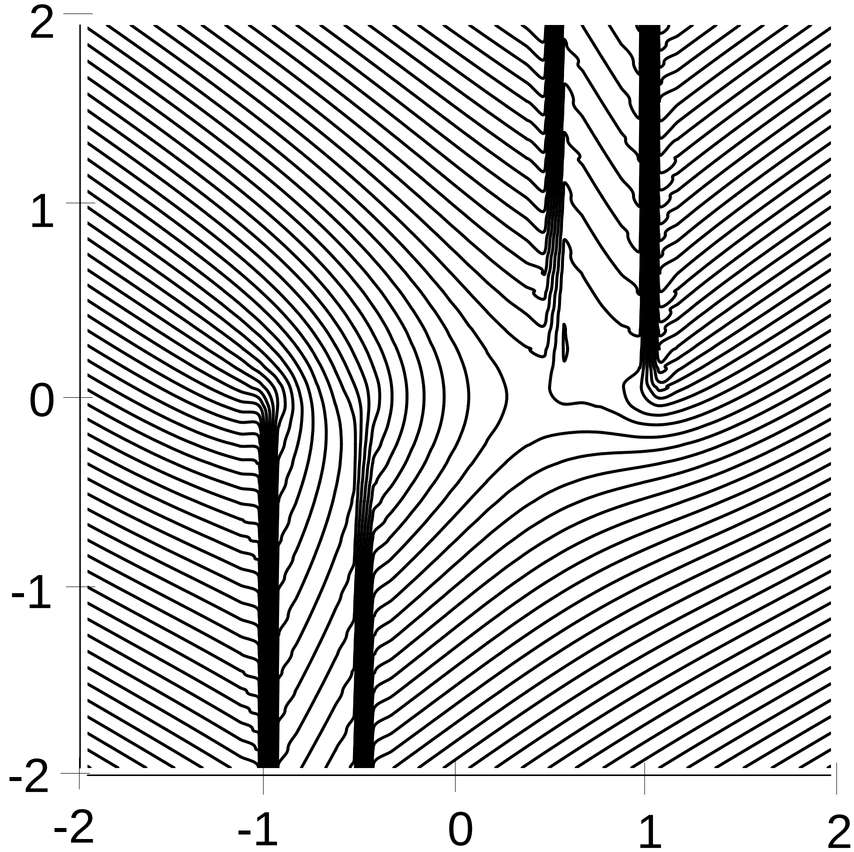

Fig. 4. Level curves of Im $\{(.9z + .4\sqrt{1/4 - z^2} + \sqrt{1 - z^2})\}$.

saddle point at $z = \sqrt{.49/1.49} \approx .57 > .5$. Perturbation analysis reveals that this zero of $w'(z)$ moves vertically upward in the $z-$plane when the distance from the interface in the physical space increases from zero. These are the features we should expect to see in the level curves and surface plot for this function.

Figure 4 shows the level curves of $v(x, y)$. The two branch cuts and the saddle point on the real axis, as well as the saddle point above the real axis between the branch cuts are all visible. This figure was produced with the following command line.

```
f1 = ContourPlot[Im[I ( .9(x + I y) +
        .4 Sqrt[.5 - x - I y]
        Sign[Arg[.5 - x - I y] + Pi/2]
        Sqrt[.5 + x + I y]
        Sign[Arg[.5 + x + I y] + Pi/2] +
        Sqrt[1 - x - I y]
        Sign[Arg[1 - x - I y] + Pi/2]
        Sqrt[x + I y + 1]
        Sign[Arg[1 + x + I y] + Pi/2]
        )],
        {x,-2,2},{y,-2,2},
        ContourLevels -> 81 ,
        PlotPoints -> 40]
```

The corresponding surface plot of $u(x, y)$ is shown in Figure 5. The saddle on the real axis between the two branch points has the same local structure as in the previous example. This saddle point is associated with the refracted wave. In addition, there is at least the suggestion of a second saddle in the trough between the two creases corresponding to the branch cuts in the first quadrant of z. The approximate location of this saddle is indicated by the arrow. Note that the height of this saddle is lower than for the former one. That means that $u(x, y)$ is smaller at this point and, thus, the integrand is exponentially smaller at this saddle point than at the one on the real axis. Relatively speaking, then, this contribution to the total wave field will be smaller than the contribution from the saddle point on the real axis. The exponential decrease is greater with increasing distance from the interface in the underlying point source problem. The term, "evanescent field" is reserved for such exponentially decaying contributions to the total field. A discussion of how this second saddle point affects the integration on the paths of steepest descents through the saddle point associated with the refracted wave is beyond the scope of this paper.

This figure was produced with the command lines

```
f2 = Plot3D[Re[I ( .9(x + I y) +
        .4 Sqrt[.5 - x - I y]
        Sign[Arg[.5 - x - I y] + Pi/2]
```

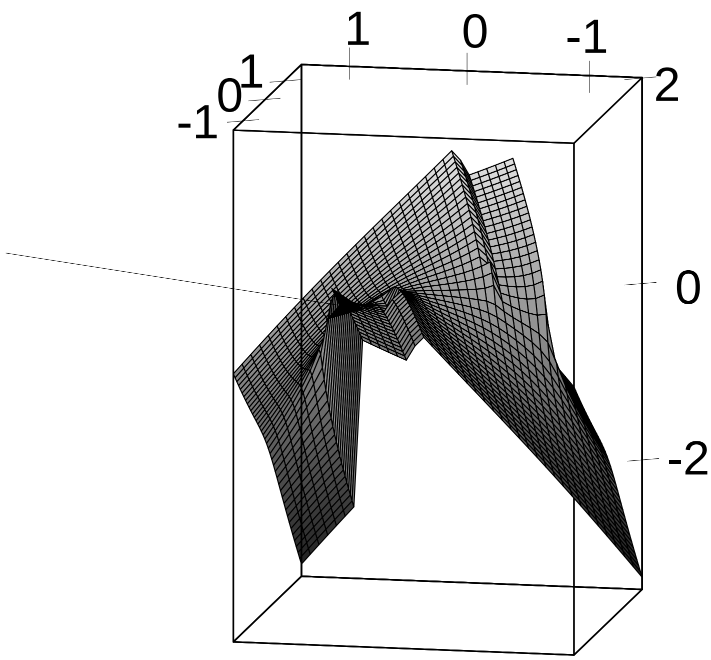

FIG. 5. Surface of Re $\{i(.9z + .4\sqrt{1/4 - z^2} + \sqrt{1 - z^2})\}$.

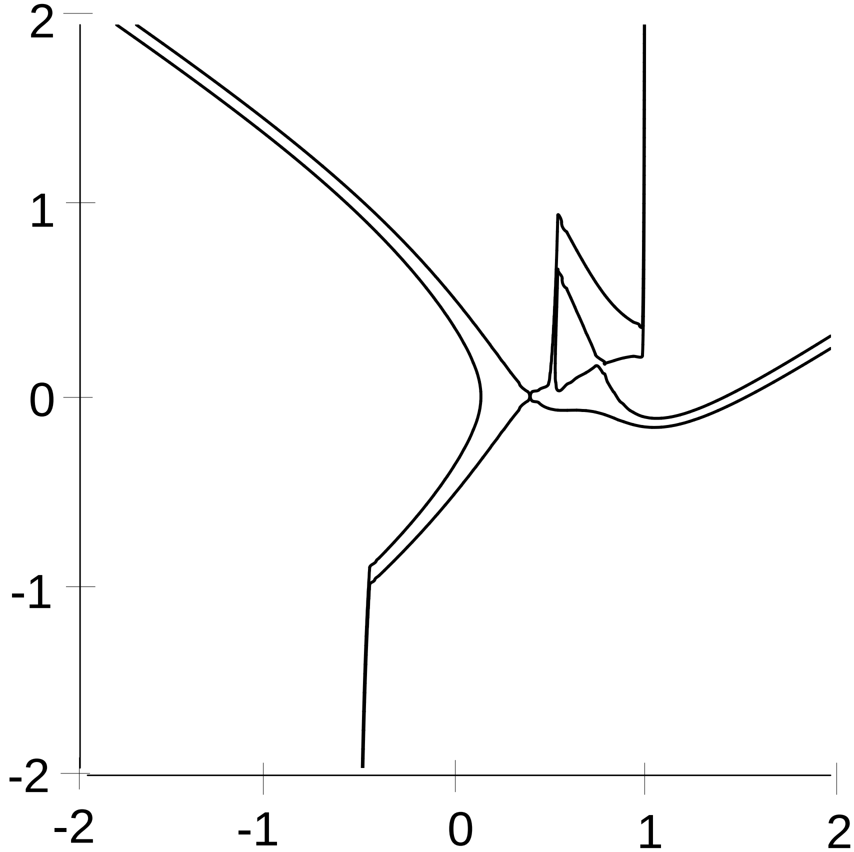

FIG. 6. Steepest ascent and descent paths for both saddle points of the exponent for transmitted waves.

```
        Sqrt[.5 + x + I y]
        Sign[Arg[.5 + x + I y] + Pi/2] +
        Sqrt[1 - x - I y]
        Sign[Arg[1 - x - I y] + Pi/2]
        Sqrt[x + I y + 1]
        Sign[Arg[1 + x + I y] + Pi/2]
        )],
        {x,-1.5,1.5},{y,-1.5,1.5},
        BoxRatios -> {4,4,6},
        PlotPoints -> 40,
        ViewPoint -> {2000,10000,-2000}]
```

Finally, for this example we show a plot of the steepest descent paths for *both* saddle points in Figure 6, produced with the following command lines

```
f1 = ContourPlot[Im[I ( .9(x + I y) +
        .4 Sqrt[.5 - x - I y]
```

```
Sign[Arg[.5 - x - I y] + Pi/2]
Sqrt[.5 + x + I y]
Sign[Arg[.5 + x + I y] + Pi/2] +
Sqrt[1 - x - I y]
Sign[Arg[1 - x - I y] + Pi/2]
Sqrt[x + I y + 1]
Sign[Arg[1 + x + I y] + Pi/2]
)],
{x,-2,2},{y,-2,2},
PlotRange -> {1.29313,1.39703},
ContourLevels -> 2 ,
PlotPoints -> 80].
```

The PlotRange command contains the two evaluations of Im $w(z)$ at the saddle points. Although the steepest ascent and descent paths through both saddle points are visible, other level curves and the spurious paths along the branch cuts are also visible. This figure could only be reliably used in conjunction with the previous figures to understand the actual steepest paths.

CONCLUSIONS

In this two part paper, through a hierarchy of progressively more difficult examples, not unlike the order presented in my course, I have described the application of Mathematica to the analysis techniques needed for the method of steepest descents. I found the use of Mathematica to be an excellent expository aid in this course. In addition, Mathematica was used in this course in the discussion of conformal mappings and the Cagniard-de Hoop method. The use of Mathematica was straightforward enough that my students learned with only a few examples to guide them, and then they used this tool routinely in homework assignments. It is clear from my experience that Mathematica can be used as a research tool in these applications as an expository tool in the classroom.

ACKNOWLEDGEMENT

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