Propagation of Elastic Waves in Transversely Isotropic Media

by

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TABLE OF CONTENTS

ABSTRACT ......................................................................................................................... ii

GLOSSARY ........................................................................................................................ iii

INTRODUCTION .................................................................................................................. 1

1. A QUALITATIVE REVIEW OF TRANSVERSELY ISOTROPIC MEDIA .......................... 2
   1.1 Shear wave birefringence ......................................................................................... 2
   1.2 The transversely isotropic model .............................................................................. 3

2. HIGH FREQUENCY WAVES IN ELASTIC ANISOTROPIC MEDIA ................................. 5
   2.1 Fundamental equations and high frequency asymptotics .......................................... 5
   2.2 The eikonal equation ............................................................................................... 8
   2.3 The transport equation ........................................................................................... 9
   2.4 Rays and group velocity ........................................................................................ 12

3. SPECIALIZATION TO THE TRANSVERSELY ISOTROPIC CASE .................................. 16
   3.1 Factorization of the eikonal equation - phase velocities ........................................ 16
   3.2 Eigenvectors computation - displacement polarizations ......................................... 18
   3.3 Group velocities and triplication effects ................................................................ 20
   3.4 Reflection and transmission at welded contact ...................................................... 22

4. RAY TRACING IN PIECEWISE HOMOGENEOUS TRISOTROPIC MEDIA .................. 25
   4.1 The model ............................................................................................................... 25
   4.2 Marching the algorithm ......................................................................................... 25
   4.3 Implementation in horizontally layered media ........................................................ 28

CONCLUSION .................................................................................................................... 30

ACKNOWLEDGMENT ........................................................................................................ 31

REFERENCES ................................................................................................................... 32

FIGURES ............................................................................................................................ 34
ABSTRACT

Shear wave splitting observed in shear (SH) and converted (P-SV) field data can be explained with help of the transversely isotropic model. Some characteristics of high frequency wave propagation in generally anisotropic media are demonstrated, such as the alignment of rays along group velocity or the orthogonality of the three displacement polarizations. Both ray and transport equations are formally derived in that general setting. However it is only by specializing to the transversely isotropic solid that explicit solutions can be obtained for displacement polarizations and group velocities. Phenomena such as triplication and scattering at interfaces are discussed in some detail. Finally a general ray tracing algorithm in piecewise homogeneous transversely isotropic media is described and implemented in the case of horizontal layers, and a typical example is discussed.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Introduced</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>17</td>
<td>a useful quantity in computing velocities</td>
</tr>
<tr>
<td>$\beta$</td>
<td>17</td>
<td>a useful quantity in computing velocities</td>
</tr>
<tr>
<td>$\gamma_1$, $\gamma_2$</td>
<td>14</td>
<td>parameters attached to a given ray</td>
</tr>
<tr>
<td>$\gamma_0^1$, $\gamma_0^2$</td>
<td>14</td>
<td>specific values of $\gamma_1$, $\gamma_2$</td>
</tr>
<tr>
<td>$d\gamma_1$, $d\gamma_2$</td>
<td>14</td>
<td>increments in $\gamma_1$, $\gamma_2$</td>
</tr>
<tr>
<td>$\delta(t)$</td>
<td>7</td>
<td>the Dirac distribution</td>
</tr>
<tr>
<td>$\delta_{ik}$</td>
<td>8</td>
<td>the Kronecker delta</td>
</tr>
<tr>
<td>$\delta_p$, $\delta_s$</td>
<td>19,20</td>
<td>useful quantities in computing polarizations</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>17</td>
<td>a discriminant</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>7</td>
<td>an infinitesimal quantity</td>
</tr>
<tr>
<td>$\epsilon_{ij}$</td>
<td>3</td>
<td>strain tensor in cartesian system</td>
</tr>
<tr>
<td>$\epsilon_{ijk}$</td>
<td>15</td>
<td>Levi-Civita density</td>
</tr>
<tr>
<td>$\epsilon_p$, $\epsilon_s$</td>
<td>20</td>
<td>useful quantities in computing polarizations</td>
</tr>
<tr>
<td>$\phi$</td>
<td>17</td>
<td>latitude angle for slowness: angle between slowness and anisotropy axis</td>
</tr>
<tr>
<td>$\theta$</td>
<td>19</td>
<td>longitudinal angle for slowness</td>
</tr>
<tr>
<td>$\lambda$, $\mu$</td>
<td>3</td>
<td>Lamé constants</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>12</td>
<td>a scalar</td>
</tr>
<tr>
<td>$\rho$</td>
<td>4</td>
<td>density</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>12</td>
<td>parameter along a ray</td>
</tr>
<tr>
<td>$\sigma_{ij}$</td>
<td>3</td>
<td>stress tensor in cartesian system</td>
</tr>
<tr>
<td>$\sigma_1$, $\sigma_2$</td>
<td>24</td>
<td>parametrization of reflector</td>
</tr>
<tr>
<td>$\tau(x)$</td>
<td>7</td>
<td>traveltime function</td>
</tr>
<tr>
<td>$\tau_M$, $\tau_{M'}$</td>
<td>23</td>
<td>traveltime for modes $M$, $M'$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>6</td>
<td>angular frequency</td>
</tr>
<tr>
<td>$\omega_0$, $\omega_1$</td>
<td>6</td>
<td>lower and upper bandlimits</td>
</tr>
<tr>
<td>$A$, $C$, $N$, $L$, $F$</td>
<td>3</td>
<td>elastic parameters for trisotropic medium (Love's notations)</td>
</tr>
<tr>
<td>$c_{ijkl}$</td>
<td>5</td>
<td>elastic tensor</td>
</tr>
<tr>
<td>$E$</td>
<td>9</td>
<td>energy density</td>
</tr>
<tr>
<td>$f(t)$</td>
<td>7</td>
<td>an arbitrary function of time</td>
</tr>
<tr>
<td>$\tilde{f}(t)$</td>
<td>7</td>
<td>Fourier transform of $f(t)$</td>
</tr>
</tbody>
</table>
\( F(x_i, r, p_i) \) \hspace{1cm} 12 \hspace{1cm} a GENERIC function of \( x_i, r, p_i \)

\( J, J(\sigma, \gamma_1, \gamma_2) \) \hspace{1cm} 15 \hspace{1cm} ray Jacobian

\( M_{ij} \hspace{0.5cm} i,j=1,2,3 \) \hspace{1cm} 7 \hspace{1cm} elements of eikonal matrix

\( n \) \hspace{1cm} 28 \hspace{1cm} components of unit normal vector

\( n_i \hspace{0.5cm} i=1,2,3 \) \hspace{1cm} 15 \hspace{1cm} unit normal vector

\( p \) \hspace{1cm} 26 \hspace{1cm} slowness vector

\( p_i \hspace{0.5cm} i=1,2,3 \) \hspace{1cm} 10 \hspace{1cm} slowness vector \( (p_i=r_i) \)

\( p_0 \) \hspace{1cm} 26 \hspace{1cm} initial slowness

\( S \) \hspace{1cm} 14 \hspace{1cm} boundary of volume \( V \)

\( d^2 S \) \hspace{1cm} 14 \hspace{1cm} differential surface area element

\( t \) \hspace{1cm} 6 \hspace{1cm} time

\( u_i \hspace{0.5cm} i=1,2,3 \) \hspace{1cm} 5 \hspace{1cm} temporal Fourier transform of \( u_i \)

\( \dot{u}_i \hspace{0.5cm} i=1,2,3 \) \hspace{1cm} 6 \hspace{1cm} \( n^{th} \) coefficient of \( WKBJS \) expansion of displacement field

\( U_i^{(n)} \hspace{0.5cm} i=1,2,3 \hspace{0.5cm} n=0,1,... \) \hspace{1cm} 7 \hspace{1cm} leading order amplitude of displacement \( (U_i=U_i^{(0)}) \)

\( U_i(x) \hspace{0.5cm} i=1,2,3 \) \hspace{1cm} 9 \hspace{1cm} \text{Quasi-} P, \text{ Quasi-S and S-Parallel} \) displacement polarizations

\( U_{QP}, U_{QS}, U_{SP} \) \hspace{1cm} 19 \hspace{1cm} displacement polarization for mode \( M, M' \)

\( U_M, U_{M'} \) \hspace{1cm} 23 \hspace{1cm} displacement amplitude

\( |U| \) \hspace{1cm} 13 \hspace{1cm} an integration volume

\( V \) \hspace{1cm} 14 \hspace{1cm} differential volume element

\( d^3 V \) \hspace{1cm} 14 \hspace{1cm} phase velocity vector

\( V_i \hspace{0.5cm} i=1,2,3 \) \hspace{1cm} 9 \hspace{1cm} \text{Quasi-} P, \text{ Quasi-S and S-Parallel} \) phase velocity amplitudes

\( V_{QP}, V_{QS}, V_{SP} \) \hspace{1cm} 17 \hspace{1cm} group velocity vector

\( W \) \hspace{1cm} 26 \hspace{1cm} initial group velocity

\( W_0 \) \hspace{1cm} 26 \hspace{1cm} components of \( W_0 \)

\( W_{10} \hspace{0.5cm} i=1,2,3 \) \hspace{1cm} 27 \hspace{1cm} group velocity components, \( \text{Quasi-P} \) mode

\( W_{QPi} \hspace{0.5cm} i=1,2,3 \) \hspace{1cm} 20,21 \hspace{1cm} group velocity components, \( \text{Quasi-S} \) mode

\( W_{QSi} \hspace{0.5cm} i=1,2,3 \) \hspace{1cm} 21 \hspace{1cm} group velocity components, \( S\text{-Parallel} \) mode

\( W_{SPI} \hspace{0.5cm} i=1,2,3 \) \hspace{1cm} 21 \hspace{1cm} group velocity components, \( S\text{-Parallel} \) mode

\( WKBJS (expansion) \) \hspace{1cm} 7 \hspace{1cm} \text{Wentzel, Kramers, Brillouin, Jeffreys}

\( x \) \hspace{1cm} 7 \hspace{1cm} a generic position vector

\( x_i \hspace{0.5cm} i=1,2,3 \) \hspace{1cm} 3 \hspace{1cm} cartesian components of position vector
INTRODUCTION

The last two years have seen a renewed interest on the part of the exploration community in elastic anisotropy. Although anisotropy effects in geological formations have been extensively documented in the past (Love 1944, Stoneley 1949, White 1955 &1981), they have been only recently studied in relation to a phenomenon of crucial importance to hydrocarbon exploration: the presence of natural fractures in reservoirs.

Recent work by Crampin (1982), Martin (1985), Thomsen & al. (1986), and others, has shown that when oriented in a preferred direction, microcracking of a rock results in global anisotropy of mechanical properties which affects wave propagation. The widely accepted model describing this situation is known as transverse isotropy and its main characteristic, in terms of wave propagation, is referred to as shear wave birefringence.

The object of this report is to review, discuss, and develop the theory of wave propagation in transversely isotropic (or trisotropic) media, with special emphasis put on the definition of a ray tracing algorithm. The logical continuation of this work will be the development and simulation of a processing scheme for data collected over fractured formations, where the main goal will be to determine the orientation and density of the fracture system.
1. A QUALITATIVE REVIEW OF TRANSVERSELY ISOTROPIC MEDIA

1.1. Shear wave birefringence:

Shear wave birefringence within geologic formations is a direct manifestation of mechanical anisotropy of one of three major types (Crampin, 1985): (i) Intrinsic, (ii) Periodic Thin Lamination (PTL), or (iii) Extensive Dilatancy Anisotropy (EDA). Intrinsic anisotropy arises from the presence of crystals of particular symmetries (olivine, for example), within an isotropic matrix. PTL anisotropy on the other hand reflects the presence of thin laminations of well contrasted lithologies, while EDA is associated with microfracturing of rocks along the principal axes of present or past regional stresses.

Although all three anisotropy types are susceptible of exhibiting birefringence, fracture induced anisotropy is not only the most suitable for detection by reflection seismology, but also the most crucial to hydrocarbon exploration. In effect, fracture networks usually determine a preferential direction for fluid flow, and any information on that matter, that is, fracture orientation and density, can be extremely valuable to the recovery process.

We describe shear wave birefringence with the help of the simplified model illustrated in Figure (1.1). A shear (SH) source is placed over a uniform homogeneous elastic medium containing a horizontal, vertically fractured layer. Beneath the source, a horizontally polarized shear wave travels downwards (Fig1.1a). Upon reaching the top of the fractured formation, it naturally splits into two distinct waves: the first and fastest has polarization parallel to the fracture system, while the other is slower and polarized perpendicular to that system (Fig1.1b). This splitting is called shear wave birefringence, and we can qualitatively explain it as follows: as the wave enters the layer, its component of displacement aligned with the fractures encounters a stiffer material than the transverse component. Since shear wave velocity is proportional to the square root of shear stiffness, the parallel component must be the fastest (Fig1.1c).

As the two events further propagate downward, they each generate a reflection at the bottom of the anisotropic layer (the reflection at the top of the formation exhibits no special feature and was ignored for clarity). The first of these reflected events to reach the surface is polarized along the fracture system, and the second one across it (Fig1.1d). A straightforward analysis of polarization can therefore determine the orientation of the fracture network. On the other hand, the time delay between the arrivals, relative to the thickness of the layer, is simply related to the degree of anisotropy, that is, to fracture density.

Birefringence is sometimes directly observable on seismic records when the anisotropic formation is thick and fractured enough to clearly separate the two events. The seismic sections in Figure (1.2) are a striking example of a case where birefringence could directly be diagnosed. There is hope, however, that simple processing techniques may enhance birefringence effects to a degree where direct detection on seismic sections is not necessary.
1.2. The transversely isotropic model

We have explained how a network of vertical parallel cracks in an elastic solid causes shear wave birefringence. The transversely isotropic model adequately describes materials whose rigidity depends on the direction of applied force with respect to a symmetry axis. That model was first proposed by Love as a modification to the linear isotropic case, and therefore very much resembles Hooke’s law, except for the introduction of three additional elastic constants.

If we introduce a natural cartesian system of coordinates \((x_i)\) where \(x_1\) is the symmetry axis, the stress-strain relation is written:

\[
\begin{align*}
\sigma_{11} &= C \epsilon_{11} + F \epsilon_{22} + F \epsilon_{33} \\
\sigma_{22} &= F \epsilon_{11} + A \epsilon_{22} + (A - 2N) \epsilon_{33} \\
\sigma_{33} &= F \epsilon_{11} + (A - 2N) \epsilon_{22} + A \epsilon_{33} \\
\sigma_{12} &= 2L \epsilon_{12} \\
\sigma_{13} &= 2L \epsilon_{13} \\
\sigma_{23} &= 2N \epsilon_{23}
\end{align*}
\]  

(1.1)

In our case, the \(x_1\) direction is the normal to the fracture planes, and we will further refer to it as the anisotropy axis. Note that the stress-strain relation is much more complicated if the symmetry axis is not one of the coordinate axis.

We first notice that when restricted to the fracture plane \((x_2, x_3)\), the stress-strain relation is purely isotropic. In effect, with deformations allowed only in that plane, system (1.1) becomes:

\[
\begin{align*}
\sigma_{22} &= A \epsilon_{22} + (A - 2N) \epsilon_{33} \\
\sigma_{33} &= (A - 2N) \epsilon_{22} + A \epsilon_{33} \\
\sigma_{23} &= 2N \epsilon_{23}
\end{align*}
\]

These equations exactly describe a two-dimensional isotropic elastic medium where \(A\) and \(N\) are the compressional and shear stiffnesses, respectively. It is this property that suggested the name “transversely isotropic” for media described by system (1.1), with the meaning of “isotropic in a plane transverse to a symmetry axis” (here the \(x_1\) axis).

The best way to understand how transversely isotropic media differ from isotropic ones is to study the constraints which, when applied to the elastic constants in (1.1), yield Hooke’s law for isotropic media in terms of Lamé constants \(\lambda\) and \(\mu\). These constant are:

\[A = C = \lambda + 2\mu\]
\[ L = N = \mu \quad (1.2) \]

\[ F = \lambda \]

Therefore the degree of anisotropy is determined by how much the constants \( A \) and \( C \) on one hand, the constants \( L \) and \( N \) on the other hand, and finally the quantities \( A - 2N \) and \( F \), respectively differ.

It is an easy and nonetheless useful exercise to give a physical interpretation for the five elastic parameters in terms of static experiments. This is done in Figure (1.3) where we have adopted Thomsen’s idea of the deck of cards: although the cards in the deck should be somewhat bonded together for a more accurate description, the idea is that a transversely isotropic medium deforms more like a deck of cards than like a solid block of paper, which would be an adequate model for an isotropic material. The absence of bonds between successive cards is an exaggerated simulation of the weak mechanical bond that exists between the two walls of a microcrack in real rocks. The direction perpendicular to a card is therefore our anisotropy axis or \( z_1 \) direction. Deformations which involve sliding the cards with respect to each other require little applied stress. On the other hand, equal amounts of deformation within the plane of a card require a much larger stress. The elastic constants simply describe how much stress must be applied to obtain a standard deformation in a given direction. Figure (1.3a) shows for example that \( N \) is the amount of shear stress required to create a unit shear deformation within a card, and is therefore larger than \( L \), which is the amount of stress required for the same deformation but across the deck, where free sliding occurs (Fig1.3b). (Actually \( L \) is zero if totally free sliding is allowed between cards). In the same line of thought, Figures (1.3c) and (1.3d) show that \( A \) is the stress required for a unit extensional deformation of a card, while \( C \) is merely the stress necessary to separate the cards a certain distance from each other. If the cards are not bonded in any way, \( C \) is exactly zero, but for a real material \( C \) is simply less than \( A \) because the fractures yield more easily than the rock matrix.

To complete this preliminary picture, it is of interest to describe the propagation of plane waves in our model. We define the propagation direction of a wave at a given point as the direction of the normal to the wavefront at that point. That direction is also that of the phase velocity, which is the velocity we refer to in this section. First let us consider waves propagating along the principal axes of the system. It is elementary to show that compressional waves can travel along fracture planes at speed \( \sqrt{A/\rho} \), and at a slower \( \sqrt{C/\rho} \) along the symmetry axis. Shear waves propagating in the fracture planes can travel at two different speeds depending on the shear deformation they produce: if the deformation is in the fracture plane, the speed is \( \sqrt{N/\rho} \); if the deformation is perpendicular to the fracture plane, that is, along the anisotropy axis, the speed is a slower \( \sqrt{L/\rho} \). Therefore as we could expect, shear wave birefringence originates exclusively from the contrast between the two shear rigidities \( N \) and \( L \). Thus it is in theory possible to consider a simplified model where \( A \) and \( C \) would be identical, and which would still exhibit birefringence. If one decides however to retain
the generality of five independent parameters, which makes better physical sense, one finds out that three different wave-types can travel in a given direction:

- a **Quasi-P (QP)** wave with polarization roughly along propagation direction and velocity in the range \([\sqrt{C/\rho} , \sqrt{A/\rho}]\).
- a **Quasi-S (QS)** wave polarized almost transversely, traveling at speeds around \(\sqrt{L/\rho}\).
- an **S-Parallel (SP)** wave, truly transverse and polarized in the fracture plane, propagating at velocities in the range \([\sqrt{L/\rho} , \sqrt{N/\rho}]\).

Figures (1.4) and (1.5) show the azimuthal dependence of plane wave velocity and polarization on propagation direction for a specific choice of parameters. General expressions are not given here as they will later be derived and analyzed in a systematic way. The next section is a tentative to conciliate some fundamental mathematical and physical concepts of interest to the general question of wave propagation in anisotropic media.

### 2. HIGH FREQUENCY WAVES IN ELASTIC ANISOTROPIC MEDIA

#### 2.1. Fundamental equations and high frequency asymptotics:

In what follows, we consider linear elastic anisotropic solids, whose mechanical behavior is described by a linear stress-strain relation that can be considered as the most general form of Hooke’s law. The stress tensor \(\sigma_{ij}\) is expressed as the second order contraction of the deformation tensor \(\epsilon_{kl}\) with a fourth rank tensor of elastic coefficients \(c_{ijkl}\), that is, if we use Eistein’s summation convention on repeated indices:

\[
\sigma_{ij} = c_{ijkl} \epsilon_{kl} .
\]  

(2.1)

It would be a mistake, however, to consider \(c_{ijkl}\) as a general tensor because physical considerations of symmetry and energy conservation impose some strong constraints on the elastic coefficients, which are summarized in the following equations:

\[
c_{ijkl} = c_{jikl} ,
\]  

(2.2)

\[
c_{ijkl} = c_{ijkl} ,
\]  

(2.3)

\[
c_{ijkl} = c_{klij} ,
\]  

(2.4)

the meaning of these relations being that 21 independent elastic constants are enough to describe the most general linear elastic anisotropic solid.
The second law of importance is Newton's law, which expresses the mechanical equilibrium of an elementary volume of solid submitted to a stress field:

$$\sigma_{ij,j} = \rho \dddot{u}_i, \quad (2.5)$$

where $\rho$ is the density of the solid and $\dddot{u}_i$ the particle acceleration. Strain is defined from displacement as follows:

$$\epsilon_{kl} = \frac{1}{2} (u_{k,l} + u_{l,k}), \quad (2.6)$$

so that stress can be directly related to displacement according to (2.1):

$$\sigma_{ij} = \frac{1}{2} \ c_{ijkl} u_{k,l} + \frac{1}{2} \ c_{ijkl} u_{l,k} \quad (2.7)$$

Next we notice that:

$$c_{ijkl} u_{l,k} = c_{ijk} u_{k,l} = c_{ijlk} u_{l,k}$$

Here we have made use of the symmetry of the elastic coefficients in (2.3) and then merely exchanged two pairs of dummy indices. Consequently (2.7) can be rewritten in the simplified form:

$$\sigma_{ij} = c_{ijkl} u_{k,l} \quad (2.8)$$

and substituted into (2.5) to yield the elastic wave equation, which is a vector equation on displacement:

$$(c_{ijkl} u_{k,l})_{,j} = \rho \dddot{u}_i \quad (2.9)$$

We are now ready to present a high frequency formalism to solve the wave equation. First we define the time domain Fourier transform $\tilde{u}_i(\omega)$ of the displacement $u_i(t)$ with the following convention:

$$\tilde{u}_i(\omega) = \int_{-\infty}^{+\infty} u_i(t) e^{i\omega t} \ dt, \quad (2.9)$$

and, reciprocally:
\[ u_i(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{u}_i(\omega) e^{-i\omega t} \, d\omega, \]

so that (2.9) becomes:

\[ (c_{ijkl} \tilde{u}_k)_{,j} = -\rho \omega^2 \tilde{u}_i. \]  

Let us now make the following definition: if a time signal \( f(t) \) measured with absolute accuracy \( \epsilon \) is such that there exist two angular frequencies \( \omega_0 \) and \( \omega_1 \) for which:

\[ |\omega| < \omega_0 \quad \text{or} \quad |\omega| > \omega_1 \implies |f(\omega)| < \epsilon, \]

then \( f(t) \) is said to be essentially bandlimited. This property certainly applies to reflection seismic records, where the the upper bandlimit is imposed by attenuation, and the lower one by several factors including resolution, geophone response, source capabilities, and suppression of low frequency surface wave noise.

Whenever the lower bandlimit \( \omega_0 \) of the seismic signal is “large enough”, high frequency asymptotics adequately describe wave propagation and greatly simplify its study. Huygens principle, ray theoretical propagation, and wave-type decoupling are examples of the concepts that only hold in the asymptotic range. Fortunately, the high frequency approach holds as long as no significant smooth change of elastic parameter occurs within a wavelength and experience has proved that this criterion is generally met in exploration seismology. Discontinuity surfaces for elastic parameters within the model (reflectors) can be properly handled provided that the radius of curvature of these surfaces and the distance between any two surfaces be in excess of several wavelengths.

We now assume the validity of a high frequency or \( WKBJ \) solution to solve equation (2.10); that is, we seek a representation of the displacement field as an algebraic series in increasing powers of \( 1/i\omega \) (see Bleistein, 1985, for example):

\[ \tilde{u}_i(\omega,x) = e^{i\omega \tau(x)} \sum_{n=0}^{\infty} \frac{U_i^{(n)}(x)}{(i\omega)^n}. \]  

It is interesting to note here that the time domain equivalent for the leading term in the series is the singular function of the surface \( t = \tau(x) \), that is, the distribution \( \delta(t-\tau(x)) \). This distribution delineates the front of the wave field as time progresses, and \( \tau \) should therefore be considered as the traveltime field. Lower order terms are progressively smoother terms corresponding to further integrations of the singular function and can be ignored for high frequency waves. If we consider a bandlimited signal, then the leading order in time domain represents a bandlimited function of a variable normal to the surface \( t = \tau(x) \), which peaks on the wavefront.
Substituting expression (2.11) into (2.10) and ordering the resulting sum, we obtain:

\[
\sum_{n=0}^{\infty} \frac{e^{i\omega r}}{(i\omega)^{n-2}} \left\{ c_{ijkl} \left[ U_{ik}^{(n-2)} + U_{il}^{(n-1)} r_{i,l} + U_{jl}^{(n-1)} r_{j,l} + U_{kl}^{(n-1)} r_{k,l} \right] + c_{ijkl,j} U_{ik}^{(n-1)} r_{i,l} - \rho U_k^{(n)} \right\} = 0 \quad .
\] (2.12)

2.2. The eikonal equation:

The eikonal equation is obtained by setting the leading order coefficient of the series in (2.12) to zero, that is:

\[
\left[ c_{ijkl,r,i} - \rho \delta_{ik} \right] U_i^{(0)} = 0 \quad .
\] (2.13)

This equation allows us to solve both for the traveltime \( r \) and the polarization of the displacement \( U_i^{(0)} \). It can be presented as an eigenvalue problem in the following sense: if we define the matrix \( M_{ik} \) as follows:

\[
M_{ik} = c_{ijkl,r,i,j} \quad ,
\]

then (2.13) is equivalent to the problem of finding the traveltime gradient \( r_i \) that makes the eigenvalues of \( M_{ik} \) equal to the density \( \rho \). Solving for the traveltime gradient, or \textit{slowness vector}, is done by setting the characteristic determinant \( \det (M_{ik} - \rho \delta_{ik}) \) to zero that is:

\[
\det \left| c_{ijkl,r,i,j} - \rho \delta_{ik} \right| = 0 \quad .
\] (2.14)

This form of the eikonal equation is more suitable for explicit solution, since it has only \( r \) as unknown through its spatial derivatives: the determinant is a third degree polynomial in \( \rho \), that can be formally factored to yield three equations of the form:

\[
\rho = f \left( c_{ijkl,r,i,j} \right) \quad .
\] (2.15)

Each of these relations actually represents the dispersion equation for one of the three wave-types that exist in a generally anisotropic medium.

Having solved for (2.14) in the form (2.15), one can determine the eigenvector \( U_i^{(0)} \) associated with each solution to (2.14) in the following manner: substituting (2.15) into (2.13) makes the matrix \( [M_{ik} - \rho \delta_{ik}] \) singular, so that the solution for \( U_i^{(0)} \) is a one dimensional set representing an eigenvector of \( M_{ik} \). This result holds only if there is no
double or triple root in (2.14). The double root case corresponds to the isotropic elastic solid, while the triple root case represents the fluid limit, that is, a solid without shear rigidity. In these situations, the multiple eigenvalues respectively span a two dimensional vector space and the entire space itself.

Although computing the eigenvectors, or displacement polarizations, can be cumbersome in the general case, we can still always make the following statement: displacement polarization for the three wave types are orthogonal to each other for a given propagation direction. This result from the matrix $M_{ik}$ being symmetric:

$$M_{ik} = c_{ijkl} \tau^l, \tau^j = c_{ikjl} \tau^l, \tau^j = c_{klji} \tau^l, \tau^j = M_{ki}.$$

Here we have again permuted two pairs of dummy indexes, used the symmetry (2.4), and the commutativity of scalar multiplication. As a consequence, the eigenvectors of M must be orthogonal. Finally let us introduce the notion of phase velocity: since we have given to $\tau$ the interpretation of traveltime, we expect the quantity:

$$V_i = \frac{\tau_{i, k}}{\tau_{k, k}}, \quad (2.16)$$

to represent the velocity of the wavefront at any point of the wave field. This quantity is better known as the phase velocity.

2.3. The transport equation:

As we have just seen, the eikonal equation permits one to determine the dispersion relation and thereby the phase velocity of each wave type, as well as the displacement polarization of the leading order term in the WKBJ expansion. However, no further information has been obtained about the displacement amplitude. We will therefore introduce energy density, a quantity proportional to the squared displacement amplitude, and for which we can write a conservation equation. This differential equation will tell us how amplitude is distributed within a wave field.

We start by defining energy density $E$ as the sum of elastic potential energy and kinetic energy per unit volume of solid:

$$E = \frac{1}{2} \sigma_{ij} \epsilon_{ij} + \frac{1}{2} \rho \dot{u}_i \dot{u}_i, \quad (2.17)$$

To leading order, the displacement field is:

$$u_i = U_i(x)e^{i\omega(t)}.$$

Substituting in (2.17) yields:
\[ E = \frac{\omega^2}{2} \left[ c_{ijkl} U_i U_k p_j p_l + \rho U_i U_i \right] e^{2i\omega \tau}, \]

where \( p_i = \tau_i \) is the slowness vector.

On the other hand, Newton’s law (2.5) simplifies to leading order into:

\[ c_{ijkl} U_k p_j p_l = \rho U_i, \]

which we may rewrite:

\[ \frac{\omega^2}{2} c_{ijkl} U_k U_k p_j p_l = \frac{\omega^2}{2} \rho U_i U_i, \tag{2.18} \]

and interpret as follows: kinetic and potential energy are equal for high frequency elastic waves in anisotropic media.

As a consequence, we may express energy density in one of the two following equivalent forms:

\[ E = \rho \omega^2 U_i U_i e^{2i\omega \tau}, \tag{2.19} \]

or,

\[ E = \omega^2 c_{ijkl} U_k U_k p_j p_l e^{2i\omega \tau}. \tag{2.20} \]

Let us now develop a conservation equation from the second order term of the asymptotic wave equation (2.12). This equation is:

\[ \left[ c_{ijkl} p_l p_j - \rho \delta_{ik} \right] U_i^{(1)} - c_{ijkl, j} U_i^{(0)} p_l \]

\[ - c_{ijkl} \left[ U_i^{(0)} p_j + U_{i, j}^{(0)} p_l + U_{i, l}^{(0)} p_j \right] = 0. \tag{2.21} \]

This is a vector equation for \( U_i^{(1)} \) of the form:

\[ A_{ki} U_i^{(1)} = B_k, \tag{2.22} \]

but where \( \text{det} (A_{ki}) = 0 \) according to the eikonal equation (2.14). This means that at least one eigenvalue of \( A_{ki} \) is zero, and that \( B_k \) must be orthogonal to the associated eigenvector(s) in order for (2.22) to have a solution, that is:

\[ c_{ijkl, j} U_i U_k p_l + c_{ijkl} U_{i,j} U_k p_j + c_{ijkl} U_{i, j} U_{k, j} p_l + c_{ijkl} U_i U_k p_{l, j} = 0, \tag{2.23} \]

where \( U_i \) is now one of the eigenvectors of \( A_{ki} \). The second term can be rearranged by
permutation of indexes and the use of symmetry (2.4):
\[ c_{ijkl} U_{i} U_{k} p_{j} = c_{ilkj} U_{k} U_{i} p_{l} = c_{klij} U_{k} U_{l} p_{i} = c_{ijkl} U_{k} U_{i} p_{l} \]

making the left member of (2.23) an exact divergence:
\[ \left( c_{ijkl} U_{i} U_{k} p_{l} \right),j = 0 \]

This relation can in turn be expressed in the form:
\[ (EW_{j}),j = 0 \]
(2.24)

where $EW_{j}$ is the energy density flux vector. The vector $W_{j}$, which has the dimensions of velocity, is known as the group velocity. Taking for $E$ the expression (2.19) and identifying the terms in (2.24), we get the following expression for group velocity:
\[ W_{j} = \frac{c_{ijkl} U_{i} U_{k} p_{l}}{\rho U_{n} U_{n}} \]
(2.25)

At this point it is of interest to notice that phase and group velocity are not necessarily colinear. We can however prove that group velocity can always be expressed as the vector sum of phase velocity and a vector perpendicular to phase velocity. From (2.25) we have:
\[ p_{j} W_{j} = \frac{c_{ijkl} U_{i} U_{k} p_{l} p_{j}}{\rho U_{n} U_{n}} \]

Equations (2.19) and (2.20) show that the right hand term is the ratio of potential to kinetic energy, and therefore equal to unit, hence:
\[ p_{j} W_{j} = 1 \]
(2.26)

On the other hand, from the definitions of slowness and phase velocity, we have:
\[ p_{j} = \frac{V_{j}}{V_{n} V_{n}} \]

that is, phase velocity and slowness have identical directions and reciprocal moduli. Substituting this result into (2.26) yields:
\[ V_{j} W_{j} = V_{n} V_{n} \]

which means that the orthogonal projection of group velocity on phase velocity equals phase velocity. In other words, group velocity is the sum of phase velocity and a vector perpendicular to phase velocity.
2.4. Rays and group velocity:

At this point we have derived the two differential equations necessary to solve for high frequency wave fields in anisotropic media, namely the eikonal equation (2.14) and the transport equation (2.24). The eikonal equation is in general a highly non-linear first order differential equation in $\tau$, which can be solved by the method of characteristics. Let us symbolically express the eikonal equation in the form:

$$F(z_i, \tau, p_i) = 0$$

where once again $p_i = \tau_i$. The method of characteristics allows us to rewrite this equation as a system of one dimensional first order equations along characteristic curves or rays (see Bleistein, 1985 for example). The rays are defined by:

$$\frac{dz_i}{F_{,p_i}} = \lambda d\sigma$$

where $\lambda$ is any non-zero scalar, while $\sigma$ is the ray variable. Along a ray, the three unknown $\tau$, $z$, $p$ satisfy the following differential equations:

$$\frac{d\tau}{d\sigma} = \lambda p_n F_{,p_n}$$  \hspace{1cm} (2.27)

$$\frac{dz_n}{d\sigma} = \lambda F_{,p_n}$$  \hspace{1cm} (2.28)

$$\frac{dp_n}{d\sigma} = -\lambda (F_{,z_n} + p_n F_{,\tau})$$  \hspace{1cm} (2.29)

The starting point for a ray is determined by an initial value for $z_i$, where $\tau$ and $p_i$ are specified.

We are now going to prove that rays are always directed along group velocity. To do this, let us consider the eikonal equation (2.12), which states that all the components of the vector $[\epsilon_{ijkl} p_i p_j - \rho \delta_{ik}] U_l$ are zero. This statement is equivalent to having the projections of that vector along three independent directions vanish. In particular, projection on $U_k$ itself must vanish. That condition provides an eikonal equation in the form:

$$F(z_i, \tau, p_i) = \left( \epsilon_{ijkl} p_i p_j - \rho \delta_{ik} \right) U_l U_k = 0$$  \hspace{1cm} (2.30)

The ray direction is given by (2.28), so that we obtain:
\[ F_{p_n} = c_{ijkl} \delta_{nl} p_j U_i U_k + c_{ijkl} \delta_{in} p_i U_i U_k \left( c_{ijkl} p_i p_j - \rho \delta_{ik} \right) \left( u_k U_{i,p_n} + u_i U_{k,p_n} \right) . \]

The last term is obviously zero, owing to (2.13) and to the symmetry of the eikonal matrix. Contracting the first two terms yields:

\[ F_{p_n} = c_{ijkn} p_j U_i U_k + c_{inkl} p_i U_i U_k \]  

The first term is identical to the second one as we show below by using index permutations and, once again, symmetry (2.4):

\[ c_{ijkn} p_j U_i U_k = c_{knij} p_j U_k U_i = c_{inkj} p_j U_i U_k = c_{inkl} p_i U_i U_k \]

Hence:

\[ \frac{dz_n}{d\sigma} = 2\lambda c_{inkl} U_i U_k p_l \]  

Next we compute the scalar \( dt/\sigma \) according to (2.31):

\[ \frac{dt}{d\sigma} = p_n F_{p_n} = 2\lambda c_{inkl} U_i U_k p_i p_n \]  

Using identity (2.18), this expression becomes:

\[ \frac{dt}{d\sigma} = 2\lambda \rho U_i U_i = 2\lambda \rho |U|^2 \]  

so that we can express the ray direction in terms of a velocity \( dx_n/d\tau \) by properly scaling \( dx_n/d\sigma \):

\[ \frac{dx_n}{d\tau} = \frac{dx_n/d\sigma}{dt/d\sigma} = \frac{c_{inkl} U_i U_k p_l}{\rho |U|^2} \]  

A comparison with (2.25) shows that:

\[ \frac{dx_n}{d\tau} = W_n , \]  

that is, the rays are directed along group velocity. This general result is of great importance in showing that energy is conserved along ray tubes. In particular, it allows
us to rewrite the transport equation as a differential equation along rays, as we show next.

Let us first define a ray tube as a 2-parameter family of rays: if a ray is indexed by the couple \((\gamma_1^0, \gamma_2^0)\), then the associated ray tube is a beam containing all rays indexed in the range:

\[
\gamma_1^0 - \frac{d\gamma_1}{2} < \gamma_1 < \gamma_1^0 + \frac{d\gamma_1}{2},
\]

\[
\gamma_2^0 - \frac{d\gamma_2}{2} < \gamma_2 < \gamma_2^0 + \frac{d\gamma_2}{2},
\]

where \(d\gamma_1\) and \(d\gamma_2\) are arbitrarily small quantities. Then we integrate the transport equation (2.24) along a section of ray tube defined by:

\[
\sigma_1 < \sigma < \sigma_2.
\]

Such a section of ray tube is illustrated in Figure (2.1). The transport equation becomes:

\[
\int_V (EW_i)_{i} d^3x = 0.
\]

Making use of the divergence theorem allows to rewrite (2.32) as:

\[
\int_S EW_i n_i d^2S = 0,
\]

where \(n_i\) is the unit vector normal to the surface \(S\) of the tube section. On the walls of the tube, defined by:

\[
\gamma_1 = \gamma_1^0 \pm d\gamma_1,
\]

and

\[
\gamma_2 = \gamma_2^0 \pm d\gamma_2,
\]

the product \(W_i n_i\) vanishes since the walls are by definition tangent to the rays, hence to group velocity \(W_i\). In consequence (2.33) simplifies into:

\[
\int_{\sigma=\sigma_1} EW_i n_i d^2S - \int_{\sigma=\sigma_2} EW_i n_i d^2S = 0.
\]
Since the two integrals apply to infinitesimal surfaces, they can be computed using the average value of the integrand, so that we get:

\[
\left[ E W_i n_i d^2 S \right]_{\sigma_i} = 0
\]

Next we notice that:

\[
n_i d^2 S = \epsilon_{ijk} \frac{\partial x_j}{\partial \gamma_1} \frac{\partial x_k}{\partial \gamma_2} d\gamma_1 d\gamma_2
\]

and substitute for \( W_i \) according to (2.31) in order to obtain:

\[
\left[ E \epsilon_{ijk} \frac{\partial x_i}{\partial \tau} \frac{\partial x_j}{\partial \gamma_1} \frac{\partial x_k}{\partial \gamma_2} \right]_{\sigma_i} = 0
\]

Finally, substituting for \( E \) according to (2.19) and defining a ray Jacobian \( J \) as follows:

\[
J(\sigma, \gamma_1, \gamma_2) = \epsilon_{ijk} \frac{\partial x_i}{\partial \tau} \frac{\partial x_j}{\partial \gamma_1} \frac{\partial x_k}{\partial \gamma_2}
\]

we get the transport equation as a difference equation along rays:

\[
\left[ \rho U^2 J \right]_{\sigma_i} = 0
\]

This equation can also be expressed in differential form by making \( \sigma_1 \) and \( \sigma_2 \) arbitrarily close:

\[
\frac{d}{d\sigma} \left( \rho U^2 J \right) = 0 \quad (2.34)
\]

Joined with the system (2.27) through (2.29), this equation allows to totally solve for the leading order elastic field.

At this point, we have formally solved the problem of determining a high frequency solution to the wave equation in (weakly) inhomogeneous linear elastic anisotropic media, by means of the method of characteristics. We have not, however, described practically how to factor (2.14), which is the starting point for the method, nor have we considered what happens at discontinuity surfaces for elastic parameters (reflectors), where the \textit{WKBJ} expansion fails. It also remains to provide adequate initial conditions for the prescribed equations. In order to address these problems and others, we now go back to the specific case of interest to us, namely transverse isotropy.
3. SPECIALIZATION TO THE TRANSVERSELY ISOTROPIC CASE

3.1. Factorization of the eikonal equation - phase velocities:

To a transversely isotropic solid corresponds a particular choice of elastic coefficients which we determine by term by term identification of (2.1) and (1.1):

\[ c_{1111} = C, \]
\[ c_{2222} = c_{3333} = A, \]
\[ c_{2233} = c_{3322} = A - 2N, \]
\[ c_{1133} = c_{3311} = c_{1122} = c_{2211} = F, \]
\[ c_{1212} = c_{2121} = c_{2112} = c_{1221} = L, \]
\[ c_{1313} = c_{3131} = c_{3113} = c_{1331} = L, \]
\[ c_{2323} = c_{3232} = c_{3223} = c_{2332} = N. \]

The eikonal equation (2.14) therefore becomes:

\[
\begin{vmatrix}
\rho - Lp_2^2 - Lp_3^2 - Cp_1^2 & -(F + L)p_1p_2 & -(F + L)p_1p_3 \\
-(F + L)p_1p_2 & \rho - Np_2^2 - Ap_2^2 - Lp_1^2 & -(A - N)p_2p_3 \\
-(F + L)p_1p_3 & -(A - N)p_2p_3 & \rho - Ap_2^2 - Np_2^2 - Lp_1^2 \\
\end{vmatrix}
= 0.
\]

First we multiply column 2 by \( p_3 \) and subtract it from column 3 multiplied by \( p_2 \), then multiply line 3 by \( p_3 \) and add it to line 2 multiplied by \( p_2 \). Next, expanding along the third column yields the following equation:

\[
\left( \rho - Lp_1^2 - N(p_2^2 + p_3^2) \right) \begin{vmatrix}
\rho - Cp_1^2 - L(p_2^2 + p_3^2) & -(F + L)p_1 \\
-(F + L)p_1(p_2^2 + p_3^2) & \rho - Lp_1^2 - A(p_2^2 + p_3^2) \\
\end{vmatrix}
= 0.
\]

Finally we factor the \( 2 \times 2 \) determinant to get the desired form:

\[
\left[ \rho - Lp_1^2 - N(p_2^2 + p_3^2) \right] \left[ \rho - \frac{(A + L)(p_2^2 + p_3^2) + (C + L)p_1^2 + \sqrt{\Delta}}{2} \right].
\]
\[
\left[ \rho - \frac{(A+L)(p_2^2 + p_3^2) + (C+L)p_1^2 - \sqrt{\Delta}}{2} \right] = 0 \quad , \quad (3.1)
\]

where:
\[
\Delta = \left( (A-L)(p_2^2 + p_3^2) - (C-L)p_1^2 \right)^2 + 4p_1^2(p_2^2 + p_3^2)(L+F)^2 \quad .
\]

This equation illustrates the fact that to leading order in frequency, three decoupled wavetypes, or modes, can simultaneously exist in an transversely isotropic medium. Each mode will therefore from now on be considered separately, and coupling will only reappear when we consider scattering at interfaces.

As was mentioned earlier, phase velocity \( V_i \) is the vector describing the speed at which wavefronts (i.e. the surfaces of constant traveltime), expand. It is therefore oriented normal to wavefronts and has intensity inversely proportional to the magnitude of travel time gradient:
\[
V_i = \frac{\tau_{i,n}}{\tau_{n,n}} = \frac{p_i}{p_n p_n} \quad . \quad (3.2)
\]

In anisotropic media, phase velocity depends on propagation direction. By introducing the angle \( \phi \) between that direction and the anisotropy axis \( (x_1) \), we can express the velocity moduli as follows:

for the \textit{Quasi} – \textit{P} mode:
\[
V_{QP}(\phi) = \left[ \frac{\alpha + \beta}{2\rho} \right]^{1/2} \quad ,
\]

for the \textit{Quasi} – \textit{S} mode:
\[
V_{QS}(\phi) = \left[ \frac{\alpha - \beta}{2\rho} \right]^{1/2} \quad , \quad (3.3)
\]

for the \textit{S} – \textit{Parallel} mode:
\[
V_{SP}(\phi) = \left[ \frac{N\sin^2 \phi + L\cos^2 \phi}{\rho} \right]^{1/2} \quad ,
\]

where:
\[
\alpha = (A+L)\sin^2 \phi + (C+L)\cos^2 \phi \quad ,
\]
\[
\beta = \left[ \left( (A-L)\sin^2 \phi - (C-L)\cos^2 \phi \right)^2 + 4\sin^2 \phi \cos^2 \phi (L+F)^2 \right]^{1/2} \quad .
\]
These results show that for $S-Parallel$ and $Quasi-P$ waves, phase velocity is minimum along the symmetry axis and increases as propagation departs from that direction to reach a maximum with in-plane propagation. In contrast, the $Quasi-S$ velocity must exhibit at least one extremum between $\phi=0$ and $\phi=\pi/2$ since its magnitude is identical for these two angles.

3.2. Eigenvectors computation - displacement polarizations:

Here we will make use of equation (2.13), which we first need to rewrite for the transversely isotropic case:

\[
\begin{align*}
\left(\rho - L p_1^2 - L p_3^2 - C p_1^2 \right) U_1 - (F+L) p_1 p_2 U_2 - (F+L) p_1 p_3 U_3 &= 0, \\
-(F+L) p_1 p_2 U_1 + \left(\rho - N p_2^2 - A p_2^2 - L p_1^2 \right) U_2 - (A-N) p_2 p_3 U_3 &= 0, \\
-(F+L) p_1 p_3 U_1 - (A-N) p_2 p_3 U_2 + \left(\rho - A p_3^2 - N p_3^2 - L p_1^2 \right) U_3 &= 0.
\end{align*}
\]

(3.4)

To solve for the eigenvectors, we consecutively substitute the eigenvalues determined by (3.1) into the system (3.4) which then becomes singular of order one. The solutions are therefore found in the form of one-dimensional sets which are the eigenspaces associated with the eigenvalues. Let us start with the $S-Parallel$ mode, that is, substitute:

\[
\rho = L p_1^2 + N (p_2^2 + p_3^2),
\]

into (3.4). This makes the last two equations dependent so that the system reduces to:

\[
\begin{align*}
\frac{(C-L)p_1^2 + (N-L)(p_2^2 + p_3^2)}{F+L} p_1 U_1 + p_2 U_2 + p_3 U_3 &= 0, \\
\frac{F+L}{A-N} p_1 U_1 + p_2 U_2 + p_3 U_3 &= 0.
\end{align*}
\]

These two equations can be satisfied simultaneously only if $U_1$ vanishes. This means that the $S-Parallel$ mode is polarized within the fracture planes, hence its name. Another consequence is that we can write:

\[
0 = p_2 U_2 + p_3 U_3 = p_1 U_1 + p_2 U_2 + p_3 U_3 = p_i U_i,
\]

that is, the parallel mode is a truly transverse mode since it is polarized perpendicular to propagation direction, and we can write:
\[ U_{SP} \propto (0, p_3, -p_2) \]

As we know from the symmetry of (3.4), the other two eigenvectors must be orthogonal to \( U_{SP} \), and we therefore look for them in the form:

\[ U_{QP} \propto (\epsilon_P p_1, p_2, p_3) \]

\[ U_{QS} \propto (\epsilon_S p_1, p_2, p_3) \]

By substituting:

\[ \rho = \frac{(A + L)(p_2^2 + p_3^2) + (C + L)p_1^2 + \sqrt{\Delta}}{2} \]

into (3.4) we obtain:

\[ \epsilon_P = \frac{(F + L)\sin^2 \phi}{\rho V_{QP}^2 - L \sin^2 \phi - C \cos^2 \phi} \]

while by substituting:

\[ \rho = \frac{(A + L)(p_2^2 + p_3^2) + (C + L)p_1^2 - \sqrt{\Delta}}{2} \]

we obtain:

\[ \epsilon_S = \frac{(F + L)\sin^2 \phi}{\rho V_{QS}^2 - L \sin^2 \phi - C \cos^2 \phi} \]

Finally we introduce a second angle \( \theta \) as described in Figure (3.1) and normalize the three eigenvectors:

\[ U_{SP} = \frac{1}{|U_{SP}|} (0, \sin \theta, -\cos \theta) \]

\[ U_{QP} = \frac{(\delta_P, \cos \theta \sin \phi, \sin \theta \sin \phi)}{\sqrt{\sin^2 \phi + \delta_P^2}} \]

\[ U_{QS} = \frac{(\delta_S, \cos \theta \cos \phi, \sin \theta \cos \phi)}{\sqrt{\cos^2 \phi + \delta_S^2}} \]  \hspace{1cm} (3.5)

where:

\[ \delta_P = \frac{(F + L)\sin^2 \phi \cos \phi}{\rho V_{QP}^2 - L \sin^2 \phi - C \cos^2 \phi} \]

and:

\[ -19- \]
\[ \delta_S = \frac{(F+L)\sin \phi \cos^2 \phi}{\rho V^2_{QS} - L \sin^2 \phi - C \cos^2 \phi} . \]

We note that \( \delta_S \) has a removable singularity at \( \phi = \pi/2 \) and \( \delta \rho \) has one at \( \phi = 0 \).

It is of interest here to remark that polarizations are actually independent of density. Figure (3.2) shows how displacement polarizations relate to propagation and anisotropy directions.

### 3.3. Group velocities and triplication effects:

We have derived in (2.25) an expression for group velocity involving displacement polarizations, slowness vector, and elastic coefficients. It is simpler, and therefore preferable, to use a direct calculation based on (2.27) and (2.28). In effect, we have proved that:

\[ W_n = \frac{dx_n}{d\tau} = \frac{F_{,p_n}}{p_i F_{,p_i}} , \]

where \( F(p_n) = 0 \) is the eikonal equation, which we will take in the form (3.1). Again we will consider each mode separately; for the Quasi-\( P \) wave, we have:

\[ F(p_n) = \rho - \frac{(A+L)(p_2^2 + p_3^2) + (C+L)p_1^2 + \sqrt{\Delta}}{2} . \]

The computation of group velocity is straightforward if one recognizes that:

\[ p_n F_{,p_n} = \alpha + \beta , \]

and expresses \( p_n \) in terms of angles \( \theta \) and \( \phi \):

\[ W_{QP1} = \cos \phi \frac{\beta (C+L) + 2 \sin^2 \phi (L+F)^2 - (C-L) \left[ (A-L) \sin^2 \phi - (C-L) \cos^2 \phi \right]}{\sqrt{2 \rho \beta^2 (\alpha + \beta)}} , \]

\[ W_{QP2} = \sin \phi \cos \theta \frac{\beta (A+L) + 2 \cos^2 \phi (L+F)^2 + (A-L) \left[ (A-L) \sin^2 \phi - (C-L) \cos^2 \phi \right]}{\sqrt{2 \rho \beta^2 (\alpha + \beta)}} . \]
\[ W_{QP3} = \frac{\beta (A + L) + 2 \cos^2 \phi (L + F)^2 + (A - L) \left( (A - L) \sin^2 \phi - (C - L) \cos^2 \phi \right)}{\sqrt{2 \rho \beta^2 (\alpha + \beta)}}. \]

In a quite similar way, we obtain the group velocity for the \textit{Quasi}-\textit{S} mode:

\[ W_{QS1} = \frac{\beta (C + L) - 2 \sin^2 \phi (L + F)^2 + (C - L) \left( (A - L) \sin^2 \phi - (C - L) \cos^2 \phi \right)}{\sqrt{2 \rho \beta^2 (\alpha - \beta)}}, \]

\[ W_{QS2} = \frac{\beta (A + L) - 2 \cos^2 \phi (L + F)^2 - (A - L) \left( (A - L) \sin^2 \phi - (C - L) \cos^2 \phi \right)}{\sqrt{2 \rho \beta^2 (\alpha - \beta)}}, \]

\[ W_{QS3} = \frac{\sin \phi \cos \theta}{\sqrt{2 \rho \beta^2 (\alpha - \beta)}}. \]

and finally for the \textit{S}-\textit{Parallel} mode:

\[ W_{SP1} = \frac{L \cos \phi}{\sqrt{\rho (N \sin^2 \phi + L \cos^2 \phi)}}, \]

\[ W_{SP2} = \frac{N \sin \phi \cos \theta}{\sqrt{\rho (N \sin^2 \phi + L \cos^2 \phi)}}, \quad (3.6) \]

\[ W_{SP3} = \frac{N \sin \phi \sin \theta}{\sqrt{\rho (N \sin^2 \phi + L \cos^2 \phi)}}. \]

Here notice that group velocity amplitude is independent of the angle \( \theta \), as expected from the symmetry of the solid. Actually writing group velocity in terms of axial and radial components would remove any \( \theta \) dependence that is, the spatial variability in group velocity can be totally described with the angle \( \phi \). Figure (3.3) shows a polar plot of group velocity for the three wavetypes. The driving parameter for these curves is \( \phi \), and we note that the \textit{Quasi}-\textit{S} velocity exhibits singular points. These singularities are related to the phenomenon of triplication, which arises because energy and wavefronts do not propagate at the same velocity in anisotropic media.

Let us now describe the phenomenon of triplication. For this, we will use the property that group velocity is perpendicular to the slowness surface, which is the surface formed by the extremity of the slowness vector as its direction is varied through all possible azimuths within a homogeneous medium. The equation of this surface is nothing else but the eikonal equation for a given mode, and we will rewrite it as in section 2.4 in the form:
\[ F(p_i) = 0 \]

On the other hand, according to equations (2.27) and (2.28), group velocity, can be expressed in the form:

\[ W_i = \frac{dz_i/d\tau}{p_n F_{p_n}} \]

This shows that group velocity is aligned on the \( p \)-gradient of the function \( F(p_i) \), and hence perpendicular to the slowness surface \( F(p_i) = 0 \).

Whenever the slowness surface is convex, the ray network emanating from it is single-folded. It is only when the slowness surface is concave that this network becomes multiple-folded, as illustrated in Figure (3.4). Typically, each concave cusp in the surface is source to a three-fold ray family limited by two surface caustics, whereby the name “triplication”. Only the \textit{Quasi-} \( S \) mode occasionally exhibits triplications, when severe anisotropy is considered.

Since the slowness surface has a symmetry of revolution around the anisotropy axis, triplication points can be determined from the analysis of the trace of that surface in an axial plane. In such a plane, the slowness surface has an equation that can be expressed in polar coordinates \((p, \phi)\) in the form:

\[ p = f(\phi) \]

For example, in the \textit{Quasi-} \( S \) case, we have according to (3.3):

\[ f(\phi) = \left[ \frac{2\rho}{\alpha(\phi) - \beta(\phi)} \right]^{1/2} \]

where \( \alpha \) and \( \beta \) are the functions defined in (3.3).

For the curve to exhibit concavity, its curvature must vanish whenever concave and convex arcs meet together. Hence triplication is present if and only if the equation:

\[ \frac{d}{d\phi} \left( \frac{1}{f} \frac{df}{d\phi} \right) = 0 \]

admits real solutions. Finding the possible triplication angles from this formulation is not trivial however, since it amounts to finding the roots of a higher order real polynomial.

3.4. Reflection and transmission at a welded contact:

So far we have considered the propagation of elastic waves in media where elastic properties vary "smoothly", that is, where the relative change of parameters within a wavelength remains small. This excludes, of course, proper handling of sharp discontinuities in elastic parameters or reflectors, which must be dealt with separately. The classical and efficient way to do this is to formally express the wave field on both sides of the reflector and to apply some kind of boundary conditions. In the case of two
rock layers, these conditions are the continuity of displacement and traction at the interface, which describe a welded contact. From them, one obtains the scattering directions and amplitudes for the various wavetypes.

The incident medium is defined as the side of the interface where the incident field, that is, the field that would exist without the presence of the reflector, is known. The transmission medium therefore represents the other side of the interface. In each region, all three wavetypes should a priori be present, and the displacement field is thus represented as the superposition of all modes. The continuity of displacement at the interface is then written as follows:

\[ \sum_M U_M e^{i\omega \tau_M} = \sum_{M'} U_{M'} e^{i\omega \tau_{M'}} \]  

where \( M \) and \( M' \) stand for the various modes in the incident and transmission media. From this we can conclude that travel times have to be equal for incident and scattered events at the interface:

\[ \tau_M = \tau_{M'} \quad \forall \ M, M' \]  

This equality can be supported by the fact that (3.8) holds for any value of angular frequency \( \omega \). Substituting this identity in (3.8), we get a second important continuity condition on amplitudes:

\[ \sum_M U_M = \sum_{M'} U_{M'} \]  

Let us then assume that the interface is locally continuous, and that the reflector is parametrized by two independent spatial variables \( \sigma_1 \) and \( \sigma_2 \). Expanding (3.9) along the interface, we obtain:

\[ \nabla_\sigma \tau_M = \nabla_\sigma \tau_{M'} \]  

which is Snell’s law in its general form. An equivalent statement is that the component of slowness tangent to the interface is identical for incident and scattered waves.

Let us now briefly discuss how to apply this result to our problem; contrarily to the isotropic case, Snell’s law does not reduce to a simpler form, and the scattered slownesses must be evaluated numerically. We start by computing the component of slowness tangent to the interface for the incident field. According to Snell’s law, all the scattered waves share the same tangential slowness as the incident one, hence the problem reduces to finding the normal component of slowness for each scattered mode.

For the \( SP \) mode, this amounts to solving a second degree real polynomial equation, which is done explicitly. If the roots are real, one distinguishes the reflected solution from the transmitted one by computing the associated group velocity vector and determining whether it points towards the desired scattering medium. Note that in general the sign of the normal slowness is not enough to discriminate between reflection
and transmission. In particular, the two real roots may have the same sign and the corresponding slownesses point towards the same medium; in such case, it is the group velocity that shows without any ambiguity which solution is reflected and which one is transmitted (Fig. 3.5). If the roots are complex, one must choose the solution that is evanescent in the scattering medium.

For the QS and QP modes, the scattering problem reduces to a fourth degree real polynomial equation, which must be solved numerically. The roots are then classified according to the mode they represent (two QP and two QS), which is a simple task unless all four roots are complex. Then the adequate solution is selected for each mode separately in exactly the same manner as for the SP mode.

Knowing the slownesses associated with both incident and scattered waves, we can determine the associated displacement polarizations from equations (3.5). Then we project them onto the principal directions of anisotropy (one axial and two radial), to obtain the strain tensor which to leading order is defined by:

$$\epsilon_{ij} = \frac{1}{2} \left(U_i p_j + U_j p_i\right).$$

Following this operation, we are able to compute the stresses according to equations (1.1). Then we obtain the tractions at the interface by contracting the stress tensor with the unit vector normal to the reflector previously expressed in terms of principal axes. Finally, the tractions are projected back on the reference coordinate system so that continuity conditions can be applied. The whole procedure is independent of the reference system, which allows us to treat the most general situations, and, although many intermediate operations are involved, it is not computationally intensive; the use of trigonometric functions can, in particular be entirely avoided. The reflection and transmission coefficients are defined here as the ratios of reflected or transmitted displacement amplitudes to the incident one. The continuity of displacement and traction therefore result in a 6x6 system of linear equations which is solved using Gaussian elimination.

To conclude this chapter, we have joined a few examples illustrating how the scattering angles relate to the incident ones for various modes. Figure (3.6) emphasizes the effect of fracture azimuth; scattering angles do not seem to be seriously affected by the fracture azimuth in this example. Figure (3.7) shows the effect of fracture dip on the scattering angle; as can be expected, incidence and scattering angles are identical whenever the fracture network is parallel or perpendicular to the interface. Intermediate fracture orientations however show a significant deviation between these angles, especially at large incidence angles. Figure (3.8) presents an example of reflection and transmission coefficients at the interface between two transversely isotropic solids (only the amplitudes are shown).
4. RAY TRACING IN PIECEWISE HOMOGENEOUS “TRISOTROPIC” MEDIA

4.1. The model:

We consider here piecewise homogeneous media, constructed by a superposition of uniform layers separated by smooth \( (C_1) \) boundaries. Within each layer, the five elastic parameters \( A, C, N, L, F \) and density \( \rho \) are constant and the anisotropy direction is fixed. There is a hierarchy of models to be treated according to the complexity of the reflectors geometry. The simplest one is referred to as \( 1.5-D \): the boundaries between layers are flat and parallel, that is, the variability in elastic properties is one-dimensional although three-dimensional propagation is accounted for; this is the tabular model described later in Section 4.3. In contrast with the isotropic case, \( 2-D \) or \( 2.5-D \) ray tracing is not possible, except if strong restrictions are put on the respective orientation of fractures and reflectors. Such restrictions make those models of little interest. The reason for this limitation is that in most situations, the ray segments that link source and receiver do not remain coplanar, as we will later show. The next model after \( 1.5-D \) is therefore the full \( 3-D \) model, where no geometrical restrictions apply.

We must also discuss source models, which are essential in shaping the wave field. They also probably involve the most difficult and least understood physical mechanisms in wave propagation. In the case of anisotropic media, the situation is complicated even more by effects such as triplication (see Van der Hilden for example). We will resolve here to simplicity by assimilating sources to ideal punctual forces and placing them in isotropic solids. This means that the sources are located at the surface of the top layer, which is isotropic. Provided that the observation point is situated several wavelengths away from the source, the incident field is adequately described by well known far field expressions (see White, 1982, for example). The associated wavefronts are spherical and provide an ideal initial surface for the ray tracing process.

4.2. Marching the algorithm:

Given one of the models described above, ray tracing is straight forward. In effect, if we elect to choose travelt ime to be the driving parameter along rays \( (\sigma = r) \), equations (2.31) through (2.33) simplify into:

\[
\frac{\partial z_i(r, \gamma_1, \gamma_2)}{\partial r} = W_i(r, \gamma_1, \gamma_2) \quad , \tag{4.1}
\]

\[
\frac{\partial p_i(r, \gamma_1, \gamma_2)}{\partial r} = 0 \quad , \tag{4.2}
\]

while the transport equation (2.38) becomes:

\[
\frac{\partial}{\partial r} \left( U^2(r, \gamma_1, \gamma_2) J(r, \gamma_1, \gamma_2) \right) = 0 \quad , \tag{4.3}
\]

where \( W_i \) is the group velocity and \( J \) the ray Jacobian defined by:
\[ J(r, \gamma_1, \gamma_2) = \epsilon_{ijk} W_i \frac{\partial x_i}{\partial \gamma_1} \frac{\partial x_k}{\partial \gamma_2} \]  \hspace{1cm} (4.4)

Equation (4.2) shows that within a layer the slowness is constant: \( p_i = p_{i0} \). Consequently the same is true of group velocity, which can be computed from slowness using equations (3.6). This also means that (4.1) integrates immediately into:

\[ x_i(r, \gamma_1, \gamma_2) = x_{i0}(\gamma_1, \gamma_2) + W_{i0}(\gamma_1, \gamma_2)(r - r_0) \]  \hspace{1cm} (4.5)

hence the rays are straight lines directed along group velocity and starting at a prescribed initial point \( x_{i0} \). Finally, the displacement amplitude \( U \) is determined from its initial value \( U_0 \) as follows:

\[ U(r, \gamma_1, \gamma_2) = U_0(\gamma_1, \gamma_2) \left( \frac{J_0(\gamma_1, \gamma_2)}{J(r, \gamma_1, \gamma_2)} \right)^{1/2} \]  \hspace{1cm} (4.6)

The initial values for \( r, p, W, x, U \) and \( J \) are given either from source radiation characteristics or from boundary conditions at an interface. As to the Jacobian \( J \), it can be directly computed from the geometrical spreading of neighboring rays: if the current ray is parametrized by \( (\gamma_1, \gamma_2) \), we can shoot four more rays with parameters \( (\gamma_1 \pm d\gamma_1, \gamma_2) \) and \( (\gamma_1, \gamma_2 \pm d\gamma_2) \) and numerically evaluate the two vectors \( \partial x_i/\partial \gamma_1 \) and \( \partial x_k/\partial \gamma_2 \) at a fixed value of \( r \).

Typically the algorithm can be summarized as follows: one indicates which reflectors are to be involved in tracing the rays, and the travel mode is specified in each layer involved, as well \( \{S-Parallel, Quasi-P, Quasi-S, \text{or, in an isotropic layer, SH,} \ P \text{ or SV} \} \). All rays are started on a small sphere of radius \( r \) around the source, parametrized by two spherical angles \( \gamma_1 \) and \( \gamma_2 \). On that sphere, the radiation pattern \( U_0(\gamma_1, \gamma_2) \) is known within a scaling factor, and the traveltime is constant and equal to \( r_0 = r/c \) where \( c \) is the velocity of shear or compressional waves depending on the desired case. The starting point \( x_0 \) and the initial slowness \( p_0 \) are determined by the specified take-off angles \( \gamma_1 \) and \( \gamma_2 \), while group velocity \( W_0 \) is normal to the sphere at \( x_0 \) and has intensity \( c \). The ray Jacobian \( J_0 \), is simply related to the radius of the sphere and the parameters \( d\gamma_1 \) and \( d\gamma_2 \).

Next we are able to determine the extremity of the first ray segment by finding the intersection of the line defined by (4.5) with the first reflector involved, that is, we determine \( r \) so that \( x \) is on the first reflector. Then in order to compute the new amplitude \( U \), one must evaluate the ray Jacobian at the extremity of the ray. For this purpose, the four neighboring rays are traced to the same traveltime as the ray of interest, that is, their extremities are defined by:
\[ x_{j_1} = x_{j_0} (\gamma_1 \pm \Delta \gamma_1, \gamma_2) + W_{j_0} (\gamma_1 \pm \Delta \gamma_1, \gamma_2) (r - \tau_{0_1}) , \] (4.7)
\[ x_{k_1} = x_{k_0} (\gamma_1, \gamma_2 \pm \Delta \gamma_2) + W_{k_0} (\gamma_1, \gamma_2 \pm \Delta \gamma_2) (r - \tau_{0_2}) , \]

where \( \Delta \gamma_1 \) and \( \Delta \gamma_2 \) are small incremental angles and \( \tau_{0_1} = \tau_{0_2} = \tau_0 \). Following this, \( J \) can be evaluated by finite difference as follows:

\[ J(\tau, \gamma_1, \gamma_2) \approx \epsilon_{ijk} W_{10} \frac{x_{j_1} - x_{j_2}}{2 \Delta \gamma_1} \frac{x_{k_3} - x_{k_4}}{2 \Delta \gamma_2} , \] (4.8)

and the incident amplitude \( U \) at the boundary can therefore be computed from (4.6).

To proceed to the next layer, we must apply boundary conditions which will provide the initial values for the next ray segment. The starting point is already known. The scattered slowness can be determined from the incident slowness by means of the procedure described in section (3.4). Once this has been done, the scattered group velocity is computed from the slowness using equations (3.6). As to the traveltime, it is continuous through the interface. The four extra rays must also be continued through the interface, since they are needed in order to further evaluate the geometrical spreading. The amplitude of the scattered wave is calculated using the system of Zoeppritz equations discussed at the end of chapter 3. The initial value for the ray Jacobian is determined as follows: we use the fact that ray tubes are continuous across reflectors, that is, the areas of the surfaces formed by intersection of the incident and scattered ray tubes with the interface are identical. Expressing this continuity condition is straightforward as in the isotropic case. We first define the elementary area of wavefront \( a_0 \) as:

\[ a_0 = \frac{\partial x}{\partial \gamma_1} \times \frac{\partial x}{\partial \gamma_2} \bigg|_\tau d\gamma_1 d\gamma_2 . \]

Then, according to (4.4) and Figure (4.1) we recognize that:

\[ Jd\gamma_1 d\gamma_2 = \left| W \right| a_0 \cos \theta_0 , \]

that is, the quantity \( Jd\gamma_1 d\gamma_2 \) represents the cross-sectional area of the ray tube, scaled by group velocity. On the other hand, the area \( a \) of the intersection of the ray tube with the reflector is:

\[ a = \frac{a_0 \cos \theta_0}{\cos \theta_1} , \]

and substituting for \( J \), we have:
\[ a = \frac{J d\gamma_1 d\gamma_2}{|W| \cos \theta_1} \]

Then \( \cos \theta_1 \) can be expressed as:

\[ \cos \theta_1 = \frac{W \cdot n}{|W|} \]

so that we finally obtain:

\[ a = \frac{J d\gamma_1 d\gamma_2}{W \cdot n} \]

Since \( a \) must be continuous at the interface, we can formulate a continuation equation for \( J \) in the following way:

\[ \frac{J_{\text{inc.}}}{W_{\text{inc.}} \cdot n} = \frac{J_{\text{scat.}}}{W_{\text{scat.}} \cdot n} \quad \text{at the interface} \]

We have therefore determined all the initial values required to proceed in solving for traveltime and amplitude in the scattering medium. To further trace the ray segment in that medium, we again use (4.5) to find the intersection of the ray with the following reflector and determine the associated traveltime. Then we need to compute the new ray Jacobian which we do by continuing the previous two extra rays to the proper traveltime, and proceed as described in (4.7) and (4.8). From the value obtained, the amplitude of the field at the extremity of the new segment is evaluated using (4.6). This process is iterated until the last reflector is finally reached, at which point the displacement polarization for the received mode can be computed with the aid of equations (3.5). When the last ray segment emerges at the surface, free boundary scattering is accounted for in computing the total field at the receiver. This insures, among other things, that the reciprocity relation between source and receiver is satisfied.

4.3. Implementation in horizontally layered media:

The algorithms described in the two previous sections has been implemented for a simple tabular model. Except for the top layer which is isotropic, arbitrary fracture azimuth and dip are allowed in each layer. Both source and receivers are located on the surface. The receivers are organized in line. The source need not be aligned with the receivers. Source parameters include polarity and sweep characteristics that simulate land vibrators. Multiple raypaths are allowed so that several events can be represented on a single shot profile. Each raypath is described by a succession of interfaces leading from source to receiver, as well as the sequence of propagation modes desired for the wave within each layer intersected by the path.

Although we previously described ray tracing in some detail, it remains to discuss how to determine the take-off angles that lead a ray along a given path from source to
receiver. We do this in the simplest manner: for each raypath, we trace a map of surface arrival points for a coarse set of take-off angles. This map is not expensive to generate since it only requires geometric, not dynamic ray tracing. Once the map is created, the take off angles are interpolated for each receiver location. Figure (4.2) shows such an interpolation map.

To conclude this section, we present computational results for a typical model: a fractured layer embedded in two isotropic shoulders (Fig. 4.3). In the anisotropic layer, the fractures are oriented N45°W and dip at 70° to the SW. The elastic parameters are specified using Thomsen’s notations: \( V_p \) and \( V_s \) are the axial (slow) \( P \) and \( S \) velocities, \( \gamma \) the relative \( S \) anisotropy, \( \epsilon \) the relative \( P \) anisotropy, and \( \delta \) the "anellipticity" parameter (in the sense that when \( \delta = \epsilon \), the \( QP \) slowness surface becomes an ellipsoid of revolution while the \( QS \) surface becomes a sphere). The relations between Thomsen’s and Love’s parameters are as follows:

\[
V_p = \sqrt{C/\rho} \quad , \quad V_s = \sqrt{L/\rho} \quad ,
\]

\[
\gamma = \frac{N-L}{2L} \quad , \quad \delta = \frac{(F+L)^2-(C-L)^2}{2C(C-L)} \quad , \quad \epsilon = \frac{A-C}{2C} \quad .
\]

The source simulates a horizontal vibrator polarized \( NS \). The source signature after autocorrelation is a Klauoder wavelet obtained from an 8 second linear sweep with a half second taper and a 5–40Hz bandpass. The recording array is a split spread \( NS \) line of 61 three component geophones (\( NS \), \( EW \) and vertical components). Five events were traced: the \( S-S \) reflection at the base of the first layer, and the four possible combinations of \( S \) modes reflected at the bottom of the fractured formation. These last events can be identified by the chronology of their propagation modes from source to receiver; they are: \( S-SP-SP-S, S-SP-QS-S, S-QS-QS-S, S-QS-SP-S \). Note that \( S-SP-SP-S \) and \( S-QS-QS-S \) represent the two split shear waves associated with birefringence. Figure (4.4) is a perspective plot of the raypaths for the five events. A striking feature is that the ray segments do not remain in the vertical plane containing the source-receiver line; this justifies why we earlier rejected the idea of \( 2-D \) ray tracing in anisotropic media. Such modeling remains reserved to other analytic or semi-analytic methods.

The sections in Figures (4.5) through (4.7) show the displacement field as recorded in the \( NS \), \( EW \), and vertical geophones respectively (please note that the scaling varies from plot to plot). The first arrival around 3 seconds is of course the \( S-S \) reflection. Although that event propagates exclusively in an isotropic medium, the scattering conditions at the interface with the underlying fractured layer significantly affect its polarization. If the second layer were isotropic, the \( S-S \) arrival would have a horizontal polarization in the same direction as that of the source. Instead here, polar plots showing horizontal displacement versus time, or hodograms, demonstrate that the \( S-S \) event has a non zero \( EW \) component upon arrival at the surface (Fig. 4.8). Unfortunately, this polarization "anomaly" strongly depends on the scattering coefficients at the interface, which are offset-dependent. Therefore it is not in itself an indicator of fracture orientation.
Now to the reflections from the base of the fractured layer. At small offsets, the prevailing events in terms of energy are the split $S-SP-SP-S$ and $S-QS-QS-S$ waves. The converted events can however be seen at wider offsets, especially on the in-line and vertical displacement sections (Fig.4.5 and 4.7). Actually, the latest event at large negative offsets is $S-QS-SP-S$, while the latest one at large positive offsets is $S-SP-QS-S$. Figure (4.9) shows hodograms in the time window 3.5–6 seconds, for a set of receiver locations (please note again that the scaling varies from plot to plot). It is remarkable that these plots exhibit the same pattern throughout the entire section: the first trend is $N45^\circ W$ and corresponds to the $S-SP-SP-S$ wave, which travel fastest. The second trend is $N45^\circ E$ and represents the slower $S-QS-QS-S$ reflection. This suggests that the study of polarization patterns may indeed be a straight forward way of determining fracture orientation. If the fractured layer is thinner or its anisotropy weaker however, the two split shear arrivals partly superpose in time; this interaction, which can also be made worse by lowering the frequency content of the signal, eventually seriously degrades the polarization patterns. Figure (4.10) illustrates what happens to the zero-offset hodogram as the thickness of the fractured layer progressively decreases: the trends described earlier can be inferred for thicknesses down to 200m, which is about eight wavelengths in this example: a rather large number. This gives a fairly good idea of the limitations of direct birefringence observation. For relatively thin anisotropic layers, some kind of signal deconvolution should be applied in order to extend the detectability of shear wave splitting.

CONCLUSION

We have presented here the main characteristics of high frequency wave propagation in transversely isotropic media, which must be understood in order to develop modeling and interpretation. Analytical expressions have been derived for phase and group velocities as well as displacement polarizations for the three wavetypes: $SP$, $QP$, and $QS$. In contrast with the isotropic case, the computation of some essential quantities has to be left implicit. Considering the weakness of anisotropy in sedimentary rocks, triplication effects should not be given too much attention, since they are unlikely to be present.

Ray tracing was discussed and implemented in the case of tabular media to help understand complex propagation effects in fractured media. It was shown in particular that even when horizontal reflectors are considered, the raypaths between source and receiver do not remain coplanar. The polarization patterns of split shear waves, predictable from the study of simplified zero offset models, are shown to persist for large offsets and non vertical fractures. On the other hand, the study of a specific example suggests that, in most practical situations, a direct detection of birefringence cannot be achieved, and that additional processing is needed.
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Figure (1.1): Schematic illustration of shear wave splitting.
Figure (1.2): Evidence of birefringence on converted “P-SV” field data (Courtesy Tom Davis & Martial Martin, 1987).
Figure (1.3): Physical interpretation of the five elastic parameters using Thomsen's ideal model of the deck of cards. Dotted lines indicate shape of solid before deformation.
Figure (1.4): Cross-section of the velocity surfaces for the three modes in an axial plane, showing the dependance of phase velocity on propagation direction. The actual surface has a symmetry of revolution around the anisotopy axis, here shown horizontal ($A=10$, $C=3$, $N=4$, $L=1$, $F=1$, $p=1$).
Figure (1.5): Polarization anomalies due to anisotropy: only the S-parallel mode is shown to be a truly transverse mode. Propagation azimuth is the angle between propagation direction (slowness) and anisotropy axis (the "1" axis). Same elastic parameters as in Figure (1.4).
Figure (2.1): A section of ray tube.
Figure (3.1): Definition of the polar angles $\theta$ and $\phi$ used throughout chapter 3.
Figure (3.2): Geometrical construction of some essential vector quantities.
Figure (3.3): Cross-section of the wave surfaces of the three modes in an axial plane, showing triplication effects associated with the Quasi-$S$ mode.
Figure (3.4): Symbolic illustration of triplication: The ray network becomes threefold at each cusp in the slowness surface.
Figure (3.5): Illustration of how to separate between reflection and transmission based on group velocity (ray direction) rather than on the direction of slowness.
Figure (3.6): Effect of fracture azimuth on scattering at an interface. Azimuth is the angle computed in the reflector plane, between the fracture direction and tangential slowness. Here *Quasi* – *P* to *Quasi* – *P* reflection is shown for the same set of parameters as in Figure (1.4), and the dip of the fractures relative to the reflector is 80 degrees.
Figure (3.7): Effect of fracture dip on scattering at an interface. Dip is the relative angle between the plane tangent to the reflector and the fracture planes. Quasi-$S$ to Quasi-$S$ reflection is shown here for the same set of parameters as in Figure (1.4), and fracture azimuth is $N45^\circ E$. 
Figure (3.8): Scattering amplitudes of the $S$-$Parallel$ mode. Incident medium parameters are: $A=10$, $C=3$, $N=4$, $L=1$, $F=1$, $\rho=1$, fractures at $N45^\circ E$. Transmission medium parameters are: $A=8$, $C=4$, $N=2$, $L=1$, $F=1$, $\rho=1$, fractures at $N45^\circ W$. Fractures are vertical in both media.
Figure (4.1): Schematic illustration of the parameters used in chapter 4 in determining a continuation formula for the ray jacobian.
Figure (4.2): Example of interpolation map used in determining ray take-off angles. Cross hair shows source location, and dots represent ray emergence points for a coarse set of take-off angles.
Figure (4.3): Description of three layer model used in chapter 4.
Figure (4.4): Perspective ray tracing plot for the example discussed, showing all five raypaths as well as an outline of the layer boundaries. Note the out of plane propagation of energy.
Figure (4.5): Shot profile of in-line displacement field for the example discussed in chapter 4.
Cross Line Displacement

Figure (4.6): Shot profile of cross-line displacement field for the example discussed in chapter 4.
Figure (4.7): Shot profile of vertical displacement field for the example discussed in chapter 4.
Figure (4.8): Hodograms of the reflection at the interface between the first, isotropic layer, and the second, fractured layer. Note that the polarization of the arrival does not follow that of the source (North). Also note that the polarization “anomaly” is offset-dependant.
Figure (4.9): Hodograms of the set of reflections at the base of the fractured layer. Note that the polarization patterns are essentially offset-independant.
Figure (4.10): Zero offset hodograms for the example discussed: here the thickness of the fractured layer is decreased from 1500m down to 100m.