Optimal wave focusing for imaging and microseismic event location
Farhad Bazargani and Roel Snieder, Center for Wave Phenomena, Colorado School of Mines

SUMMARY

Focusing waves inside a medium has applications in geophysics in areas such as imaging and microseismic event location. The goal in focusing is to concentrate the wave energy at a specific time and location inside a medium. Various techniques have been devised and used to achieve this goal. Time-reversal (TR) is a well-researched method that has been used routinely to focus waves. The ability of TR methods to focus wave fields inside heterogeneous media is bounded by limitations caused by imperfect acquisition, attenuation, and the diffraction limit. To go beyond these limitations, we present a solution by formulating wave focusing as an optimization problem. Solving this optimization problem gives the needed signals for transmission to the medium to get the best focus. These signals are optimized for the configuration of the injection points, velocity of the medium, and the focusing target.

INTRODUCTION

The objective in wave focusing is to determine the waveforms that, when transmitted through a medium, create a wavefield that concentrates at a specific time and location. Wave focusing is conceptually related to the problem of imaging, and hence finds important applications in areas such as exploration geophysics.

Time-reversal (TR) is a focusing technique that is robust and effective in heterogeneous media. The method relies on the time-reversal invariance of the wave operator and spatial reciprocity (Fink et al., 2002; Snieder, 2004). The TR process consists of two basic steps. In the first, the wavefield generated at a source in the medium is recorded using receivers surrounding the source. In the second step, the recorded wavefields are time-reversed, retransmitted through the medium, and propagated back to refocus approximately at the original source location.

TR methods are applied in key areas of geophysics on both global and exploration scales (Lu, 2002; Larmat et al., 2006, 2010). In exploration seismology, TR focusing is used in microseismic event location (Lu and Willis, 2008; McMechan, 1982; Xuan and Sava, 2010), fracking and reservoir monitoring (Shapiro, 2008), salt-flank imaging and redatuming seismic data (Lu, 2002), and migration (Berkhout, 1997; McMechan, 1983). Despite these broad applications, the TR process has important limitations. In theory, for a broadband pulse emitted by an ideal point source, the returning field refocuses on a spot with dimensions on the order of the smallest wavelength (Abbe diffraction limit (Fink, 1997)). From the experimental point of view, it is not possible to record and retransmit the wavefield everywhere on a surface that fully encloses the source. In practice, the wavefield is sampled at spatially sparse locations with a limited recording aperture. This imperfect acquisition causes an increase in the dimension of the point-spread function (Fink, 2006). Another problem is that in real applications of TR-focusing, the media are dissipative and TR invariance of the wave equation does not hold valid in dissipative media.

We propose an alternative approach to wave focusing wherein the problem is cast as an optimization problem. As discussed above, TR focusing methods are not optimal for real problems because of imperfect acquisition, attenuation, and the diffraction limit. The motivation for this research is to improve upon the existing techniques especially where such techniques do not perform optimally in dealing with limitations encountered in real focusing problems.

FOCUSING AS AN OPTIMIZATION PROBLEM

Consider $N$ sources at distinct locations $r_i$ wherein each source is capable of transmitting predefined signals. The objective is to design signals such that, upon transmission, focuses optimally at a specific time and location inside the medium. Each source must work in concert with the others by injecting a signal that is tailored in amplitude and shape according to the specific medium, source geometry, and focusing target in order to achieve the best spatio-temporal focus.

If we denote the signal injected by the source at $r_i$ as $a_i(t)$, then the superposed wavefield recorded at an arbitrary location $r$ inside the medium is

$$u(r,t) = \sum_{i=1}^{N} a_i(t) * g(r, r_i, t),$$  

where $*$ is the convolution operator, and $g(r, r_i, t)$ is the impulse response (Green’s function) recorded at location $r$ and corresponding to an impulsive source at location $r_i$. Note that if the velocity of the medium is known, these Green’s functions can be computed.

Convolution in the time domain corresponds to multiplication in the frequency domain. Therefore, considering the problem in the frequency domain, each frequency component of the wavefield $u$ in equation 1 can be restated as a weighted sum of the corresponding frequency component $G(r, r_i, \omega)$ of the Green’s functions

$$u(r, \omega) = \sum_{i=1}^{N} a_i(\omega) G(r, r_i, \omega),$$  

where the weights $a_i(\omega)$ are the Fourier components of the injected signals.

The problem can now be restated as how to optimally determine the weights $a_i(\omega)$ in equation 2 so that the superposed field $u$ in the time domain focuses at a desired location of focus $r_f$ and at a desired time of focus (usually taken to be $t_f = 0$). Put another way, the goal is to have $u(r, t)$ as close as possible...
to $\delta(r-r_f)\delta(t)$, where $\delta$ denotes the Dirac delta function. In the frequency domain, this amounts to making each $u(r,\omega)$ as close as possible to $\delta(r-r_f)$. This goal can be achieved, for example, by minimizing an objective function defined as

$$J = \int |u(r,\omega) - \delta(r-r_f)|^2 dr. \quad (3)$$

This condition is known as the *deltaness criterion* in the context of the method of Backus and Gilbert (BG) in inverse theory (Backus and Gilbert, 1968; Aki and Richards, 1980; Aster et al., 2012). Inserting equation 2 in objective function 3 and minimizing with respect to each $a_i(\omega)$ gives a linear system of equations of the form

$$\Gamma a = g^*, \quad (4)$$

where $\Gamma$ is the $N \times N$ Gram matrix (Parker, 1994) with elements defined as

$$\Gamma_{ij} = \int G(r_i, \omega) G^*(r_j, \omega) dr, \quad i, j = 1, 2, ..., N. \quad (5)$$

$g$ is an $N \times 1$ vector with components

$$g_i = G(r_f, r_i, \omega), \quad i = 1, 2, ..., N, \quad (6)$$

and the symbol $*$ denotes complex conjugate. The linear system in 4 can be solved for the $N \times 1$ vector $a$ for each frequency. These $a_i(\omega)$ vectors constitute the Fourier coefficients for the signals that must be transmitted by each source at $r_i$ to achieve an optimal focus at $r_f$.

The Backus and Gilbert (BG) focusing method, introduced above, is closely related to TR focusing, i.e., a particular choice of the Gram matrix $\Gamma$ in equation 4 can reduce the BG method to TR focusing. Replacing $\Gamma$ by the identity matrix $I$ gives

$$a = g^*. \quad (7)$$

Now, since complex conjugation in the frequency domain is equivalent to time-reversal in the time domain, the new system of equations 7 describes exactly the same process as time-reversal in the time domain. Therefore, TR is a special case of the more-general BG method. Replacing $\Gamma$ by the identity matrix amounts to cancelling the cross-talk between sources and having each source work independently to inject time-reversed Green’s functions. The off-diagonal elements of the Gram matrix $\Gamma$ hold crucial information about the configuration of the wave-focusing experiment, i.e., the relative positions of the sources with respect to the propagation medium and the focusing target. Each element plays a role in determining how the sources must work in tandem to inject the signals that achieve the optimum focusing at the target.

**SYNTHETIC DATA EXPERIMENT**

We perform a numerical focusing experiment wherein we apply the BG method to focus wavefields at a target and compare the result with the focus achieve by TR. The configuration of the numerical experiment is shown in Figure 1. The triangles represent the injection points (sources), the locations where the signals are emitted and numerically propagated to focus at the target, denoted by the red circle. The velocity model used for wave propagation is a 2D model shown in Figure 1. Wave propagation is simulated using an explicit finite-difference approximation of the 2D acoustic isotropic wave equation with absorbing boundary condition.

To form $\Gamma$ according to equation 4, we require an approximation of the impulse response of the medium $g(r, r_i, t)$. To compute the impulse response, at each injection point we inject a bandlimited Ricker wavelet with peak frequency of 16 Hz and forward propagate the wavefield in time. These wavefields are then transformed to the frequency domain and used in equation 5 to compute the elements of the $5 \times 5$ matrix $\Gamma$ for each frequency. At this point, the right-hand side of equation 4 is also known because $g_i = G(r_f, r_i, \omega) = G(r, r_i, \omega)|_{r=r_f}$.

Solving equation 4 for each frequency gives the Fourier coefficients $a_i(\omega)$ of the signals that, upon injection, create wavefields that optimally focus at the target. Figure 2a shows one such signal computed for injection point 2. The signal, of the same injection point computed using the TR method, is shown...

**Figure 2:** Signals required to be injected by source 2 shown in Figure 1 computed by (a) the proposed focusing method, and (b) time-reversal method.
Optimal wave focusing

Figure 3: Results of the numerical focusing experiment. The plots show the snapshots of propagating wavefields at the time of focus for (a) BG (the proposed method) and (b) time-reversal.

in Figure 2b for comparison. Note the higher frequency content of the optimally computed signal (Figure 2a) compared to the time reversed signal (Figure 2b).

Figures 3a and 3b show snapshots of the wavefield associated with the BG and TR, respectively. As is evident in these snapshots, the BG method has outperformed TR in achieving a more compact spatial focus in this synthetic experiment.

DISCUSSION

Alternative deltaness criteria

The deltaness criterion defined as minimizing objective function 3 is not the only and probably not the best option. Other formulations of the deltaness criterion have been suggested and used (Backus and Gilbert, 1968; Aki and Richards, 1980; Aster et al., 2012).

The focused fields shown in Figure 3 illustrate a possible motivation for trying other deltaness criteria. Even though our method shows a better spatial focus in Figure 3a compared to TR in Figure 3b, energy is present around the focus in the side lobes. To reduce this side energy, we can use an objective function in which energy at distances farther away from the focusing target is penalized. One such objective function suggested by (Backus and Gilbert, 1968) is

\[ W = \int (r - r_f)^2 |u(r, \omega)|^2 \, dr, \]  

subject to the constraint

\[ \int |u(r, \omega)| \, dr = 1, \]  

where the weight factor \((r - r_f)^2\) in the integrand of expression 8 is responsible for penalizing the side energy. Note that depending on the specific requirements of a wave focusing problem, other weight factors can also be used.

Minimizing the energy in the side lobes could be essential for some applications of wave focusing, e.g., in imaging. In other applications such as microseismic event location it might not be as important.

Application in imaging

The BG approach can also be used to enhance reverse-time-migration (RTM) imaging. Here, we show how the two ideas are connected and propose a method to combine them.

In the space-time domain, RTM is formulated as

\[ m_{\text{mig}}(r) = \int [g(r_s, r, t - t) \ast d(r_g, r_s, t)] \bigotimes_{t=0}[w(t) \ast g(r, r_s, t)] \, dg, \]  

where subscripts s and g stand for source and receiver, respectively, \(\bigotimes_{t=0}\) denotes the zero-lag cross-correlation, \(m_{\text{mig}}(r)\) represents the reflectivity distribution perturbed from the background medium, \(g(r, r_s, t)\) denotes the Greens function for a specified background medium with a source at \(r_s\) and a receiver at \(r\), and \(d(r_g, r_s, t)\) is the reflected wave recorded by the geophone at \(r_g\) due to a source with wavelet \(w(t)\) shot at \(r_s\). The integration is over the data-space geophone variable denoted by \(r_g\).

By rearranging equation 10, Schuster (2002) introduces generalized diffraction-stack migration as an alternative implementation and interpretation of RTM. In generalized diffraction-stack migration, the migration image \(m_{\text{mig}}(r)\) can be interpreted as the zero-lag cross-correlation of the shot gather data \(d(r_g, r_s, t)\) with a focusing kernel defined as

\[ f(r_s, r, r_g, t) = g(r_g, r, t) \ast g(r, r_g, t). \]  

We show in Appendix A that, using an approach similar to the BG method described above, we can optimize the focusing kernel 11 for each shot and for specific targets in the image. Using an optimized focusing kernel in implementation of the generalized diffraction-stack migration could enhance the resolution of the migration image and reduce the acquisition footprint in the image.

Least-Squares Migration (Nemeth et al., 1999) is an effective imaging method that deals with imaging problems resulting
from imperfect acquisition. In this respect, one interesting question to address is how the proposed optimized generalized-diffraction stack migration scheme compares with the least-squares migration.

**Application in microseismic event location**

TR based methods are commonly used in seismology for event location (Larmat et al., 2010). Such methods have the same limitations as TR focusing and therefore, the BG approach can be useful in seismic event location. For example, note that \( g(r_f, r, \omega) \) in equation 6 is the impulse response of a source at \( r \), recorded at \( r_f \), and reciprocity implies that \( g(r_f, r, \omega) = g(r, r_f, \omega) \). In the context of seismic event location, the Green’s function \( g(r, r_f, \omega) \) represents the microseismic or earthquake data recorded in the field at each receiver location.

The presence of noise in field data is an important matter that needs careful attention in applying BG focusing to event location. With noisy data, equation 4 must be modified as

\[
\Gamma a = g^* + \eta, \quad (12)
\]

where \( \eta \) denotes the noise vector. In this case, \( \Gamma \) must be computed according to equation 5 using a velocity model, which is ideally accurate. Even when the velocity model is not accurate, using the Gram matrix computed based on this inaccurate velocity is better than ignoring it altogether as is done in time-reversal. The stability of the solution \( a \) in equation 12 depends on the condition number of the Gram matrix, which is ideally independent on the configuration of the receivers (injection points) and the medium. In general, a combination of some sort of regularization and alternative delunness criteria might be used to stabilize equation 12.

Another important consideration related to applying the BG focusing method to microseismic event location is the source mechanism. In the numerical examples shown in this paper, we have assumed a point source with an isotropic radiation pattern. The source mechanism for a microseismic event is, however, a double couple with its characteristic radiation pattern for both P and S waves. Therefore, it is more appropriate to use a deluness criterion that takes assumed features of the source mechanism for microseismic events into account. The BG focusing technique can also be used for determining the optimal way to process and combine all different components of recorded microseismic data in order to achieve more accurate results in microseismic monitoring.

**ACKNOWLEDGEMENTS**

This research was supported by the sponsors of the Center for Wave Phenomena at the Colorado School of Mines.

**APPENDIX A: OPTIMIZATION OF THE RTM FOCUSING KERNEL**

We use the Backus and Gilbert method to optimize the RTM focusing kernel for each receiver in a shot gather for a specific target in the model space.

For weak scattering, the scattered data recorded at receiver location \( r_f \) with a source at \( r_s \) can be written in the frequency domain in terms of the Born approximation to the Lippmann-Schwinger equation (Schuster, 2002):

\[
d(r_s, r_f) = \int G(r_s, r) W G(r, r_f) m(r) dr, \quad (A-1)
\]

where \( G(r_s, r) \) is the Green’s function for the Helmholtz equation for a specified background medium with a source at \( r \) and receiver at \( r_f, G(r, r_f) \) is the Green’s function for the same medium with a source at \( r \) and receiver at \( r_f, m(r) \) represents the reflectivity distribution perturbed from the background, and \( W \) represents the source wavelet function. The integration is over the model space.

Using RTM equation 10 written in the frequency domain for geophones at locations \( r_f \), where \( i = 1, 2, \ldots, N \), the reflectivity value at a target location \( r_f \) can be expressed as a weighted sum of data

\[
m_{\text{mig}}(r_f) = \sum_{i=1}^{N} a_i d(r_s, r), \quad (A-2)
\]

where the complex weights \( a_i \) are

\[
a_i = G^*(r_s, r_f) W^* G'(r_f, r), \quad i = 1, 2, \ldots, N. \quad (A-3)
\]

Note that \( a_i \) in A-3 is the complex conjugate of the RTM focusing kernel defined in equation 11.

Inserting \( d(r_s, r_f) \) from A-1 into A-2 and changing the order of integration and summation gives

\[
m_{\text{mig}}(r_f) = \int \left[ \sum_{i=1}^{N} a_i G(r_s, r) W G(r, r_f) \right] m(r) dr. \quad (A-4)
\]

The bracketed factor in A-4 is an averaging kernel that we would ideally like to closely approximate a Dirac delta function with spatial support at \( r_f, i.e.,

\[
\sum_{i=1}^{N} a_i G(r_s, r) W G(r, r_f) \rightarrow \delta(r - r_f). \quad (A-5)
\]

Our goal here is to determine new coefficients \( a_i' \) (instead of \( a_i \) defined in A-3) such that the deluness criterion A-5 is satisfied as closely as possible. This goal can be achieved, for example, by minimizing an objective function of the form

\[
J = \int \left[ \sum_{i=1}^{N} a_i' G(r_s, r) W G(r, r_f) - \delta(r - r_f) \right]^2 dr, \quad (A-6)
\]

where the integration is carried out over the model space.

Minimizing A-6 gives a linear system of the form

\[
\Gamma a' = a, \quad (A-7)
\]

where \( \Gamma \) is an \( N \times N \) matrix with elements

\[
\Gamma_{i,j} = \int G(r_s, r) G^*(r_{s,j}, r) G(r_s, r_f) G^*(r_f, r_f) dr,
\]

and \( a \) is an \( N \times 1 \) vector with elements defined in A-3.

Solving equation A-7 gives the \( N \times 1 \) vector \( a' \) of optimized kernels that can be used as enhanced migration focusing kernels in generalized diffraction-stack migration.
Optimal wave focusing

REFERENCES