The extraction of the Green's function and the generalized optical theorem

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ABSTRACT

The relationship between the Green's function and the average cross-correlation of a diffuse field is shown to be equivalent to the generalized optical theorem. The correspondence to the usual optical theorem holds for the extraction of the imaginary component of the Green's function at the source for general linear scalar systems from the autocorrelation of field fluctuations. The imaginary component of the Green's function at the source is finite and shown to be related to the loss in energy (acoustic waves) and probability (quantum mechanics) due to radiation and attenuation.

Key words: interferometry, autocorrelation of noise

1 INTRODUCTION

The extraction of the Green's function from ambient fluctuations within a diffuse field has received considerable attention for different research areas (Weaver, 2005; Larose *et al.*, 2006b). It has been shown that the extraction of the Green's function can be carried out for a wide variety of systems (Snieder *et al.*, 2007; Wapenaar *et al.*, 2006). For scalar systems that are invariant to time-reversal, theory relates the imaginary part of the Green's function to a bilinear combination of the Green's function (Snieder *et al.*, 2007):

$$Im\,(G(\mathbf{r}_A,\mathbf{r}_B)) = \oint G(\mathbf{r}_A,\mathbf{r})O(\mathbf{r})G^*(\mathbf{r}_B,\mathbf{r})dS\ , \qquad (1)$$

where $O(\mathbf{r})$ is an operator that follows from theory, and Im denotes the imaginary part. A similar expression holds for systems that are not invariant for time reversal, but for such systems a volume integral with the same functional form as the surface integral should be added to the right hand side of this expression (Snieder *et al.*, 2007). Equation (1) forms the basis for extraction of the Green's function from ambient fluctuations. When spatially uncorrelated noise with power spectrum $|S(\omega)|^2$ excites a field $u(\mathbf{r})$, then the Green's function follows from (Larose *et al.*, 2006b; Snieder *et al.*, 2007; Sánchez-Sesma & Campillo, 2006)

$$Im\,(G(\mathbf{r}_A,\mathbf{r}_B)) \propto \frac{\langle u(\mathbf{r}_A)u^*(\mathbf{r}_B)\rangle}{|S(\omega)|^2}\ , \qquad (2)$$

where $< \cdots >$ denotes a source-average. In practice this source average is replaced by an average over a set of non-overlapping windows (Larose *et al.*, 2006b). The right-hand side corresponds, in the frequency domain, to the cross correlation of the field measured at locations $\mathbf{r}_A$ and $\mathbf{r}_B$, respectively. By setting $\mathbf{r}_A = \mathbf{r}_B$ the right-hand side reduces to the autocorrelation of the field fluctuations, while the left-hand side gives the imaginary component of the Green's function at the source. This quantity measures the return of waves to the source location and can be used to study localization (Larose *et al.*, 2004; Larose *et al.*, 2006a).

This relationship between the imaginary part of the Green's function at the source and radiated energy has been shown earlier for the special case of elastic waves in a homogeneous space (Weaver, 1985; Sánchez-Sesma *et al.*, 2007). The presence of a free surface and of scatterers has been explored for both scalar and elastic systems (Sánchez-Sesma *et al.*, 2007). This work provides a generalization of these results to a more general class of scalar linear systems that can be both heterogeneous and dissipative.

2 THE IMAGINARY PART OF GREEN'S FUNCTION AT THE SOURCE

Consider, first, the damped oscillator: $m\ddot{x}+\gamma\dot{x}+sx=f$, with m, γ and s the mass, damping and stiffnes parameters, respectively. Using the Fourier convention $f(t)=(2\pi)^{-1}\int f(\omega)\exp(-i\omega t)d\omega$, the Green's function satisfies the relation $G=1/\left(s-i\gamma\omega-m\omega^2\right)$ in the frequency domain, which implies that $Im\,(G)=\gamma\omega|G|^2$. The imaginary part of the Green's function thus is proportional to the rate of dissipation by this one-degree-of-freedom system. We next derive a general expression that relates the imaginary part of the Green's function at the source to both the radiated and dissipated power lost by a unit harmonic source.

Consider the following scalar partial differential equation for a field u that is of Nth order in the time derivatives

$$\left(a_N(\mathbf{r},t)*\frac{\partial^N}{\partial t^N}+\cdots+a_2(\mathbf{r},t)*\frac{\partial^2}{\partial t^2}+a_1(\mathbf{r},t)*\frac{\partial}{\partial t}\right) \times u(\mathbf{r},t)=H(\mathbf{r},t)*u(\mathbf{r},t)+q(\mathbf{r},t)\ . \qquad (3)$$

The asterisk denotes convolution in time, while the operator H contains the space derivatives of the field equation. The field is excited by the forcing $q(\mathbf{r},t)$. Using the above Fourier convention, this equation corresponds in the frequency domain to

$$\sum_{n=1}^{N}a_n(\mathbf{r},\omega)\left(-i\omega\right)^n u(\mathbf{r},\omega)=H(\mathbf{r},\omega)u(\mathbf{r},\omega)+q(\mathbf{r},\omega)\ . \qquad (4)$$

Henceforth the derivation is in the frequency domain, and we suppress the frequency-dependence. The operator H is not necessarily self-adjoint over the volume V under consideration, and, following ref. (Snieder *et al.*, 2007), we define the bilinear form L by $\int_V\left(f\left(Hg\right)-\left(Hf\right)g\right)dV\equiv\oint_{\partial V}L(f,g)dS$, where the surface integral is over the surface ∂V that bounds the volume V under scrutiny. Now assume that the $q(\mathbf{r})$ are spatial delta functions $\delta(\mathbf{r}-\mathbf{r}_{A,B})$ at locations $\mathbf{r}_A$ and $\mathbf{r}_B$, respectively, with harmonic time variation. The fields u_A and u_B are then the Green's functions $G(\mathbf{r},\mathbf{r}_{A,B})$. Using reciprocity ($G(\mathbf{r}_A,\mathbf{r}_B)=G(\mathbf{r}_B,\mathbf{r}_A)$) and setting $\mathbf{r}_A=\mathbf{r}_B=\mathbf{r}_0$ it follows from expression (16) of ref. (Snieder *et al.*, 2007) that, for the general system (3):

$$\begin{aligned} &Im\left(G(\mathbf{r}_0,\mathbf{r}_0)\right)= \\ &-\textstyle\sum_{n\ odd}(-1)^{(n+1)/2}\omega^n\int_V Re\left(a_n\right)|G(\mathbf{r}_0,\mathbf{r})|^2dV \\ &-\textstyle\sum_{n\ even}(-1)^{n/2}\omega^n\int_V Im\left(a_n\right)|G(\mathbf{r}_0,\mathbf{r})|^2dV \\ &-\tfrac{i}{2}\oint_{\partial V}L\left(G^*(\mathbf{r}_0,\mathbf{r}),G(\mathbf{r}_0,\mathbf{r})\right)dS \\ &+\textstyle\int_V G(\mathbf{r}_0,\mathbf{r})Im\left(H\right)G^*(\mathbf{r}_0,\mathbf{r})dV\ , \end{aligned} \qquad (5)$$

where Re denotes the real part. In order to see its meaning we next present three examples.

3 DAMPED ACOUSTIC WAVES

These waves satisfy the following partial differential equation

$$\kappa\frac{\partial^2 p}{\partial t^2}=\nabla\cdot\left(\frac{1}{\rho}\nabla p\right)+q\ , \qquad (6)$$

now p is pressure, ρ mass density, and κ compressibility. For attenuating media, κ is complex and frequency-dependent. In the notation of the general equation (3), $a_2=\kappa$, and $H=\nabla\cdot(\rho^{-1}\nabla)$; all other a_n vanish. Using Green's theorem, the bilinear form L for this example is given by $L(f,g)=\rho^{-1}\left(f(\partial g/\partial n)-(\partial f/\partial n)g\right)$. We assume that the boundary ∂V is a sphere in the far field and that radiation boundary conditions hold on this boundary; that is, $\nabla p=ik\mathbf{n}p$. In this case $(-i/2)\oint L(p^*,p)dS=\omega\oint(\rho c)^{-1}|p|^2dS$, where $c=\omega/k$ is the speed of sound, and the general equation (5) reduces to

$$\begin{aligned}\omega^{-1}Im\left(G(\mathbf{r}_0,\mathbf{r}_0)\right)&=\omega\textstyle\int_V Im\left(\kappa\right)|G(\mathbf{r}_0,\mathbf{r})|^2dV \\ &+\oint_{\partial V}\tfrac{1}{\rho c}|G(\mathbf{r}_0,\mathbf{r})|^2dS\ .\end{aligned} \qquad (7)$$

The first term in the right-hand side gives the energy dissipated within the volume V, while the last term accounts for the energy flux through the boundary ∂V. The imaginary part of the Green's function at the source thus accounts for the two ways in which energy injected into the medium by the source leaves the volume V.

Note that the volume V need not be the volume of all space; it can be an arbitrary sub-volume, and its boundary ∂V need not be a physical boundary. When V is a volume with radius much less than the extinction length $l_{ext}=\left[Im\left(\sqrt{\rho\kappa}\right)\omega\right]^{-1}$, most of the energy is lost through radiation through the boundary ∂V, and the second term in the right-hand side dominates. In contrast, where the volume defines a region much larger than the extinction length, the first term in the right hand side of expression (7) dominates.

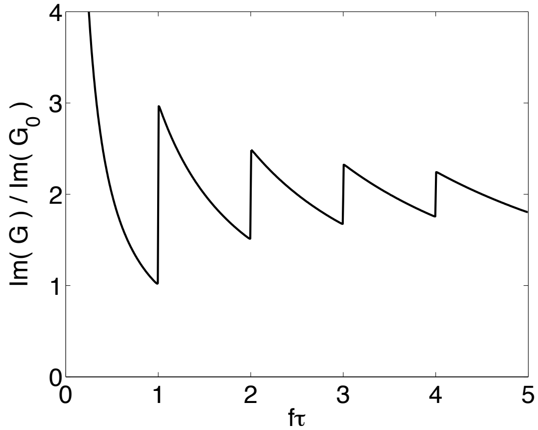

Figure 1. The imaginary part of the Green's function of the acoustic layer at the source, normalized with the corresponding value for the free-space Green's function, as a function of normalized frequency $f\tau$, where $\tau = 2h/c$.

4 QUANTUM MECHANICS

For this case, the field is governed by Schrödinger's equation:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi \ . \tag{8}$$

In the notation of equation (3), $H = -(\hbar^2/2m)\nabla^2 + V$, $a_1 = i\hbar$, and all other a_n are equal to zero. Using Green's theorem it follows that $L(\psi^*, \psi) = -(\hbar^2/2m)\left(\psi^*(\partial\psi/\partial n) - (\partial\psi^*/\partial n)\psi\right)$. This quantity is related to the probability density current $\mathbf{J} = (\hbar/2mi)\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$ (Merzbacher, 1970). Inserting these results in expression (5) gives

$$-\frac{2}{\hbar} Im\left(G(\mathbf{r}_0, \mathbf{r}_0)\right) = \oint_{\partial V} \mathbf{J}_G \cdot d\mathbf{S} \ , \tag{9}$$

where $\mathbf{J}_G$ is the probability density current associated with the Green's function G. The right-hand side gives the probability that the particle leaves the volume V. Here, the imaginary part of the Green's function is not equal to the energy loss; but instead corresponds to a loss in probability.

When waves with power spectrum $|S(\omega)|^2$ strike the scatterer, the imaginary component of the Green's function follows from the autocorrelation of the field fluctuations (Snieder *et al.*, 2007):

$$-\frac{2}{\hbar^2} Im\left(G(\mathbf{r}_0, \mathbf{r}_0)\right) = \frac{k}{m|S(\omega)|^2} \left\langle |\psi(\mathbf{r}_0, \omega)|^2 \right\rangle \ . \tag{10}$$

The combination of expressions (9) and (10) shows that the probability that particles leave the volume V is related to the fluctuations in the probability density $\left\langle |\psi(\mathbf{r}_0, \omega)|^2 \right\rangle$.

5 A SIMPLE ACOUSTIC MODEL

As a prototype of the potential applications, consider the problem of acoustic waves propagating within a layer of thickness h. At the upper and lower edge, the pressure gradient is assumed to vanish ($\partial p/\partial n = 0$); hence the reflection coefficient is equal +1 at the boundaries. It follows from the method of images that the wave-field at the source location for a source placed at the top of the layer is given by

$$G(0,0) = \frac{2\rho}{4\pi}\left(\lim_{r\to 0} \frac{e^{ikr}}{r} + 2\frac{e^{2ikh}}{2h} + 2\frac{e^{4ikh}}{4h} + \cdots\right) . \tag{11}$$

The overall factor of 2 results because the source radiates only into the plate rather than in all directions, while the factor 2 for the backscattered waves arises from the interaction of the waves with the surface (where the receiver is located). The first term in the right-hand side gives the direct wave at the source in infinite space, corrected with a factor 2 that accounts for the boundary condition. Since this wave is singular, it is written in the form of a limit. The subsequent terms in expression (11) come from the successive waves that bounce back and forth within the layer.

The first term is twice the Green's function in a homogeneous space $G_0(r) = \rho \exp(ikr)/4\pi r$. While at the source ($r = 0$) this Green's function is singular, the imaginary part of this Green's function, given by $Im\left(G_0(r)\right) = \rho \sin(kr)//4\pi r$, tends to the finite value $\rho k/4\pi$ as $r \to 0$.

For the layer model, the imaginary component of the Green's function is given by

$$Im\left(G(0,0)\right) = \frac{2\rho k}{4\pi}\left(1 + 2\frac{\sin(2kh)}{2kh} + 2\frac{\sin(4kh)}{4kh} + \cdots\right) . \tag{12}$$

This function, normalized to the corresponding value $Im(G_0(0,0)) = \rho k/4\pi$ for the free-space Green's function, is shown in figure 1 as a function of normalized frequency $f\tau$, where $\tau = 2h/c$ is the two-way vertical travel time in the layer. The peaks in the response curve correspond to the resonances of the layer at frequencies $f_n = n/\tau = nc/2h$, with n an integer. Since energy is radiated sideways in the layer, the peaks have nonzero width. According to expression (2), the imaginary part of the Green's function can, in general, be extracted from field fluctuations using the relation

$$Im\left(G(\mathbf{r}_0, \mathbf{r}_0)\right) \propto \frac{\left\langle |u(\mathbf{r}_0)|^2 \right\rangle}{|S(\omega)|^2} \ . \tag{13}$$

The right-hand side depends on the autocorrelation$\left\langle |u(\mathbf{r}_0)|^2 \right\rangle$ of recorded field fluctuations. This means that properties of the medium, in this case the travel-time h/c through the layer, can be inferred from the autocorrelation of field fluctuations.

6 DISCUSSION

This theorem, which concerns the scattering amplitude $A_k(\mathbf{n}, \mathbf{n}')$ of scattered waves with wave number k, and $\mathbf{n}$ and $\mathbf{n}'$ unit vectors representing the directions of the incoming and outgoing waves, respectively, states that (Heisenberg, 1943)

$$Im\left(A_k(\mathbf{n}_A, \mathbf{n}_B)\right) = \frac{k}{4\pi} \oint A_k(\mathbf{n}_A, \mathbf{n}) A_k^*(\mathbf{n}_B, \mathbf{n}) d^2\mathbf{n}. \quad (14)$$

This expression has the same form as equation (1) for the extraction of the Green's function. The only difference is that equation (1) holds in the space domain, while the generalized optical theorem (14) is formulated in the wave-number domain. The optical theorem follows from expression (14) by setting $\mathbf{n}_A = \mathbf{n}_B = \mathbf{n}_0$:

$$\sigma_e = \oint |A_k(\mathbf{n}_0, \mathbf{n})|^2 d^2\mathbf{n} \; = \frac{4\pi}{k} Im\left(A_k(\mathbf{n}_0, \mathbf{n}_0)\right) \; . \quad (15)$$

where σ_e is the extinction cross section, and $A_k(\mathbf{n}_0, \mathbf{n}_0)$ is the scattering amplitude in the forward direction. This theorem, which relates the radiation loss by scattering to the properties of the forward-scattered wave (Newton, 1976), is equivalent to the general expression (5) that relates the imaginary component of the Green's function at the source to the loss of generalized energy.

For uncorrelated noise sources with power spectrum $|S(\omega)|^2$ on a large sphere surrounding the scattering object (Wapenaar *et al.*, 2005) , equation (14) can be written analogously to equation (2) as

$$Im\left(A_k(\mathbf{n}_A, \mathbf{n}_B)\right) = \frac{k \langle \psi^s(\mathbf{n}_A) \psi^{s*}(\mathbf{n}_B) \rangle}{4\pi |S(\omega)|^2}, \quad (16)$$

where $\psi^s(\mathbf{n})$ is the scattered field in the direction $\mathbf{n}$ excited by the noise. By setting $\mathbf{n}_A = \mathbf{n}_N = \mathbf{n}_0$, the extinction cross section is given by

$$\sigma_e = \frac{4\pi}{k} Im\left(A_k(\mathbf{n}_0, \mathbf{n}_0)\right) = \frac{\langle |\psi^s(\mathbf{n}_0)|^2 \rangle}{|S(\omega)|^2}. \quad (17)$$

The scattering amplitude and scattering cross section thus can be inferred from the auto-correlation of field fluctuations. Note the resemblance between expressions (10) and (17).

Discussion. – The generalized optical theorem and the equations for the extraction of the Green's function have the same functional form. Both equations express the same physical principle, but in different spaces. The generalized optical theorem is given in wave-number space while the imaginary part of Green's function is expressed in position space. There is an even deeper connection with scattering theory, because these expressions are equivalent to relations used in the Marchenko equation for inverse scattering (e.g., ref. (Budreck & Rose, 1992)).

The optical theorem and the general expression for the imaginary part of Green's function at the source account for generalized energy loss at the source. We use the term *generalized energy*, because the imaginary component of the Green's function is related to integrals that are quadratic in the field representing energy for acoustic waves and probability in quantum mechanics.

In general, one cannot measure the Green's function at the source because it is singular. The imaginary part of the Green's function, however, is regular. As shown in expression (13), this quantity can be inferred from the autocorrelation of measured field fluctuations. The autocorrelation of observed field fluctuations has been used for monitoring a volcano and a fault zone during an earthquake (Sens-Schönfelder & Wegler, 2006) and to study the coherent backscattering effect for ultrasound (Larose *et al.*, 2006a). The example in the previous section illustrates that the formal relationship, presented here, between the autocorrelation of the field fluctuations and the imaginary part of the Green's function can be used to infer properties of the medium using field fluctuations recorded by a single sensor.

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