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The formation of caustics in two- and three-dimensional media

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ABSTRACT

In terms of ray theory, the focus point (also related to caustics and triplications) is the point in space where the ray position is stationary for perturbations in the initial condition. Criteria for the formation of caustics are presented. With ray perturbation theory, a condition for the development of triplications is defined for plane wave sources and for point sources. This theory is then applied on two cases of slowness media; 1-D slowness perturbation models and 2-D Gaussian random media. The focus position in 1-D slowness models is proportional to the inverse of the square root of relative slowness fluctuations. For Gaussian random media, the distance at which caustics generate is dependent on the relative slowness perturbation in a power of minus two thirds. It is shown with snaphots of propagating plane wavefields that caustics develop as predicted by theory. The theory for caustic formation can be generalised to three dimensions.

1 INTRODUCTION

In terms of ray theory, the concept of caustics is understood as the focus point in space through which rays go. The consequence of the generation of caustics in a wave field is, in the ray geometrical limit, that the amplitude in the wave field is infinitely high at the focus point because the geometrical spreading factor is zero at the caustic point (Aki and Richards 1980, Menke and Abbot 1990). This phenomenon is investigated by several authors; White *et al.* (1988) use limit theorems for stochastic differential equations on the equation of dynamic ray tracing to predict when caustics start to develop in Gaussian random media. Kravtsov (1988) gives a thorough description of caustics. Brown and Tappert (1986) use Chapman's method to write explicitely the variation of 2-D and 3-D wavefields in the vicinity of focus points. They set up three properties of transient wavefields away from caustics; the most important characteristic of transient waves through caustics is that the triplication will generate after the ballistic wavefield due to causality.

A new theory for caustic formation is presented. This theory is based on ray perturbation theory but is formally equivalent to dynamical ray theory as used in White *et al.* (1988) because the normal derivative of the equations in ray perturbation theory is identical to the equation of dynamic ray tracing (Pulliam and Snieder 1998). In contrast to the treatment of White *et al.* (1988) this application is not restricted to random media.

In section 2, the general theory for caustic formation of wave fields emitted by plane wave sources and point sources is presented. The theory is then applied on a 1-D slowness perturbation medium and a 2-D Gaussian random medium for both plane wave sources and point sources. The results for the 2-D Gaussian random medium are similar to those found in White *et al.* (1988). In section 3, the theory for caustic formation is tested on numerical experiments where a plane wave field propagates in a 1-D slowness perturbation field and in a 2-D Gaussian random medium.

2 THEORY

We demonstrate how the focal length of converging wave fields in 2-D slowness perturbation fields can be computed. First, we derive the general theory for two distinct source geometries; the plane wave (plw) source and

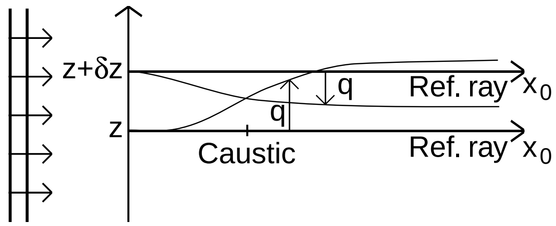

Figure 1. Definition of the geometric variables for an incoming plane wave in a 2-D medium with a constant reference slowness. There is one horizontal reference ray at z and another one at $z + \delta z$. The caustic develops at the intersection point of the two perturbed rays.

the point source (ps). Second, we apply this theory on two case studies; 1-D slowness perturbation fields and 2-D Gaussian random media. The presented theory for caustic formation can be generalised to three dimensions.

2.1 General Theory

We make use of ray perturbation theory (Snieder and Sambridge 1992) and separate the ray into a reference ray and a perturbed ray. The slowness field $u = u_0 + u_1$, is decomposed into the reference slowness field u_0 and the slowness perturbation field u_1. The reference slowness u_0 is kept constant in this work which means that the reference ray is a straight line. The perpendicular deflection from the reference ray to the perturbed ray at propagation distance x_0 is denoted by $q(x_0)$.

First, the case of an incoming plane wave is treated. Imagine two horizontal reference rays with slightly different initial positions. One reference ray is at the position z while the other reference ray is at the position $z + \delta z$. See Fig. 1 for a definition of the geometrical variables. For each reference ray there is a perturbed ray due to the slowness perturbation in the medium. The condition for caustics, that is to say that the two perturbed rays intersect, gives the following equation:

$$q(x_0, z + \delta z) + \delta z - q(x_0, z) = 0, \tag{1}$$

or

$$\frac{\partial q}{\partial z}(x_0) = \lim_{\delta z \to 0} \frac{q(x_0, z + \delta z) - q(x_0, z)}{\delta z} = -1. \tag{2}$$

Snieder and Sambridge (1992) show how the perpendicular ray deflection $q(x_0)$ from the reference ray can be computed given the slowness perturbation u_1;

$$q(x_0) = \int_0^{x_0} G(x_0, x)[\partial_\perp \left(\frac{u_1}{u_0}\right)](x)dx, \tag{3}$$

with $\partial_\perp$ the component of the gradient perpendicular to the reference ray so that

$$q(x_0) = \int_0^{x_0} G(x_0, x)\frac{\partial}{\partial z}\left(\frac{u_1}{u_0}\right)(x)dx, \tag{4}$$

for a horizontal reference ray. The Green's function

$$G(x_0, x) = \left\{ \begin{array}{ll} 0 & \text{if } x_0 < x \\ x_0 - x & \text{if } x_0 > x \end{array} \right\}, \tag{5}$$

has the boundary conditions, $G(0, x) = \dot{G}(0, x) = 0$. The condition for caustics in Eq. (2) contains the partial derivative of $q(x_0)$ with respect to z. Using Eq. (4) together with the condition for caustics in Eq. (2) at given z, we find that caustics are formed at x_0 when

$$\int_0^{x_0} G(x_0, x)\frac{\partial^2}{\partial z^2}\left(\frac{u_1}{u_0}\right)(x)dx = -1. \tag{6}$$

Second, the point source case is considered. We investigate the generation of caustics developing for rays that leave a point source with the azimuth φ. Assume again that two reference rays with slightly different initial positions are emitted from the source. One reference ray is sent in the direction $\varphi + \delta\varphi/2$, while the other reference ray is emitted in the direction $\varphi - \delta\varphi/2$. The distance between the reference rays is given by $x_0\delta\varphi$. The condition that the two perturbed rays cross each other leads to the following equation:

$$q(x_0, \varphi + \frac{1}{2}\delta\varphi) + x_0\delta\varphi - q(x_0, \varphi - \frac{1}{2}\delta\varphi) = 0, \tag{7}$$

or

$$\frac{1}{x_0}\frac{\partial q}{\partial \varphi}(x_0) = -1. \tag{8}$$

Using $\partial_\perp = (1/x)(\partial/\partial\varphi)$ in Eq. (3) the perpendicular ray deflection to the reference ray is derived. Hence

$$q(x_0) = \int_0^{x_0} G(x_0, x)\frac{1}{x}\frac{\partial}{\partial\varphi}\left(\frac{u_1}{u_0}\right)(x)dx. \tag{9}$$

The Green's function in Eq. (9) for the reference ray with the azimuth φ is the same as in the case of incoming plane waves which is stated in Eq. (5). With Eq. (9) combined with the condition of caustics in Eq. (8) at given z, we get that caustics generate at x_0 when

$$\frac{1}{x_0}\int_0^{x_0} G(x_0, x)\frac{1}{x}\frac{\partial^2}{\partial\varphi^2}\left(\frac{u_1}{u_0}\right)(x)dx = -1. \tag{10}$$

The second derivative of u_1/u_0 with respect to the transverse coordinate is an important quantity. It reflects that it is the curvature of the relative slowness perturbation that generates caustics. For example, negative $\partial^2/\partial z^2(u_1/u_0)$ and $\partial^2/\partial\varphi^2(u_1/u_0)$ lead to focusing of wave fields, whereas in areas with defocusing effects the two quantities are positive.

2.2 A medium with 1-D Slowness Perturbations

The focus position of a plane wave propagating in a medium with a constant reference slowness field u_0 and 1-D slowness perturbations $u_1(z)$ can be computed analytically. The reference ray in such a medium is a straight at given z. The condition for caustics in the case of incident plane waves given by Eq. (6) can be used to determine when caustics start to generate at the offset x^{plw}_{caus} at given z. The integration in Eq. (6) is carried out from 0 to x^{plw}_{caus}. Hence

$$x^{plw}_{caus}(z) = \sqrt{\frac{-2}{\frac{\partial^2}{\partial z^2}(\frac{u_1}{u_0})(z)}}. \tag{11}$$

The focal distance x^{ps}_{caus} of wave fields emitted by point sources is easily derived from the condition for caustics in Eq. (10). The second derivative $\partial^2/\partial\varphi^2 = x^2\partial^2/\partial z^2$ which permits an evaluation of the integration in Eq. (10) in the range from 0 to x^{ps}_{caus}. Thus

$$x^{ps}_{caus}(z) = \sqrt{\frac{-6}{\frac{\partial^2}{\partial z^2}(\frac{u_1}{u_0})(z)}}. \tag{12}$$

The distance between the source and receiver is denoted L. If $x^{plw/ps}_{caus}(z) < L$, triplications will be present in the recorded wave field.

2.3 Gaussian random media

Secondly, we discus the formation of caustics in Gaussian random media. The auto-correlation function $F(r)$ of a Gaussian random medium is given by

$$\begin{aligned} F(r) &= < u_1(\mathbf{r}_1)\mathrm{u}_1(\mathbf{r}_2) > \\ &= (\epsilon \mathrm{u}_0)^2 \exp(-\frac{\mathrm{r}^2}{\mathrm{a}^2}), \end{aligned} \tag{13}$$

where ϵ is the rms value of the relative slowness perturbations, a denotes the correlation length (or roughly the length-scale of slowness perturbations) and $r = |\mathbf{r}_1 - \mathbf{r}_2|$. Notice that the reference slowness is biased in a realisation of a finite Gaussian random model (e.g. Müller *et al.* 1992). However, this artifact does not lead to focusing or defocusing effects because the shift in reference slowness only affects the slowness field on average, but not the derivatives of the slowness.

According to Eq. (2) caustics develop in a plane wave field when $\partial q/\partial z = -1$. This implies that on average in a random medium caustics develop when

$$< \left(\frac{\partial q}{\partial z}\right)^2 (x_0) >= 1, \tag{14}$$

where $< \ldots >$ is the expectation value. For this reason the following quantity is used to monitor the formation of caustics.

$$\begin{aligned} H^{plw}(x_0) &\equiv < \left(\frac{\partial q}{\partial z}\right)^2 (x_0) >, \\ &= \frac{1}{u_0^2}\int_0^{x_0}\int_0^{x_0} G(x_0,x')G(x_0,x'') \\ &\quad \times < \frac{\partial^2}{\partial z^2}u_1(x')\frac{\partial^2}{\partial z^2}u_1(x'') > dx'dx'' \end{aligned} \tag{15}$$

The monitor is zero at the source position and $H^{plw}(x_0) = 1$ when caustics start to develop at the offset x_0 according to Eq. (14)

We follow the same method as used in Roth *et al.* (1993) to evaluate the right-hand side of Eq. (15). First, the expectation value of the slowness perturbation field differentiated with respect to z twice at the offset x' and x'', respectively, in Eq. (15) is expressed in a simple form containing the characteristic parameters for the Gaussian random medium. The following expression is evaluated on the horizontal reference ray z_0.

$$\begin{aligned} &< \frac{\partial^2}{\partial z^2}u_1(x',z)\frac{\partial^2}{\partial z^2}u_1(x'',z) > |_{z=z_0} \\ =&< \frac{\partial^4}{\partial z'^2\partial z''^2}u_1(x',z')u_1(x'',z'') > |_{z'=z''=z_0} \\ &= \frac{\partial^4 F(r)}{\partial z'^2 \partial z''^2}|_{z'=z''=z_0}. \end{aligned} \tag{16}$$

The auto-correlation function $F(r)$ is differentiated twice with respect to z' and z'' in Eq. (16), which gives

$$\frac{\partial^4 F(r)}{\partial z'^2\partial z''^2}|_{z'=z''=z_0} = \frac{3}{r^2}\left(F''(r) - \frac{F'(r)}{r}\right)|_{z'=z''=z_0}. \tag{17}$$

The prime and double prime of $F(r)$ signifies a single and double differentiation with respect to r. Using the auto-correlation function $F(r)$ in Eq. (13) for Gaussian random media the left-hand side of Eq. (16) is finally written as

$$\begin{aligned} &< \frac{\partial^2}{\partial z^2}u_1(x',z)\frac{\partial^2}{\partial z^2}u_1(x'',z) > |_{z=z_0} \\ &= 12\frac{(\epsilon u_0)^2}{a^4}\exp(-\left(\frac{r}{a}\right)^2). \end{aligned} \tag{18}$$

The right-hand side of the monitor for plane waves in Eq. (15) can be simplified further. Define

$$f(r) \equiv< \frac{\partial^2}{\partial z^2}u_1(x',z)\frac{\partial^2}{\partial z^2}u_1(x'',z) > |_{z=z_0}, \tag{19}$$

where $r = |x' - x''|$ and

$$\begin{aligned} \eta(x',x'') &= G(x_0,x')G(x_0,x'') \\ &= x_0^2 + x'x'' - x_0(x'+x''). \end{aligned} \tag{20}$$

We then derive from Eq. (15) that

$$\begin{aligned} &\int_0^{x_0}\int_0^{x_0} G(x_0,x')G(x_0,x'') \\ &\times < \frac{\partial^2}{\partial z^2}u_1(x')\frac{\partial^2}{\partial z^2}u_1(x'') > |_{z=z_0}dx'dx'' \\ &= \int_0^{x_0}\int_0^{x_0} \eta(x',x'')f(|x'-x''|)dx'dx'', \end{aligned} \tag{21}$$

for x' and x'' smaller than x_0. Using the integration technique in Roth *et al.* (1993) the expression for the monitor in Eq. (15) is simplified further. The details of this integration method are explained in appendix A; here we just give the results. The double integration in Eq. (21) from 0 to x_0 is changed to an integration from 0 to x_0 of the function $f(r)$ in Eq. (19) multiplied by a summation of two integrations of $\eta(x',x'')$ in Eq. (20) from r to x_0 and from 0 to $x_0 - r$, respectively. In brief, Eq. (21) is written as

$$\int_0^{x_0} drf(r)[\int_r^{x_0} \eta(x', x'-r)dx' + \int_0^{x_0-r} \eta(x', x'+r)dx']. \quad (22)$$

The solution to the two integrations of $\eta(x', x'')$ inside the rectangular brackets are computed analytically.

$$\int_r^{x_0} \eta(x', x'-r)dx' = \int_0^{x_0-r} \eta(x', x'+r)dx' = \frac{1}{3}x_0^3 - \frac{1}{2}x_0^2 r + \frac{1}{6}r^3. \quad (23)$$

The expression for the function $f(r)$ in Eq. (19) and for the integration of $\eta(x', x'')$ in Eq. (23) are used together with the expression for the monitor in Eq. (15). Hence the monitor for plane waves propagating in a Gaussian random medium simplifies to

$$H^{plw}(x_0) = 12\frac{\epsilon^2}{a^4}\int_0^{x_0}(\frac{2}{3}x_0^3 - x_0^2 r + \frac{1}{3}r^3) \times \exp(-\left(\frac{r}{a}\right)^2)dr. \quad (24)$$

By letting x_0 go to zero in Eq. (24) it is easy to verify that $H^{plw}(0) = 0$.

Assume first that the propagation length is less than the correlation length, i.e. $x_0/a < 1$. The exponential function is set to unity in this regime and the integration of the right-hand side of Eq. (24) is carried out directly. Hence

$$H^{plw}(x_0) = 3\epsilon^2 \left(\frac{x_0}{a}\right)^4 \ll 1, \quad (25)$$

which reflects that caustics are not formed in this regime.

Suppose instead that the propagation distance is much greater than the correlation length, i.e. $x_0/a \gg 1$. We can then compute the analytical solution of the monitor in Eq. (24) by letting the range of integration go to infinity because the exponential in the integrand approaches zero for $r \gg a$. Thus

$$\begin{aligned} H^{plw}(x_0) &\approx 12\frac{\epsilon^2}{a^4}\int_0^{\infty}(\frac{2}{3}x_0^3 - x_0^2 r + \frac{1}{3}r^3) \times \exp(-\left(\frac{r}{a}\right)^2)dr \\ &= 12\frac{\epsilon^2}{a^4}(\frac{\sqrt{\pi}a x_0^3}{3} - \frac{a^2 x_0^2}{2} + \frac{a^4}{6}) \\ &\approx 4\sqrt{\pi}\epsilon^2\left(\frac{x_0}{a}\right)^3. \end{aligned} \quad (26)$$

We have made use of the assumption that $x_0/a \gg 1$ to eliminate the last two terms in the brackets of Eq. (26). Let L denote the source-receiver offset. We then derive the non-dimensional number L/a from Eq. (26) in the case that caustics develop at $x_0 < L$. Hence by using $H^{plw}(L) \geq 1$, we get that

$$\frac{L}{a} \geq \frac{\epsilon^{-2/3}}{(4\sqrt{\pi})^{1/3}} = 0.52 \quad \epsilon^{-2/3}. \quad (27)$$

For a point source the generation of caustics can be evaluated along similar lines. The monitor $H^{ps}(x_0)$ is defined in the same way as the monitor for plane waves except that the condition for caustics formation in Eq. (8) for point sources is applied. Thus

$$\begin{aligned} H^{ps}(x_0) &\equiv < \left(\frac{1}{x_0}\frac{\partial q}{\partial \varphi}\right)^2 (x_0) >, \\ &= \frac{1}{x_0^2 u_0^2}\int_0^{x_0}\int_0^{x_0} G(x_0, x')G(x_0, x'')\frac{1}{x'x''} \\ &\times < \frac{\partial^2}{\partial \varphi^2}u_1(x')\frac{\partial^2}{\partial \varphi^2}u_1(x'') > dx'dx''. \end{aligned} \quad (28)$$

According to Eq. (8) caustics develop at the offset x_0 when the monitor in Eq. (28) is equal to one. The mean value $< \frac{\partial^2}{\partial \varphi^2}u_1(x')\frac{\partial^2}{\partial \varphi^2}u_1(x'') >$ is related to $< \frac{\partial^2}{\partial z^2}u_1(x')\frac{\partial^2}{\partial z^2}u_1(x'') >$ by using the chain rule that

$$\frac{\partial}{\partial z} = \frac{\partial \varphi}{\partial z}\frac{\partial}{\partial \varphi} = \frac{1}{x}\frac{\partial}{\partial \varphi}, \quad (29)$$

because $z = x\varphi$ for small values of φ. Thus

$$< \frac{\partial^2}{\partial \varphi^2}u_1(x')\frac{\partial^2}{\partial \varphi^2}u_1(x'') > = (x'x'')^2 < \frac{\partial^2}{\partial z^2}u_1(x', z)\frac{\partial^2}{\partial z^2}u_1(x'', z) > |_{z=z_0} \quad (30)$$

The procedure used for the derivation of the monitor for plane waves is repeated for the monitor for point sources. The only difference from the previous example is that the function $\eta(x', x'') = G(x_0, x')G(x_0, x'')x'x'' = x_0^2 x'x'' + (x'x'')^2 - x_0(x' + x'')x'x''$. The final result of the rather long derivation of the monitor for caustic formation in the point source case is given by

$$H^{ps}(x_0) = 12\frac{\epsilon^2}{a^4}\int_0^{x_0}(\frac{1}{15}x_0^3 - \frac{4}{3}x_0 r^2 + \frac{4}{3}r^3 + \frac{7}{15}\frac{r^5}{x_0^2}) \times \exp(-\left(\frac{r}{a}\right)^2)dr. \quad (31)$$

By letting x_0 go to zero it can be shown that $H^{ps}(0) = 0$.

Similar to Eq. (25), it can easily be shown that triplications due to point source wavefields do not generate when the length-scale of heterogeneity is greater than the source-receiver distance. Assume instead that $x_0/a \gg 1$ and carry on just like in the case of incident plane waves. The analytical expression of the right-hand side of Eq. (31) is given by

$$H^{ps}(x_0) = \frac{2\sqrt{\pi}}{5}\epsilon^2\left(\frac{x_0}{a}\right)^3. \quad (32)$$

The non-dimensional number L/a for the condition that caustics develop in the recorded wave field is derived from Eq. (32), thus the condition that $H^{ps}(L) \geq 1$ gives that

$$\frac{L}{a} \geq \left(\frac{5}{2\sqrt{\pi}}\right)^{1/3}\epsilon^{-2/3} = 1.12 \quad \epsilon^{-2/3}. \quad (33)$$

It is instructive to compare Eq. (27) for plane waves

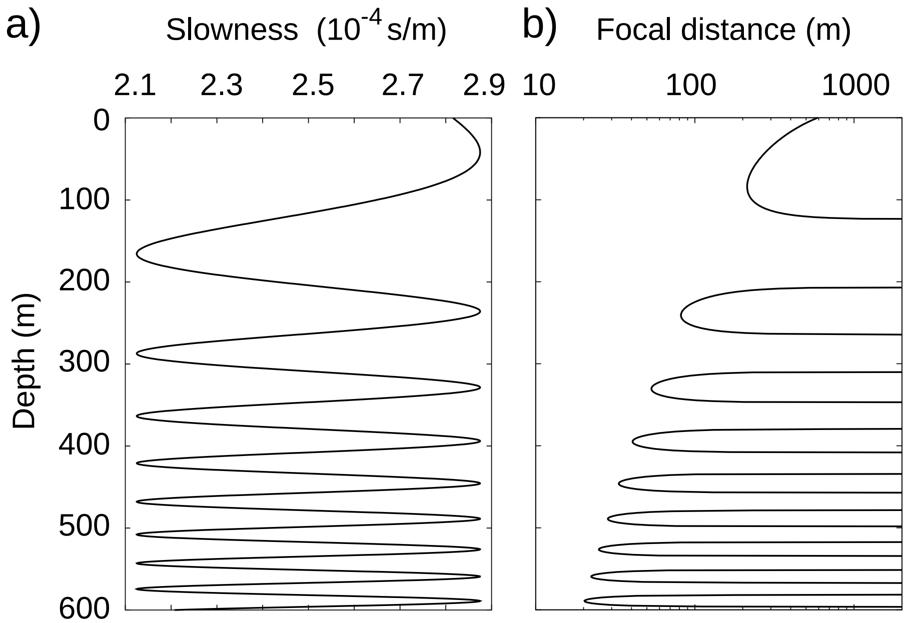

Figure 2. The 1-D slowness field with $\epsilon = 0.035$ is shown in a). The focal distance (solid line) of a plane wave field is calculated as function of depth in b). Notice that the incoming plane wave field is focusing in regions with positive slowness perturbations and it is being defocused when the slowness perturbation is negative.

and Eq. (33) for point sources with estimates obtained by White *et al.* (1988). They use limit theorems for stochastic differential equations on the equation of dynamic ray tracing in Gaussian random media to calculate the probability that a caustic occurs at a certain propagation distance. In Fig. 4 and 5 of White *et al.* (1988), they demonstrate universal curves for the probability of caustic formation as a function of the universal distance defined as

$$\tilde{\tau} = (8\pi)^{1/6}\epsilon^{2/3}\frac{L}{a}, \tag{34}$$

where we have made a change of symbol from White *et al.* (1988) by using ϵ for the relative rms value of slowness perturbations and L for the propagation distance of the wave field. This means that in the theory of White *et al.* (1988) caustics develop when the non-dimensional number L/a is given by

$$\frac{L}{a} = \frac{\tilde{\tau}}{(8\pi)^{1/6}}\epsilon^{-2/3}. \tag{35}$$

This expression has the same dependence on ϵ as the condition for caustics in Eq. (27) and (33).

According to Fig. 4 and 5 of White *et al.* (1988) the highest probability for generation of caustics for plane waves is found for $\tilde{\tau} = 0.9$ and for point sources $\tilde{\tau} = 1.9$. By inserting the appropriate value of $\tilde{\tau}$ into the factor $\tilde{\tau}/(8\pi)^{1/6}$ from Eq. (35), we find that the factor is 0.53 and 1.11 for the case of plane waves and point sources, respectively. Comparing these two numbers with the corresponding factors in Eq. (27) and (33), we see that there is a good agreement between the work of White *et al.* (1988) and our work.

Although we have derived the condition for caustics due to plane waves and point sources in two dimensions, the theory for caustics can be generalised to three dimensions. In 3-D, the equation for the ray perturbations q_1 and q_2 in the directions perpendicular to the ray decouples for a homogeneous reference model and a coordinate system that does not rotate around the reference ray (see Eq. (50) of Snieder and Sambridge 1992). The condition for caustics in Eq. (2) and (8) for plane waves and point sources, respectively, can be applied to the ray perturbation in two orthogonal directions separately. For example as it is shown in appendix B, the non-dimensional number L/a for the point focus in 3-D Gaussian random media is given by Eq. (27) for plane waves and by Eq. (33) for point sources. Notice that in 3-D a caustic is not necessarily the same as a point focus. A caustic can in that case also be a line of focus points, whereas a focus point, as the word says, is located at a point.

3 NUMERICAL EXAMPLES

In this section, numerical examples of caustic formation of plane wave fields are shown for two distinct media; A 1-D medium with the slowness perturbation field described by $u_1(z) = \sqrt{2}\epsilon u_0 \sin((z + z_0)^4/k)$, and a 2-D Gaussian random medium with the slowness perturbation field described by Eq. (13). The quantity u_0 is the reference slowness which is constant for all numerical experiments shown in this paper. The rms value of relative slowness fluctuations is denoted by ϵ. The parameters z_0, k and ϵ are appropriate values to adjust the media such that the development of triplications is significant.

In Fig. 2a, the 1-D slowness medium with $z_0 = 350$ m, $k = 1.5\ 10^{10}$ m^4, $u_0 = 2.5\ 10^{-4}$ s/m and $\epsilon = 0.035$ is plotted. It is seen in that figure that the slowness field changes more and more rapidly as function of z. In Fig. 2b, the focal distance of a plane wave field propagating in the 1-D slowness medium as shown in Fig. 2a is computed by using Eq. (11). The offset from the source position is plotted on the abscissa while the depth at which caustics start to develop is plotted on the ordinate. The focal distance of the converging, plane wave field is shown with the solid line. Notice that there are zones with defocusing of the plane wave field which is manifested in Fig. 2b between 120 m and 205 m, between 265 m and 305 m, between 350 m and 380 m, etc... In these zones the wave field propagates through a zone with a positive curvature of the relative slowness perturbation, so caustics do not develop. Thus the focal distance is infinite. The curvature of the relative slowness fluctuations increases as function of z, so the focal distance of the converging wave field decreases as depth increases.

In Fig. 3, snapshots of a plane wave field propagating through the 1-D slowness perturbation field with the same value of z_0, k, u_0 and ϵ as for the 1-D medium in Fig. 2a are shown. The snapshots are produced with a finite difference solution of the acoustic wave equation. The central frequency is 1000 Hz, and the applied source function is a Ricker wavelet. The snapshots are registered for every 5 ms with the first snapshot at the source position and the last snapshot at about 100 m offset. Positive amplitudes are dark while negative amplitudes are bright. The first triplications are visible in the snapshot at $t = 10$ ms ($\sim$ 40 m) at depths below 500 m as the wavefront contains large positive amplitudes. In the snapshots for $t = 15$ ms ($\sim$ 60 m), 20 ms ($\sim$ 80 m) and 25 ms ($\sim$ 100 m), the triplications generated in the wave field are very clear as they give rise to a half bowtie-formed wave after the ballistic wavefront. Comparing the theoretically predicted minimum focal distance of the converging wave field in Fig. 2b with the offset at which triplications start to develop in the wave field in Fig. 3, we find that there is good agreement between the presented theory for caustic formation and the numerical 1-D experiment.

In Fig. 4, snapshots of a plane wave propagating in a 2-D Gaussian random medium with the reference slowness $u_0 = 2.5\ 10^{-4}$ s/m, the relative slowness fluctuation $\epsilon = 0.025$ and the correlation length $a = 7.1$ m are presented. The central frequency is 1000 Hz, while the Ricker wavelet is applied as source impulse. The 10 snapshots are computed for every 2.5 ms, where the

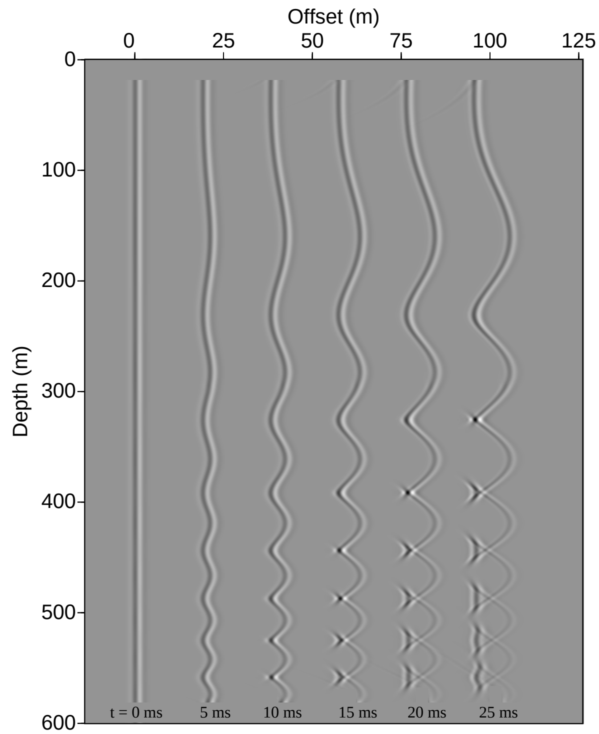

Figure 3. Snapshots of plane wave propagation in the 1-D slowness perturbation model with $\epsilon = 0.035$. The absolute travel times $t = 0, 5, 15, 20, 25$ ms are marked at the respective wavefronts. Caustics become very clear after the ballistic wavefronts for $t = 15, 20, 25$ ms.

first snapshot is taken at the initial wavefront and the last snapshot is taken at about 90 m offset. In the upper and lower part of the plane wavefronts in Fig. 4, a circular wave due to diffraction at the edge of the numerical grid can be seen. Inserting the appropriate value for ϵ into Eq. (27) the non-dimensional number $L/a = 6.1$ for the development of triplications in the Gaussian random medium is found. The expectation value of the offset at which caustics start to generate is then $L = 43$ m for $a = 7.1$ m. In Fig. 4, no triplications are observed in the wavefront at $t = 0$ ms, 2.5 ms and 5 ms, i.e. at approximately 0 m, 10 m, and 20 m, respectively. Then for $t = 7.5$ ms ($\sim$ 30 m) and 10 ms ($\sim$ 40 m) the multipathing that is associated with the formation of caustics can be seen as a minor wave field after the ballistic wavefront. This generation of minor wave fields delayed compared to the ballistic wavefront is neither due to uncertainties in the FD-code nor due to scattering effects (for the employed wave $\lambda/a \approx 0.5$) but because of caustic formation. For the last 5 snapshots at $t = 12.5$ ms ($\sim$ 50 m), 15 ms ($\sim$ 60 m), 17.5 ms ($\sim$ 70 m), 20 ms ($\sim$ 80 m) and 22.5 ms ($\sim$ 90 m) triplications are developing rather strongly after the wavefront. The maximum amplitude variation along the wave field for

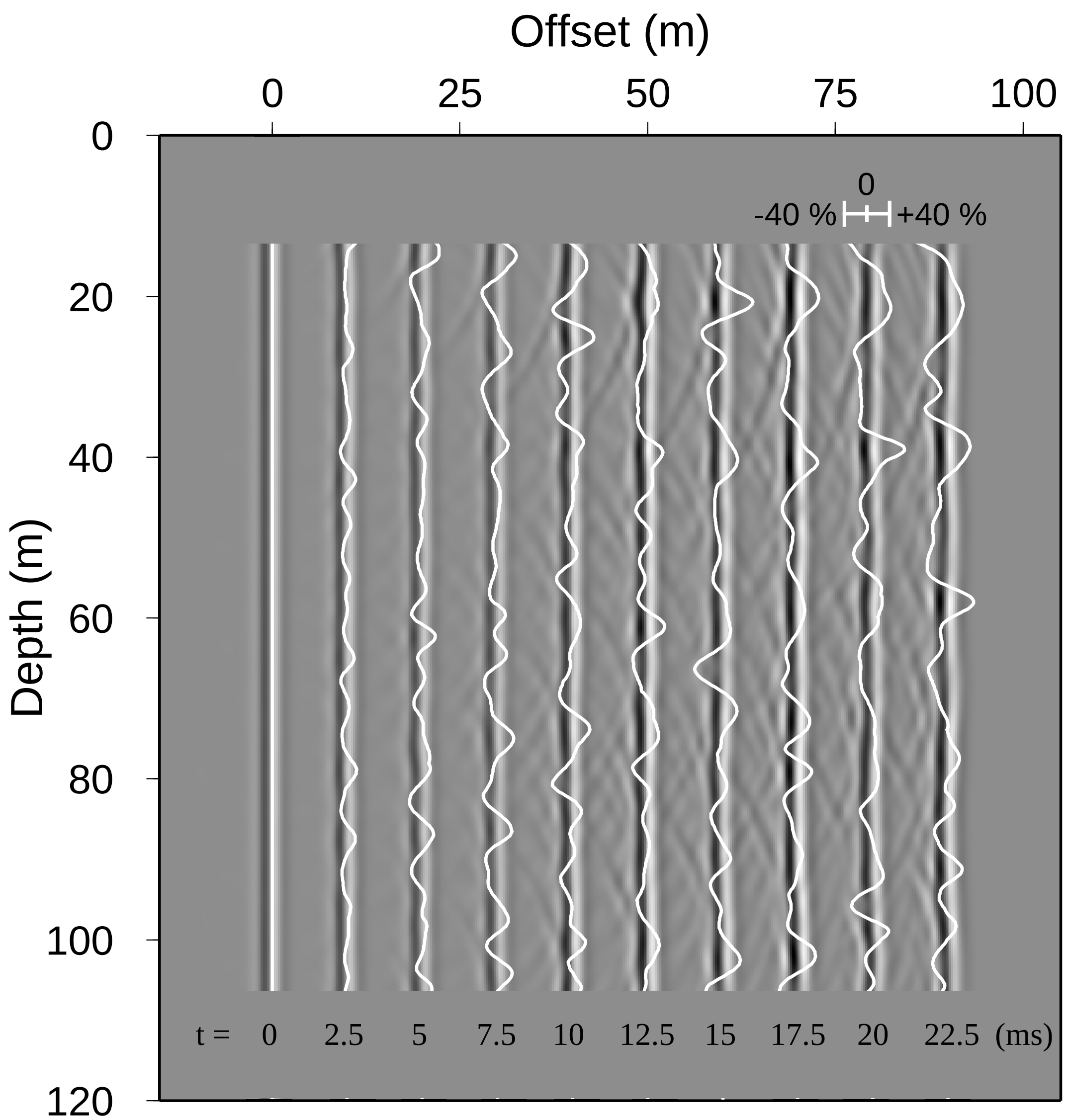

Figure 4. Snapshots of plane wave propagation in the 2-D Gaussian random model with $\epsilon = 0.025$ and $a = 7.1$ m. The absolute travel times $t = 0$, 2.5, 5, 7.5, 10, 12.5, 15, 17.5, 20, 22.5, 25 ms are marked at the respective wavefronts. Caustics develop in the wave fields for t larger than or equal to 7.5 ms. The maximum amplitude variation along the wave field for each wavefront is shown with the white solid line. Notice that the colour in the wavefronts gets darker when the maximum amplitude is at a peak. The bar in the upper right corner shows the percentage variation of the maximum amplitude in the perturbed slowness model compared with the reference amplitude computed in the constant reference slowness model.

each wavefront is plotted with the white solid line in Fig. 4. For the initial wavefront at $t = 0$ ms, the amplitude is constant, while the maximum amplitude along the wave field varies with increasing extrema for the wavefronts for larger t. The bar in the upper right corner of Fig. 4, shows the relative percentage of the amplitude variations in the perturbed slowness model compared with the reference amplitude computed for the homogeneous reference slowness model. Notice that the largest positive values of the maximum amplitude along the wavefronts correspond to the parts of the wavefronts with darkest colour while the negative amplitude variations are shown with bright colours. Witte *et al.* (1996) use the kinematic ray-tracing equation to construct a ray-diagram for a Gaussian random medium with fixed $\epsilon = 0.03$, but with different values of the correlation length a. Using Eq. (27) with $\epsilon = 0.03$, gives the non-dimensional number $L/a = 5.4$. Looking at the top panel in Fig. 4 of Witte *et al.* (1996) for $L/a = 10$, it is seen that the first caustics generate at $z \approx$ 5-6 which corresponds well with the theoretical value computed with Eq. (27).

4 CONCLUSIONS

In this paper, we develop a theory for the formation of caustics. The theory is based on ray perturbation theory, but is equivalent to a similar approach by White *et al.* (1988) where the equation of dynamic ray tracing is used to predict when triplications develop in Gaussian random media.

We have applied the theory for the generation of caustics in two case studies (i.e. 1-D slowness perturbations fields and 2-D Gaussian random media) where the plane wave source and the point source are taken into account. The theory for caustic formation can be generalised to wave fields propagating in 3-D. We find that the formation of caustics for 1-D slowness perturbation fields depends on the inverse of the square root of the relative slowness perturbation, while for Gaussian random media the formation of caustics is dependent upon the relative slowness perturbation in a power of minus two thirds.

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APPENDIX A: THE INTEGRATION IN EQUATION (21)

In this appendix, the step from Eq. (21) to Eq. (22) is demonstrated. The right hand side of Eq. (21) is written as

$$\int_0^{x_0}\int_0^{x_0} \eta(x',x'')f(|x'-x''|)dx'dx'', \tag{A1}$$

where

$$f(|x'-x''|) = < \frac{\partial^2}{\partial z^2}u_1(x',z)\frac{\partial^2}{\partial z^2}u_1(x'',z) > |_{z=z_0}, \tag{A2}$$

and

$$\eta(x',x'') = G(x_0,x')G(x_0,x''). \tag{A3}$$

The integration over x'' in Eq. (A1) is split into one integration over x'' from 0 to x' and another integration

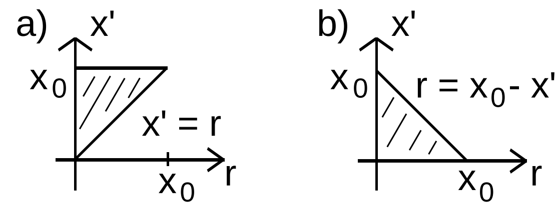

Figure A1. The integration technique which is found in Roth *et al.* (1993). The area of integration for $\int_0^{x'} \eta(x',x'-r)f(r)dr$ and $\int_0^{x_0-x'} \eta(x',x'+r)f(r)dr$ in Eq. (A6) is shown in a) and b), respectively.

over x'' from x' to x_0. Thus Eq. (A1) is rewritten as

$$\int_0^{x_0} dx' \quad [\int_0^{x'} \eta(x',x'')f(x'-x'')dx'' + \int_{x'}^{x_0} \eta(x',x'')f(x''-x')dx'']. \tag{A4}$$

Define now $r = x'-x''$ and $r = x''-x'$ for the first and second integration over x'' in the brackets of Eq.(A4) and carry out a change of variables in the two integrations over x'' inside the brackets. The result is given by

$$\int_0^{x_0} dx' \quad [-\int_{x'}^{0} \eta(x',x'-r)f(r)dr + \int_0^{x_0-x'} \eta(x',x'+r)f(r)dr]. \tag{A5}$$

or

$$\int_0^{x_0} dx' \quad [\int_0^{x'} \eta(x',x'-r)f(r)dr + \int_0^{x_0-x'} \eta(x',x'+r)f(r)dr]. \tag{A6}$$

The integration over x' multiplied with the first and second integration over r in the brackets of Eq. (A6) corresponds to the triangle in the r-x' plane as shown in Fig. A1a and A1b, respectively. By changing the order of integration in Eq. (A6), but still keeping in mind that the double integration over r and x' must be over the triangles as shown in Fig. A1a and A1b, it is possible to rewrite the Eq. (A6) in the following way

$$\int_0^{x_0} dr \quad [\int_r^{x_0} \eta(x',x'-r)f(r)dx' + \int_0^{x_0-r} \eta(x',x'+r)f(r)dx']. \tag{A7}$$

After rearranging the term $f(r)$ outside the integration over x', the result is finally given by

$$\int_0^{x_0} drf(r) \quad [\int_r^{x_0} \eta(x',x'-r)dx' + \int_0^{x_0-r} \eta(x',x'+r)dx'], \tag{A8}$$

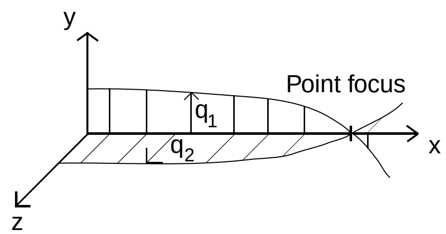

Figure A2. Estimation of the point focus in a 3-D medium.

which is the equation that is stated in Eq. (22).

APPENDIX B: CAUSTICS IN 3-D

Imagine that a plane wave field is propagating along the x-axis with the decoupled ray deflections q_1 and q_2 parallel to the y-axis and z-axis, respectively. See Fig. A2 for the geometry. Using the results from Snieder and Sambridge (1992), the decoupled differential equations for the ray deflection coordinates are then given by

$$\frac{d^2}{dx^2}q_i = \hat{q}_i \cdot \nabla \left(\frac{u_1}{u_0}\right), \tag{B1}$$

where $i = 1, 2$. The ray deflections are gathered together in the ray deflection vector $\mathbf{q}= (0, q_1, q_2)$. The condition for caustic formation in Eq. (2) is applied on each ray deflection coordinate. Hence

$$\nabla\mathbf{q}(x_o) = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}, \tag{B2}$$

for a point focus at the offset x_0. In order to determine when caustics develop in a 3-D Gaussian random medium, the expectation value of $\nabla\mathbf{q}(x_o) \cdot \nabla\mathbf{q}(x_o)$ is computed. Thus according to Eq. (B2) for a point focus, caustics develop when

$$\begin{aligned} < \nabla\mathbf{q} \cdot \nabla\mathbf{q} > (x_o) &=< \left(\frac{\partial q_1}{\partial y}\right)^2 (x_o) + \left(\frac{\partial q_2}{\partial z}\right)^2 (x_o) > \\ &=< \left(\frac{\partial q_1}{\partial y}\right)^2 (x_o) > + < \left(\frac{\partial q_2}{\partial z}\right)^2 (x_o) >= 2, \end{aligned} \tag{B3}$$

at the offset x_0. For 2-D Gaussian random media, the following result is derived;

$$H^{plw}(x_0) \equiv< \left(\frac{\partial q}{\partial z}\right)^2 (x_0) >= 1, \tag{B4}$$

when a caustic develops at the offset x_0. According to Eq. (26) $H^{plw}(x_0) = 4\sqrt{\pi}\epsilon^2(x_0/a)^3$ for $x_0/a \gg 1$. This result can be used for each ray deflection q_i separately, so the monitors $H_1^{plw}(x_0)$ and $H_2^{plw}(x_0)$ for q_1 and q_2, respectively, are defined as

$$\begin{aligned} H_1^{plw}(x_0) &\equiv < \left(\frac{\partial q_1}{\partial y}\right)^2 (x_0) > \qquad \text{and} \\ H_2^{plw}(x_0) &\equiv < \left(\frac{\partial q_2}{\partial z}\right)^2 (x_0) >, \end{aligned} \tag{B5}$$

where $H_1^{plw}(x_0) = H_2^{plw}(x_0) = 4\sqrt{\pi}\epsilon^2(x_0/a)^3$. Combining Eq. (B3) with the monitors defined in Eq. (B5) it is computed that

$$H_1^{plw}(x_0) + H_2^{plw}(x_0) = 2, \tag{B6}$$

or

$$4\sqrt{\pi}\epsilon^2 \left(\frac{x_0}{a}\right)^3 = 1, \tag{B7}$$

for a caustic at the offset x_0 in a 3-D Gaussian random medium which is also the result found for 2-D Gaussian random media. Similarly, the result for caustic formation due to a point source in 3-D Gaussian random media is unaltered from the result in the 2-D case.

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