2014 Project Review
Consortium Project on Seismic Inverse Methods for Complex Structures

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Optimal wave focusing in acoustic media

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ABSTRACT
Focusing waves inside a medium has applications in various science and engineering fields, e.g., in medicine, nondestructive evaluation, ocean acoustics, and geophysics. The goal in focusing is to concentrate the wave energy at a specific time and location inside a medium. Various techniques have been devised and used to achieve this goal. Time-reversal is a method that is routinely used to focus acoustic and seismic waves. One important geophysical application of time-reversal focusing is in seismic source imaging. However, the method is not optimal for source imaging when acquisition is incomplete. Here, we propose a new technique wherein wave focusing is cast as an optimization problem. The new technique mitigates the adverse effects that incomplete acquisition has on time-reversal focusing and source imaging.

Key words: source imaging, focusing, optimization, time-reversal, acoustic emission

1 INTRODUCTION
The objective in wave focusing is to determine the waveforms that, when transmitted through a medium, create a wavefield that concentrates at a specific time and location. Wave focusing is conceptually related to the problem of imaging, and hence finds important applications in areas such as seismology and exploration geophysics.

Several methods for focusing have been devised, including those based on inverse scattering (Haddadin and Ebbini 1998; Broggini et al. 2012; Behura et al. 2012), phase conjugation (Parvulescu 1961), and time-reversal (Fink 1997). Time-reversal (TR) is a well-established focusing technique that is robust and effective in heterogeneous media. The method relies on the time-reversal invariance of the wave operator and spatial reciprocity (Fink et al. 2002; Snieder 2004).

A time-reversal mirror (TRM) is an array of transducers, each capable of recording, time-reversing (last-in first-out), and retransmitting signals into the medium. The TR process consists of two basic steps (Figure 1). In the first step, the wavefield generated by a source in the medium is recorded using a TRM. (To completely reconstruct the wavefield by the TR method, one needs a closed TRM surrounding the source.) In the second step, the recorded waveforms are time-reversed, retransmitted through the medium, and propagated back to refocus at the original source location. In a dissipative medium, time-reversal invariance is not satisfied. Spatial reciprocity alone, nevertheless, explains the robustness and efficiency of the TR process in many applications involving dissipative media (Fink 2006).

TR focusing has been implemented in a variety of physical scenarios. It is applicable to both physical and numerical (back-propagation) experiments. In both, one deals with propagation of a time-reversed field, but the propagation is real in a physical problem and simulated in a back-propagation numerical problem (Fink 2006). Applications involving physical TR include medical imaging (Robert and Fink 2008), lithotripsy, underwater acoustics, and nondestructive testing (Fink 1997; Larmat et al. 2010).

TR back-propagation methods are applied in key areas of geophysics on both global and exploration scales. In global seismology, TR techniques are used for studying earthquake source mechanism and location, for monitoring nuclear explosions, and in environmental applications of geophysics (Lu 2002; Larmat et al. 2006, 2010). In exploration seismology, TR focusing is used in microseismic event location (McMechan 1982; Lu and Willis 2008; Xuan and Sava 2010) and reservoir monitoring (Shapira 2008); in salt- flank imaging and redatuming seismic data (Lu 2002); and in reversed time migration (McMechan 1983; Berkhout 1997; Schuster 2002).

Despite these broad applications, the TR process has important theoretical and practical limitations. In theory, for a broadband pulse emitted by an ideal point
Figure 1. Illustration of a time-reversal experiment. (a) Forward propagation step: waves excited by a source travel through the complex medium and are recorded at stations marked as circles. (b) Back-propagation step: the recorded signals are reversed in time and rebroadcast into the medium at the corresponding stations. The waves then propagate through the medium and converge on the original source location.

source, the returning field refocuses on a spot with dimensions on the order of the smallest wavelength (Abbe diffraction limit). This is because evanescent waves containing source details smaller than the involved wavelengths cannot be sensed in the far-field. The loss of this information causes the resolution of the process to be bounded by the diffraction limit (Fink, 1997).

In practice, the wavefield is sampled at spatially sparse and limited locations. Also, it is not practical to surround the source with a full-aperture TRM, so a finite-aperture TRM is used instead. This incomplete acquisition results in a distortion in the shape of the point-spread function (Fink, 2006).

Another problem is that in real applications of TR-focusing, the media are dissipative and time-reversal invariance of the wave equation does not hold valid in dissipative media. However, as shown by Fink (2006), even in a dissipative medium, the TR process always maximizes the output amplitude at the focal time although it does not impose any constraints on the field around the focus. For example, side lobes can be observed around the source.

Several studies have been devoted to investigate these limitations and alleviate their effect to improve the TR procedure. Tanter et al. (2000, 2001), Aubry et al. (2001), and Montaldo et al. (2003) present the spatio-temporal inverse-filter method, a new focusing technique based on the inverse of the wave propagator between a source and elements in a TRM. For a lossless medium, the inverse filter method yields the same result as that of the TR method, but in the presence of attenuation, the spatio-temporal inverse filter methods are more effective than is the TR method.

Research on the connection between medium complexity and the size of the focal spot has shown a direct relationship between the complexity of the medium and the resolution in TR focusing; the more complicated the medium between the source and the TRM, the sharper the focus (Blomgren et al., 2002; Fink, 2008; Vellekoop et al., 2010). This is because a finite-aperture TRM acts an antenna that uses complex environments to appear wider than it actually is, resulting in a focusing capability that is less dependent on the TRM aperture. In media made of random distribution of sub-wavelength scatterers, a time-reversed wave field can interact with the random medium to regenerate not only the propagating but also evanescent waves required to refocus below the diffraction limit (super-resolution). Schuster et al. (2012) demonstrate a method that uses evanescent waves generated by scatterers in the near-field region of seismic sources to achieve super-resolution.

We propose an alternative approach to wave focusing wherein the problem is cast as an optimization problem. As discussed above, the mainstream focusing methods currently in use are not optimal for real problems because of imperfect acquisition, attenuation, and the diffraction limit. The motivation for this research is to improve upon the existing techniques especially where such techniques do not perform optimally in dealing with the limitations posed on wave focusing by incomplete acquisition.

The organization of the paper is as follows: We first lay out the theoretical foundation of our new approach to wave focusing (section 2) and show how to design waveforms that will optimally focus at a desired known location and time inside a medium. Next we discuss how the new method, named Backus-Gilbert focusing (BG) is connected to some other focusing techniques like time-reversal and show that such other techniques are special cases of the more general solution to wave focusing that our approach provides (section 3). In the
next step, we show how, with a modicum of modification and reinterpretation, the BG theory can be adapted for application in source imaging problems (section 4). We start by adapting the BG theory for imaging a single point source, and then in appendix A we generalize the theory for an arbitrary spatio-temporally distributed source without knowing the properties of the source itself. Section 5 is devoted to two simple but representative numerical examples that demonstrate the application of the BG method in imaging of a point source and then a dipole source. In section 6 we discuss various aspects of the method and elaborate further on some explicit and implicit assumptions that are used in the construction of the new method and discuss their significance. Finally, in appendix B we show an alternative time-domain formulation of the idea that underlies the BG method.

2 FOCUSING AS AN OPTIMIZATION PROBLEM

Consider an acoustic medium with known velocity and \( N \) receiver stations at distinct locations \( x_i \), wherein each station records signals for \( T \) seconds for times \( 0 < t < T \). Suppose that at some time \( \tau \in [0, T] \) an impulsive point source at location \( x = \xi \) goes off and generates an acoustic wavefield that is eventually sampled by the \( N \) receivers as \( d_i(t) \) for \( i = 1, 2, ..., N \).

In the time-reversal method, we focus acoustic energy at \( x = \xi \) and at \( t = T - \tau \) by rebroadcasting the shifted time-reversed signals \( d_i(T - t) \) at each station. However, because of the incomplete acquisition geometry of the receivers, the focus created by this TR process is suboptimal.

Our objective is to design signals such that, upon transmission, focus optimally at the target location \( \xi \) and at the time \( T - \tau \). To achieve the best spatio-temporal focus, each station must work in concert with the others by injecting a signal that is tailored in amplitude and shape according to the medium properties, location of the focusing target, and the geometry of the stations.

If we denote the signal injected by the station at \( x_i \) as \( a_i(t) \), then the superposed acoustic scalar wavefield recorded at an arbitrary location \( x \) inside the medium is

\[
\phi(x, t) = a_i(t) * G(x, t; x_i, 0), \quad \forall i \in \{1, 2, ..., N\},
\]

where we have used the Einstein’s notational convention for summation over repeated indices, \( * \) denotes the convolution operator, and \( G(x, t; x_i, 0) \) is the Green function (impulse response) recorded at location \( x \) and corresponding to an impulsive source at time \( t = 0 \) and location \( x_i \). Note that if the medium is known, these Green functions can be computed.

Convolution in the time domain corresponds to multiplication in the frequency domain. Therefore, considering the problem in the frequency domain, using the Fourier convention \( f(x, \omega) = \int f(x, t) \exp(i\omega t) \, dt \), each frequency component of the wavefield \( \phi(x, t) \) in equation 1 can be restated as a weighted sum of the corresponding frequency component \( G(x; x_i, \omega) \) of the Green functions, i.e.,

\[
\phi(x, \omega) = a_i(\omega) G(x; x_i, \omega), \quad \forall i \in \{1, 2, ..., N\},
\]

where the weights \( a_i(\omega) \) are the Fourier components of the signal that must be injected by the station at \( x_i \).

The problem can now be restated as how to optimally determine the unknown weights \( a_i(\omega) \) in equation 2 so that the superposed field \( \phi \) in the time domain focuses at a desired location of focus \( \xi \) and at a desired time of focus \( T - \tau \). Put another way, the goal is to have \( \phi(x, t) \) as close as possible to \( \delta(x - \xi)\delta(t - T + \tau) \), where \( \delta \) denotes the Dirac delta function. In the frequency domain, this goal can be achieved if we let each frequency component \( \phi(x, \omega) \) approach \( e^{i\omega(T - \tau)} \delta(x - \xi) \) by minimizing an objective function defined as

\[
J = \int_W |\phi(x, \omega) - e^{i\omega(T - \tau)} \delta(x - \xi)|^2 \, dx,
\]

where \( W \) denotes a subset of the medium that contains the target location (Figure 2).

Inserting equation 2 in objective function 3 and minimizing with respect to \( a_i(\omega) \) results in a linear system of equations of the form

\[
\Gamma(\omega)a(\omega) = e^{i\omega(T - \tau)} \mathbf{g}^*(\omega),
\]

where \( \Gamma \) is an \( N \times N \) Gram matrix (Parker, 1994) defined as

**Figure 2.** Acoustic velocity model and the configuration of the numerical experiments of section 4. The white diamonds show the location of the stations, the white square \( W \) depicts a small subset of the medium that contains the focusing target represented by the white circle.
by its elements as
\[ \Gamma_{ij}(\omega) = \int_W G(x; x_i, \omega) G^*(x; x_j, \omega) \, dx, \quad \forall i, j \in \{1, 2, ..., N\}, \]  
(5)

\( a(\omega) \) is an \( N \times 1 \) vector that is the unknown of the equation, \( g \) is an \( N \times 1 \) vector with elements
\[ g_i(\omega) = G(\xi; x_i; \omega), \quad \forall i \in \{1, 2, ..., N\}, \]  
(6)

and the superscript * denotes complex conjugation. The linear system in \( \mathbb{R}^N \) can be solved for the vector \( a(\omega) \) for each frequency separately. These \( a(\omega) \) vectors constitute the Fourier coefficients for the signals that the stations must transmit to obtain an optimal focus at location \( \xi \) and time \( T - \tau \).

The condition that is implied by minimizing the objective function \( 3 \) is known as the deltares criterion in the context of the method of Backus and Gilbert (BG) in inverse theory [Backus and Gilbert 1968]. A useful description of this method is provided by Aki and Richards [1980] and Aster et al. [2012]. Hence, we adopt the term Backus-Gilbert focusing for the method described above for designing optimal signals for wave focusing.

### 3 CONNECTION WITH TIME-REVERSAL AND DECONVOLUTION

The Backus and Gilbert focusing method (BG), introduced in section 2, provides a more general solution to the wave-focusing problem compared to the other focusing techniques such as TR. In fact, there is a close mathematical relationship between these methods. To see the connection between BG and TR, let us replace \( \Gamma \) in equation 3 by the identity matrix \( I \) to get
\[ a(\omega) = e^{i\omega(T-\tau)} g^* (\omega). \]  
(7)

The complex conjugation and multiplication of \( g \) in equation 3 by \( e^{i\omega(T-\tau)} \) for all frequencies amounts to in-place time-reversal of the corresponding signals \( G(\xi; t - \tau; x_i, 0) \) in the time domain. In other words, using the identity matrix as the crudest approximation for \( \Gamma \) results in the new system of equations 7 which describes exactly the same process as time-reversal in the time domain. Replacing \( \Gamma \) by the identity matrix amounts to ignoring the cross-talk between stations and having each station work independently to inject the time-reversed Green functions. Therefore, we might say that TR is a special cases of the more-general BG with a gross approximation of \( \Gamma \) as identity matrix.

In a similar way, we can show that BG is related to the deconvolution method (DC) presented by Ulrich et al. [2012]. To show this relationship, let us set the off-diagonal elements of \( \Gamma \) equal to zero (\( \Gamma_{ij} = 0 \) for \( i \neq j \)). Solving the system of equations 4 for \( a \) gives
\[ a_i(\omega) = \frac{e^{i\omega(T-\tau)} G^*(\xi; x_i, \omega)}{\int_W G(x; x_i, \omega) G^*(x; x_i, \omega) \, dx}, \quad \forall i \in \{1, 2, ..., N\}. \]  
(8)

In DC, the same frequency components of the signals to be back-propagated for focusing are computed as
\[ a_i(\omega) = \frac{e^{i\omega(T-\tau)} G^*(\xi; x_i, \omega) e^\epsilon}{G(\xi; x_i, \omega) G^*(\xi; x_i, \omega) + e}, \]  
(9)

where \( \epsilon \) is a regularization term, on the order of \( |G(\xi; x_i, \omega)|^2 \), added for stability of the solution. Notice the similarity between equations 8 and 9: the numerators on the right hand side of both equations are the same, and the denominators are similar except for the integration over the spatial element in equation 8 and the regularization term \( \epsilon \) in equation 9. Therefore, keeping only the diagonal elements of \( \Gamma \) reduces BG to a method very similar to DC.

It is important to note that all elements of the Gram matrix \( \Gamma \), not just the diagonal elements, hold crucial information about the configuration of the wave-focusing experiment, i.e., the relative positions of the stations with respect to the propagation medium and the focusing target. Each element of \( \Gamma \) plays a role in determining how the stations must work together to inject the signals that achieve the optimum focusing at
the target. The function of the off-diagonal elements of $\Gamma$ is to adjust the signal emitted by each station with respect to the signal of the other stations for the optimum focusing. Whereas in TR these off-diagonal elements are ignored by the crude approximation $\Gamma = I$ and therefore using the full $\Gamma$ in BG can improve the TR focusing result.

4 APPLICATION IN SOURCE IMAGING

Since the beginning of modern seismology, understanding earthquake sources has been a focus of research. More recently, exploration geophysicists have become interested in studying the source of the micro-earthquakes that are generated during hydraulic fracturing of rocks (Rentsch et al., 2007). Techniques based on time-reversal focusing are now routinely used for seismic event location and source imaging (Larmat et al., 2010). The effectiveness of such techniques is, therefore, constrained by the same limitations (e.g., imperfect acquisition) that bound the efficacy of TR focusing. This means that the BG methodology proposed here to enhance TR focusing can be useful in source-imaging applications.

In a source-imaging problem, we would like to focus the energy of the wavefield, that is sampled (often sparsely and incompletely) as data by a limited number of receivers, back to its origin. When source imaging is viewed as a focusing problem, the location of the source is considered as the focusing target, and the activation time of the source is considered as the time of focus.

The formulation of BG presented in section 2 seems to require exact knowledge of the target location $\xi$ and time $\tau$. This means that in utilizing BG for source imaging, where the location of the source (focusing target $\xi$) and its activation time (focusing time $\tau$) are not known a priori, equation 4 may not be used directly. Nevertheless, as we show below, with a slight modification and reinterpretation of the BG formulation, explicit knowledge of the source location and time can be rendered unnecessary. In short, the BG method can be used in source imaging, because the required information about the source is encoded and implicitly available in the recorded data.

Let us begin by assuming that the source, that we intend to image, is an impulse $\delta(x - x_i) \delta(t - \tau)$. In this context, the data $d_i(t)$ recorded by a receiver at $x_i$ due to our impulsive source is the Green function $G(x_i, t; \xi, \tau)$. The reciprocity of the Green function allows for expressing this data in the frequency domain as

$$d_i(\omega) = e^{i\omega \tau} G(\xi; x_i, \omega).$$

Now, using 6 and 10, equation 4 can be rewritten as

$$\Gamma(\omega) a(\omega) = e^{i\omega \tau^T} d^*(\omega),$$

where $d(\omega)$ is an $N \times 1$ vector with elements defined by equation 10.

Note that the right hand side of this equation is now completely known. The significance of equation 11 is that it allows us to use the BG formalism for imaging an impulsive source with no knowledge of the source time and location. The actual location and time of the source can be found, eventually, by solving equation 11 for the optimized signals $a(\omega)$, injecting them by the receivers, and scanning the resulting wavefield for the source image. In other words, after solving equation 11 for the optimized signals, the procedure for finding the source time and location (imaging the source) would be exactly similar to the usual practice in time-reversal source imaging.

The argument above relied on our initial assumption that the source was impulsive. However, as shown in appendix A, this argument can be generalized to hold true for any arbitrary spatio-temporal acoustic source, meaning that equation 11 can be used regardless of the source being impulsive or not.

5 NUMERICAL EXPERIMENTS

Here we test the ideas presented above by performing two numerical source-imaging experiments. We first apply the BG method to image a point source and compare the result with the same image produced by other tech-
In the first experiment, to simulate the data \(d_i(t)\) we generate a source wavefield by injecting a Ricker wavelet with peak frequency of 64 Hz at time \(\tau = 50\) ms and location \(\xi = (400\) m, 400 m\) within \(W\). We sample this source wavefield by the receiver at \(x_i\) for \(T = 0.4\) s. After simulating the data, for the rest of the experiment, we pretend that we do not know the actual location and time of the source.

To form the Gram matrix \(\Gamma\) in equation 11, we require the Green functions \(G(x, t; x_i, 0)\), \(\forall i \in \{1, 2, ..., N\}\). We approximate these Green functions by injecting a band limited spike with frequencies between 5 Hz and 150 Hz at each receiver location \(x_i\), and propagate the wavefield for \(T = 0.4\) s. These wavefields are then Fourier transformed to the frequency domain and used in equation 11 to compute the elements of the \(8 \times 8\) complex matrix \(\Gamma\) independently for all frequencies within the bandwidth of the experiment. In computing the integral of equation 11 the oscillatory integrand is tapered near the edges of integration window \(W\) to avoid dominant contribution from the end points.

At this point, we can form the system of equations 11 for each frequency independently and solve the system for \(a_i(\omega)\), the Fourier coefficients of the optimized signals \(a_i(t)\). These optimized signals are then broadcast by the receivers to generate the optimal wavefield \(\phi(x, t)\) that will focus to create the image of the source at location \(\xi \in W\) and at time \(T - \tau\). As with TR, the last step is to scan the wavefield \(\phi(x, t)\) to detect and extract the source image. (The source image can be detected using, for example, its high energy.) After detecting the source image, the actual values of \(\xi\), \(\tau\), and also the spatio-temporal characteristics of the source can be inferred from that image.

Figures 3 and 4 summarize the results of the first experiment. Figure 3 shows the normalized optimal signals \(a_i(t)\) and the normalized time-reversed data \(d_i(t)\) for receivers 1 through 8. The signals in each columns have been normalized by dividing the amplitude of each sample by the maximum absolute value of the amplitude of all traces in the same column. The optimization process has produced signals (left column) that are different from their corresponding time-reversed data (right column) in both amplitude and shape. For example, the small amplitude events in the time-reversed traces (e.g., the energy encircled in green) correspond to the reflection energy that is reflected from the discontinuity at 500 m in Figure 2. Note how the same reflected events (e.g. the energy encircled in red) are amplified by BG in the optimally computed signals.

Figure 4 shows the exact source in the first experiment. More specifically, it depicts the portion of the source wavefield enclosed within \(W\) at the activation time of the point Ricker wavelet that was used to simulate the source. Figure 4 shows the image of this source produced by TR. Instead of a compact and round spot, the TR image of the source is distorted. This distortion is a consequence of the incomplete acquisition geometry of the receivers. For recovering a spot-like image of the point source, TR requires a balanced illumination of the target from all angles. However, in this TR experiment, target illumination is imbalanced and limited to the small angle subtended by the first and last receivers.

The source image produced by BG is shown in Figure 5. BG has outperformed TR in achieving a more compact and spot-like image of the point source. This improvement is mostly the result of augmentation of the poor target illumination by BG. Figure 5 shows the entire BG wavefield \(\phi(x, t)\) at the time of focus. (The
portion enclosed by W here is the same as Figure 3). The strong burst of energy denoted by the white arrow in Figure 3 is the result of propagating the amplified reflected events in the optimized signals shown in Figure 3. When the optimized signals are propagated, this strong burst of energy travels in advance and part of it, after bouncing off the reflector at $Z = 500$ m, illuminates the target from below. Of course, BG did not create this energy out of nowhere. The energy is also present in the TR experiment, but it is much weaker. All that was done by BG was to detect this weak energy and amplify it in order to balance the target illumination. Effectively, this is equivalent of using the reflector at $Z = 500$ m as an acoustic mirror in order to boost the illumination angles.

For the sake of generality, we also imaged the point source using using SW (Figure 4h) and DC (Figure 4i) . Although the results by SW and DC are a slight improvement over TR (Figure 4b) they are both significantly less localized in space than the BG image (Figure 4l).

In the second experiment, we keep the configuration of the source and receivers and velocity the same as in the first experiment. However, instead of a point source for simulating the data, we use a dipole with time dependence given by a Ricker wavelet. The remaining steps follow exactly the same as before. Unlike a point source that has an isotropic radiation pattern, a dipole’s radiation pattern is directional. Consequently, in a source imaging experiment with incomplete acquisition, the arrangement of the receivers relative to the nodal planes and lobes of the source radiation pattern becomes an important factor. For this reason, we repeat the second experiment for 4 different orientations of the source dipole, i.e., for $\theta = 0, \pi/4, \pi/2,$ and $3\pi/4$, where $\theta$ denotes the angle of the dipole orientation measured with respect to the horizontal.

Figure 6 summarizes the outcome of these experiments. The left column shows snapshots of the wavefield associated with dipole sources with different orientations (left column) and their corresponding BG images (middle column) and TR images (right column).

**Figure 6.** Comparison of BG ans TR in imaging a dipole as an example of a distributed source with anisotropic radiation pattern. Snapshots of the wavefield associated with dipole sources with different orientations (left column) and there corresponding BG images (middle column) and TR images (right column).

Knowledge of the medium (e.g. velocity and density) is the most significant assumption made in formulating the BG method. This information is needed for computing the Green function for each receiver location. These Green functions are in turn used for computing the $\Gamma$ matrix.

As in any imaging method, the accuracy of the velocity model is very important in BG. Using an inaccurate velocity model can generally cause the image of the source to be created at a wrong time or location. More specifically, in situations like the examples of section 5, the BG method relies on amplifying the weak reflection energy in data in order to mitigate the adverse effects of incomplete acquisition. In such situations, for BG to effectively balance the illumination, the velocity model must be accurate and contain the reflectors that are associated with the main reflection energy in the data. Otherwise, BG will not be as effective as it is portrayed by examples of section 5 and, depending on the fidelity of the velocity model in predicting the reflections and how far it is from the true velocity model, the BG image of the source will be degraded and approach that of TR. (Recall that TR corresponds to the crude approximation $\Gamma \approx I$ and therefore is less sensitive to the inaccuracies in the velocity model.)

Apart from the velocity model, we also need an estimate of the source location so that we can be certain that the source is somewhere within a limited area $W$. In practice, such an estimate is usually available or
the optimum focusing is achieved when the size of focusing results. Based on our tests (not shown here) is a lower limit on how small certainty in the location of the source) is, the easier it is based the formulation of the BG method. With a limited the definition of the objective function upon which we imaging method. This is because is an important matter that needs careful attention in applying BG in source imaging. In our numerical examples of section we did not include noise. Applying the BG method for source imaging hinges upon the validity of equation which allows us to form and solve the system of equations without explicit knowledge of the source location, time, and spatio-temporal character. In a real source imaging scenario, data is always contaminated with noise and, therefore, equation must be modified as

\[ \Gamma(\omega) a(\omega) \approx e^{i\omega T} (d^r(\omega) + \eta^r(\omega)), \quad (12) \]

with \( \eta \) denoting the noise vector and \( d \) the noise free data described by equation

The stability of the solution \( a \) to equation depends on the condition number of the matrix \( \Gamma(\omega) \), which itself depends on the configuration of the receivers, the properties of the medium, and the frequency for which the \( \Gamma(\omega) \) is being computed. If \( \Gamma(\omega) \) is ill-conditioned, then some regularization technique, e.g., truncated SVD can be used to find a stable solution to equation

To test the robustness of the BG method in the presence of noise, and also to demonstrate the effect of additive noise to data, we repeated the first numerical source imaging experiment in section for data contaminated with band-limited Gaussian random noise with the same bandwidth as the data and signal-to-noise ratio of 1. The result of this test is shown in Figure 4. For this geometry, \( \Gamma(\omega) \) is not well-conditioned for low frequencies. This is because for lower frequencies (long wavelengths) the Green functions computed for adjacent receivers become similar with the result of \( \Gamma(\omega) \) becoming more singular. Therefore, to find a well-behaved solution, we regularized the system using truncated singular value decomposition (TSVD) for low frequencies.

The formulation of the BG focusing presented in this paper has so far been in the frequency domain. However, it is possible to formulate the same optimization idea in the time domain as well. The details of the time-domain formulation are shown in appendix B Solving the BG optimization problem in the time domain involves an iterative solution, where each iteration includes two wave propagation steps (see equation (12)). In this respect, time-domain Backus-Gilbert focusing (BGT) resembles least-squares migration (Nemeth et al., 1999). Considering that this iterative process must be repeated for imaging each source, implementing the BG method in the time domain could be expensive.

Solving the BG optimization in the time domain or in the frequency domain are fundamentally the same problems; in both, we minimize the same objective function. This is a consequence of the Parseval’s theorem (compare objective functions and). To illustrate this point, we repeated the first numerical source imaging experiment in section for imaging a point source with BGT. Figures (a), (b), and (c) show the result of BGT after 2, 10, and 20 iterations of the conjugated-gradient method, respectively. In early iterations (Figure (a)) the BGT focus is comparable to the TR focus (Figure (a)). However, by increasing the number of conjugate-gradient iterations, the resolution of the source image produced by BGT improves and after 20 iterations, the BGT focus (Figure (c)) approaches that of frequency-domain BG (Figure (c)).

Minor differences between the time- and frequency-domain BG focusing results must be attributed to the differences in numerical implementations of the two methods. While in the time-domain, we solve the BG optimization problem using an iterative scheme, in the frequency-domain, the BG optimization problem is solved directly by inverting the associated linear systems for each frequency. For this reason, adding noise to data can have a different effect on the frequency-domain BG than on the time-domain BG and depending on the method used for regularizing the inverse problem and the noise level, the difference between the results obtained by time-domain and frequency-domain BG can increase.

In the frequency-domain formulation, computing the matrix \( \Gamma \) requires simulating the wave propagation several times to model the Green function for each receiver location. However, a nice feature of the frequency-domain implementation is that for a fixed configuration of the receivers, \( \Gamma \) and its inverse have to be computed only once. (Recall that \( \Gamma \) on the left hand side of equation depends only on the receiver geometry and the properties of the medium and the right hand side depends only on the recorded data.) Therefore, once \( \Gamma^{-1} \) is computed and stored, we can reuse it to solve equation whenever a new source occurs. This means that for receivers with a fixed geometry, applying the BG in the frequency domain to image multiple sources can be very fast, inexpensive, and computationally efficient. However, note that in a case where the number of receivers are too large, and the number of sources to be imaged are small, it might be preferable to solve the BG problem in the time domain.
7 CONCLUSION
The ability of time-reversal (TR) methods to focus waves inside heterogeneous media is bounded by limitations such as attenuation, the diffraction limit, and imperfect acquisition. To go beyond the limitations caused by imperfect acquisition, we formulate wave focusing as an optimization problem. This technique, called the Backus-Gilbert (BG) method, can be easily applied in source imaging to obtain optimal images of an acoustic source.

The only requirements are accurate knowledge of the medium, and an estimate of the source location. The method makes no particular assumptions about the spatio-temporal character of the source. All the information that the method needs about the source is encoded in the recorded data.

Our numerical tests show that the Backus-Gilbert (BG) approach is capable of achieving a significantly better resolved image of the source compared to that achieved by TR. The advantage of BG over TR is most pronounced in source imaging experiments with limited aperture coverage and severely incomplete acquisition geometry. In fact, the aim of the method is mitigating the adverse effects of imperfect acquisition, and we emphasise that BG can not beat the diffraction limit as it ultimately relies on propagating band-limited signals from a limited number of stations to achieve an optimal focus.

The BG method can find applications in geophysics in both global and exploration scales. In global seismology, techniques based on TR focusing are routinely used to image earthquake sources (Larmat10). In exploration scale, geophysicists have interest in imaging and understanding the source mechanism of the micro earthquakes generated in hydraulic fracturing experiments. Often, in both earthquake source imaging and microseismic monitoring, the number and distribution of receivers is not enough to sample the wavefield properly. For example, in global seismology, most seismometer stations are on the continents and in microseismic surveys, the receivers are installed in one or two monitoring wells. In examples like these, TR can not perform optimally and BG could be a good candidate to compensate for the incomplete acquisition and poor aperture coverage.

The earth is, however, an elastic medium and therefore, to be able to apply BG in geophysical source imaging applications, we need to develop an equivalent elastic version of the BG method presented in this paper. We deal with this problem specifically in report CWP-792.

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REFERENCES


APPENDIX A: OPTIMIZED IMAGING OF AN ARBITRARY ACOUSTIC SOURCE

In section 4, we discussed application of our method for imaging an impulsive source at an unknown time and location. Here, we show that this method is equally applicable for imaging any source function (not just impulsive) with an arbitrary spatio-temporal characteristic.

Let \( s(x, t) \) denote the spatio-temporal source function defined over all space and time such that it can be nonzero only for \( x \in W \) and \( t \in [0, T] \). We can think of the source function as a succession of spatially impulsive sources that are applied with strength \( s(\xi, t) \) at each location \( \xi \) and write

\[
 s(x, t) = \int s(\xi, t) \delta(x - \xi) \, d\xi. \tag{A1}
\]

The data recorded by a station at \( x_i \) associated with this distribution of impulsive sources can then be written as

\[
 d_i(t) = \int s(\xi, t) * G(x_i, t; \xi, 0) \, d\xi, \tag{A2}
\]

where \( G(x_i, t; \xi, \tau) \) is the Green function with the source at \( x = \xi \). In the frequency domain, equation (A2) becomes

\[
 d_i(\omega) = \int s(\xi, \omega) G(x_i; \xi, \omega) \, d\xi. \tag{A3}
\]

Having defined the source function and the data associated with it, we can now lay out the BG optimization problem.

We assume that the scalar source wavefield at the time of focus is proportional to the source function \( s(x, t) \), with unity as the proportionality constant. With this assumption, we define our goal as finding signals \( a_i(t) \) such that the difference between the wavefield \( \phi(x, t) = a_i(t) * G(x, t; x, 0) \) and the time-reversed source function \( s(x, T - t) \) is minimum.

We achieve this in by minimizing an objective function defined as

\[
 J = \int \int_W |\phi(x, t) - s(x, T - t)|^2 \, dx \, dt, \tag{A4}
\]
which can be expressed in the frequency domain as
\[ J(\omega) = \int_W |a_i(\omega) G_i(x; x_i, \omega) - a_j(\omega) G_j(x; x_j, \omega)|^2 dx \]
for each frequency. Minimization \( A5 \) with respect to \( a_i(\omega) \) gives
\[ a_j(\omega) \int_W G(x; x_i, \omega) G^*(x; x_j, \omega) dx = e^{i\omega T} \]  
which is the frequency component of the data recorded by station at \( x_i \).

\[ s^*(\xi, \omega) = \phi(\xi, \omega) \]
where in the last step we have used the reciprocity principle for the acoustic Green function. Using equation \( A3 \), equation \( A6 \) can be written as
\[ a_j(\omega) \int_W G(x; x_i, \omega) G^*(x; x_j, \omega) dx = e^{i\omega T} d^*_j(\omega) \]
where \( \Gamma(\omega) a_i(\omega) = e^{i\omega T} d^*_i(\omega) \),

\[ (B1) \]

\[ \Gamma(\omega) a_i(\omega) = e^{i\omega T} d^*_i(\omega) \]
where \( \Gamma \) is an \( N \times N \) matrix with the same definition as given by equation \( B4 \) in section \( 2 \) and \( d(\omega) \) is an \( N \times 1 \) vector with elements defined as \( A3 \) which is the frequency component of the data recorded by station at \( x_i \).

\section*{APPENDIX B: TIME-DOMAIN FORMULATION OF THE BG METHOD}

In section \( 2 \) we formulated the BG focusing method in the frequency domain. Here, we derive the time-domain formulation of the same idea.

Let \( s(x, t) \) denote the source function defined over \( x \in W \) and \( t \in [0, T] \). Where \( W \) and \( T \), as defined in section \( 2 \) represent the optimization window and the maximum recording time, respectively. We assume that the scalar source wavefield at the time of focus is proportional to the source function \( s(x, t) \), with unity as the proportionality constant. With this assumption, the goal of the optimization is to find signals \( a_i(t) \), the signals the must be injected by a station at location \( x_i \), such that the difference between the wavefield \( \phi(x, t) = a_i(t) \ast G(x, t; x_i, 0) \) and the shifted time-reversed source function \( s(x, T - t) \) is minimum. We achieve this goal by minimizing an objective function defined as
\[ J = \int_W \int_W |\phi(x, t) - s(x, T - t)|^2 dx dt. \]
Substituting
\[ \phi(x, t) = \int dt' a_i(t') G(x, t - t'; x_i, 0) \]
into \( B1 \) and minimizing \( J \) with respect to \( a_i(t_m) \) for some \( m \) and \( n \) by setting \( \partial J/\partial a_i(t_m) = 0 \) gives
\[ \int dt \int_W dx [G(x, t - t_m; x_n, 0)] \]
\[ \int dt' G(x, t - t'; x_i, 0) a_i(t') \]
\[ = \int dt \int_W dx G(x, t - t_m; x_n, 0) s(x, T - t). \]
Replacing \( t \) with \( \tau \), \( i \) with \( j \), \( n \) with \( i \), and \( t_m \) with \( t \), equation \( B3 \) becomes
\[ \int dt \int_W dx [G(x, \tau - t; x_i, 0)] \]
\[ \int dt' G(x, \tau - t'; x_j, 0) a_j(t')] \]
\[ = \int dt \int_W dx G(x, \tau - t; x_i, 0) s(x, T - \tau) \]
\[ = \int dt \int_W dx G(x, \tau - t; x_i, 0) s(x, T - t), \]
where in the last step, we used of the reciprocity of the Green’s function, \( G(x, t; x_i, 0) = G(x_i, t; x, 0) \). This equation can be further simplified if we define a new function
\[ s'(x, t) = s(x, T - t), \]
and substitute it in equation \( B4 \) to get
\[ \int dt \int_W dx [G(x_i, \tau - t; x, 0)] \]
\[ \int dt' G(x, \tau - t'; x_j, 0) a_j(t')] \]
\[ = \int dt \int_W dx G(x_i, \tau - t; x, 0) s'(x, t). \]
Now, we define the propagation operator \( L \) such that
\[ L f(t) = \int G(x, t; x_i, 0) f_j(\tau) d\tau, \]
and the adjoint propagator \( L^\dagger \) such that
\[ L^\dagger \theta(x, t) = \int G(x_i, T - t; x, 0) \theta(\tau) d\tau, \]
where \( f(t) \) is an arbitrary vector function of time with components \( f_i(t) \), and \( \theta(x, t) \) is an arbitrary space-time function. The adjointness of \( L \) and \( L^\dagger \) defined above can be verified by showing
\[ < L^\dagger \theta(x, t) , \theta(x, t) > = < f(t) , L^\dagger \theta(x, t) >, \]
where the 1D and 2D inner products on the left and right hand side of the identity \( B9 \) are defined as
\[ < f(t) , h(t) > = \int f(t) h(t) dt, \]
and
\[ < \theta(x, t) , \psi(x, t) > = \int_W \theta(x, t) \psi(x, t) dx dt, \]
respectively.

Finally, using eqs. \( B5 \) and \( B7 \) and \( B8 \), equation \( B6 \) can be compactly written in the form of a normal equation

\[
L^\dagger L \mathbf{a}(t) = L^\dagger s(\mathbf{x}, T - t). \quad \text{(B12)}
\]

Even though the source function \( s(\mathbf{x}, t) \) in the right hand side of equation \( B12 \) is not known, \( L^\dagger s(\mathbf{x}, T - t) \) is known as it is approximately equal to the data recorded in the field, i.e.,

\[
L^\dagger s(\mathbf{x}, T - t) \approx \mathbf{d}(T - t), \quad \text{(B13)}
\]

where \( B13 \) is not an exact identity because in reality the recorded data is contaminated with noise. Now, if we approximate the first gradient \( L^\dagger s(\mathbf{x}, T - t) \) with the recorded data, then the normal equation \( B12 \) can be iteratively solved using conjugate gradient. Our numerical experiments (not shown in this report) show that using data contaminated with random or coherent noise as an estimate of the first gradient in the conjugate gradient scheme is possible and conjugate gradient can yield a solution to \( B12 \) after 20 to 30 iterations. After solving \( B12 \) for \( \mathbf{a}(t) \), to get the source image, we must inject and propagate the optimally computed signals \( \mathbf{a}(t) \) and then scan the resulting wavefield for the source image.
ABSTRACT

Time-reversal (TR) methods provide a simple and robust solution to source imaging problems. However, for recovering a well-resolved image of the source, TR requires a balanced illumination of the target from all angles. When acquisition is incomplete and a balanced illumination is not possible, the TR solution may not be adequate. In a previous paper, by formulating source imaging as an optimization problem, we presented a method named Backus-Gilbert focusing (BG) to enhance the performance of time-reversal focusing in acoustic media despite incomplete acquisition. Here, we generalize the theory of Backus-Gilbert focusing for application in elastic media.

Key words: elastic waves, source imaging, focusing, optimization, time-reversal, microseismic

1 INTRODUCTION

Since the beginning of modern seismology, understanding earthquake sources has been a focus of research. Source parameters such as time, location, focal mechanism, and rupture process provide invaluable insight for understanding fault systems and earthquakes. Exploration seismologists study microseismic events to acquire information about the inducing mechanism and the size and orientation of fractures in a hydrofracturing experiment (Rentsch et al., 2007; Baig and Urbancic, 2010; Maxwell and Urbancic, 2001; Eaton and Forouhildeh, 2011).

Conventional methods for studying seismic sources rely on the kinematic information in seismic data to invert for the source parameters (Stein and Wysession, 2003). More recently, inversion methods based on full-waveform data aim at solving for the moment tensors that completely describe the focal mechanism of the seismic source (Jost and Herrmann, 1989; Baig and Urbancic, 2010; Song and Toksöz, 2011). The stability of these full waveform inversion methods is limited by the acquisition geometry and the level of noise in data (Song and Toksöz, 2011; Eaton and Forouhildeh, 2011). In particular, the solid angle subtended by the receiver array, as viewed from the source location, plays a fundamental role in the stability of the inversion; the smaller solid angles imply less stability (Eaton and Forouhildeh, 2011).

An alternative approach to studying seismic sources is through direct imaging of the source using TR focusing methods, which are based on TR invariance of the wave operator (Fink, 1997, 2006, 2008; Larmat et al., 2006). TR based techniques are routinely used for seismic event location and source imaging (Larmat et al., 2010). The effectiveness of TR source imaging techniques is also reduced by incomplete acquisition because to resolve the source, TR requires a well-balanced illumination of the target.

To compensate for the adverse effects of incomplete acquisition on TR, Bazargani et al. (2014) presented a new method, named Backus-Gilbert focusing, in which source imaging in acoustic media is formulated as an optimization problem. The only prerequisites of the method are knowledge of the acoustic medium and an estimate of the source location. In this paper, we generalize the theory of the BG focusing for elastic media.

The organization of the paper is as follows: We first lay out the theoretical foundation of our new approach to wave focusing and show how to design waveforms to optimally image a point source with unknown parameters using sparse data (section 2). Next, in section 3, we discuss how BG is connected to TR focusing and show that TR can be regarded as a special case of the more general solution to wave focusing that BG provides. Section 4 is devoted to a numerical example where we apply the elastic BG method in imaging a double-couple point source. In section 5, we discuss various aspects of the BG method and elaborate further on some explicit and implicit assumptions that are used in the construction of the new approach and their significance. In appendix A, we show that the theory of the method presented...
2 OPTIMIZED IMAGING OF A POINT SOURCE

2.1 Notation

Before delving into the details of the BG optimization for elastic source imaging, we define notational conventions that are used throughout this paper.

1. We use Einstein’s notational convention for repeated indices: whenever an index (a subscript or a superscript) is repeated, summation over that index is implied.
2. All superscripts are associated with the receivers and take any integer value between 1 and $N$.
3. All subscripts denote spatial components.
4. Fourier transforms follow the convention $W_n = \sum_{n} \psi_n(x,t) e^{i \omega t}$, for each component $x_i$. (In practice, to determine the extent of $W$, a rough estimate of the source location $\xi$ is required.)
5. As a superscript, the asterisk * denotes complex conjugation, otherwise, it represents time convolution of two functions.

2.2 Formulation

Consider an elastic medium in which a single point source radiates seismic energy from an unknown location $\xi \in W$, where $W$ represents a subset of the medium that contains the source. (In practice, to determine the extent of $W$, a rough estimate of the source location $\xi$ is required.) We also assume that the source mechanism of the point source is described by an unknown moment tensor $M(t)$ with an unknown time dependence.

Suppose that we sample the source displacement field $u(x,t)$ by $N$ multicomponent receivers at locations $x^i$. This is a passive experiment in the sense that all receivers start recording at time $t = 0$ and stop at $t = T$ such that $T$ is large enough to allow adequate sampling of the source displacement field $u(x,t)$.

Let us denote the $n$th component of the data vector $d(t)$ recorded by the station at $x^i$ by $d_n^i$. As shown by Aki and Richards [2002], this data can be expressed as

$$d_n^i(t) = u_n(x^i,t) = M_{pq}(t) \frac{\partial G_{np}(x^i,t;\xi,0)}{\partial x_q}.$$  \hspace{1cm} (1)

where $M_{pq}$ denotes the elements of the source moment tensor $M$, and $G_{np}$ represents the elements of the elastodynamic Green tensor $G$. In the frequency domain, equation 1 becomes

$$d_n^i(\omega) = M_{pq}(\omega) \frac{\partial G_{np}(x^i,t;\xi,\omega)}{\partial x_q}. \hspace{1cm} (2)$$

The body force equivalent of our point source can be written as [Aki and Richards 2002]

$$f_n(x,t) = -M_{np}(t) \frac{\partial u(x-\xi)}{\partial x_q}, \hspace{1cm} (3)$$

where $\delta$ represents the Dirac delta function. Having defined the configuration of the experiment, the data, and the body force equivalent of the point source, we are ready to formulate the optimization problem.

According to the time-reversal process, after recording the data $d(t)$ associated with a source, a time-reversed version of the source wavefield $u(x,T-t)$ can be reconstructed by broadcasting time-reversed data $d(T-t)$ from each receiver. This process works well when the acquisition geometry of the experiment is complete and allows for adequate sampling of the source wavefield. However, with incomplete acquisition, the TR process cannot properly reconstruct the time-reversed source displacement field. Therefore, the simple process of time-reversing the data and re-injecting them is suboptimal in such situations.

Our goal is to find signals $a_n^i(t)$, for each component of each receiver, such that upon injection and propagation from the receivers, the resulting displacement field $\psi(x,t)$ correctly reconstructs the time-reversed source displacement field $u(x,T-t)$. To accomplish this goal, we define an objective function

$$J = \int_W \int_W |\psi(x,t) - u(x,T-t)|^2 dx dt. \hspace{1cm} (4)$$

where $W$ denotes a subset of the medium that contains the source, $u$ is the source displacement field, and $\psi$ is the reconstructed displacement field with its components

$$\psi_n(x,t) = G_{np}(x,t;\xi,0) \ast a_p^i(t). \hspace{1cm} (5)$$

We show in Appendix B that in the near-source region the particle displacements are proportional to the body force equivalent of the source. This makes intuitive sense because if a force is applied somewhere in an elastic medium then the particles pushed by the force are expected to move in the same direction as the force. Therefore, we can write

$$u(x,t) = C f(x,t), \hspace{1cm} (6)$$

where $C$ denotes a proportionality constant with dimensions of [displacement]/[force]. Here, for simplicity, we assume $C = 1$.

Using equations 3, 5, and 6 the objective function
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The objective function \[ J(\omega) = \sum_{\nu} J^r(\omega) \]
can be concisely expressed as
\[
J(r, \omega) = \int_W |G_{rp}(\mathbf{x}, \mathbf{t}; \mathbf{x}', 0) + a_p^r(t)|^2 d\mathbf{x}.
\]
the reciprocity principle for the components
where in the last three steps of equation 9, we used inte-
\[
J = \sum_{\nu} \int_W |G_{rp}(\mathbf{x}, \mathbf{t}; \mathbf{x}', 0) + a_p^r(t)|^2 d\mathbf{x}.
\]

in which the \(3N \times 3N\) matrix \(\Gamma_{kl}\) consists of nine \(N \times N\) submatrices \(\Gamma_{kl}\) defined as
\[
\Gamma_{kl}^{ij} = \int_W G_{r_k}^{s_i} G_{r_l}^{s_j} d\mathbf{x},
\]
the \(3N \times 1\) vector \(\mathbf{a}\) contains three \(N \times 1\) subvectors corresponding to the three components of the optimized signals
\[
\mathbf{a}_k = \begin{pmatrix} a_k^1 \\ a_k^2 \\ \vdots \\ a_k^N \end{pmatrix},
\]
and the \(3N \times 1\) vector \(\mathbf{d}\) contains three \(N \times 1\) subvectors corresponding to the three components of the recorded data
\[
\mathbf{d}_k = \begin{pmatrix} d_k^1 \\ d_k^2 \\ \vdots \\ d_k^N \end{pmatrix}.
\]
Note that in equation 14 a summation is carried out over the index \(r\).

The significance of equation 12 is that \(\Gamma\) on the left hand side of equation 12 can be computed (based on equation 14) as long as the medium is known and an estimate of the source location is available. The right hand side of equation 12 depends only on the recorded data, meaning that all the information the method requires about the unknown source is available and encoded within the data. Therefore, equation 12 can be solved for \(\mathbf{a}(\omega)\).

Even if only a subset of the data components are available, the formalism presented above is still valid and applicable. This is because equation 13 can be readily modified to form a new system that corresponds to the available data components. As a simple example, equation 13 can be modified for application in 2D as
\[
\begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = e^{i\omega T} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}^\ast,
\]
where \(\Gamma_{kl}, \mathbf{a}_k\) and \(\mathbf{d}_k\) are defined by equations 14, 16 and 16 respectively. Similarly, equation 13 can be modified for 3D cases where only a subset of data (e.g., only vertical components) is available for all or some receivers.
After solving equation 12 and finding $a(\omega)$ for all frequencies, we can compute the optimal time-domain signals $a^*(t)$. To image the source, these optimized signals must be injected and propagated by the receivers and the resulting wavefield $\psi(x, t)$ must be scanned for the image of the source.

We showed in this section that we can approach the problem of imaging an unknown point source as an optimization problem. We started our argument by assuming that there is only one point source inside the medium. However, as we show in appendix B, this argument can be generalized to hold true for any arbitrary distributed source. This is because a distributed source can be regarded as a collection of point sources. In other words, equation 12 can be used to find the optimal signals for imaging any type of source regardless of it being a point source or not.

### 3 CONNECTION WITH TIME REVERSAL

The Backus-Gilbert focusing method (BG), introduced in section 2, provides a more general solution to the wave-focusing problem compared to TR. In fact, there is a close mathematical relationship between these methods. To be able to see the connection between BG and TR, let us approximate $\Gamma$ are now ready to study the relationship between BG and TR. Let us approximate $\Gamma$ in equation 12 by the identity matrix $I$ to get

$$a(\omega) = e^{i\omega T} d^*(\omega).$$

(18)

The complex conjugation and multiplication of $d(\omega)$ in equation 18 by $e^{i\omega T}$ for all frequencies amounts to replace time-reversal of the data $d(t)$ in the time domain. In other words, using the identity matrix as the most simplistic approximation for $\Gamma$ in equation 12 results in a new system of equations (as in equation 18) which describes exactly the same process as time-reversal. By replacing $\Gamma$ with the identity matrix, we turn off the function of the submatrices of $\Gamma$. This amounts to ignoring the cross-talk between the components of the stations and make them work independently from the other stations to inject the time-reversed data. Therefore, we might say that TR is a special cases of the more-general BG with a crude approximation of $\Gamma$ as the identity matrix.

### 4 NUMERICAL EXPERIMENT

We apply the elastic BG method in a 2D numerical experiment to image a point source with a double-couple mechanism and compare the result with the same image produced by TR. The configuration of the experiment is shown in Figure 1. The six diamonds represent the receivers, the white circle depicts the source location, and the white square shows the optimization window $W$, used in the definition of the objective function in equa-
The earth model used for the wave propagation is a heterogeneous elastic 2D model (Figure 1) consisting of three layers. Wave propagation is simulated using an explicit finite-difference approximation of the 2D elastic isotropic wave equation with an absorbing boundary condition.

The data $d_i(t)$ is generated by a horizontal point source of slip with a double-couple mechanism located at $\xi = (600 \, m, 600 \, m)$ within $W$. The time dependence of the source is given by a Ricker wavelet with peak frequency of $55 \, Hz$ and peak time $t = 100 \, ms$. We sample the displacement field generated by this source using receivers at $\mathbf{x}'$ for $T = 0.6 \, s$. After simulating the data, for the rest of the experiment, we pretend that we do not know the actual location and time of the source.

To form the Gram matrix $\Gamma$, in equation 17 we require the Green functions $G(\mathbf{x}, t; \mathbf{x}', 0)$. We approximate these Green functions by injecting a band-limited spike with frequencies between 2 $Hz$ and 140 $Hz$ at each receiver location $\mathbf{x}'$ and propagating the wavefield for $T = 0.6 \, s$. These wavefields are then Fourier transformed to the frequency domain and used in equation 17 to compute the elements of the $12 \times 12$ matrix $\Gamma$ independently for all frequencies within the bandwidth of the experiment. In computing the integral of equation 17 over $W$, the oscillatory integrand is tapered near the edges of integration window $W$ to avoid dominant contribution from the end points.

At this point, we can form the system of equations 17 for each frequency independently and solve the system for $a_i(\omega)$, the Fourier coefficients of the optimized signals $a_i(t)$. These optimized signals are then broadcast by the receivers to generate the optimal displacement field $\psi(\mathbf{x}, t)$ that will focus at the correct source location. Note that, in this 2D example, $\psi(\mathbf{x}, t)$ is a vector field with 2 components, a vertical and a horizontal component.

The last step is to scan this optimal wavefield to detect and extract the source image which has been created at location $\xi \in W$ and at the correct time associated with the source activation time. After detecting the source image, the actual source location, time, and the spatio-temporal characteristics of the source can be inferred from that image.

Possible criteria to detect the source image can include high energy or a particular type of radiation pattern which is expected for the source. For example, a seismic source with a double-couple mechanism has characteristic four-lobe radiation patterns for the radial and transverse displacement components (Aki and Richards 2002). The radial and transverse components of the displacement field can be readily computed as

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Vertical component of the optimal signals versus the timer-reversed vertical component of the data (left panel), and horizontal component of the optimal signals versus the timer-reversed horizontal component of the data (right panel). The weak reflection energies (green circles) are amplified in the optimize traces (red circles).}
\end{figure}
the curl and divergence of the displacement field, respectively.

Figures 2 and 3 summarize the results of the numerical experiment described above. Figure 2 consists of two panels. The left panel depicts the vertical component of the optimal signals in the column labeled BG, and the vertical component of the time-reversed data in the column labeled TR. Similarly, the right panel depicts the horizontal component of the optimal signals in the column labeled BG, and the horizontal component of the time-reversed data in the column labeled TR. The signals in each column (of both panels) have been normalized by dividing all samples by the maximum absolute value of the amplitude of all traces in the same column.

The optimization process has produced signals that are different from their corresponding time-reversed data in both amplitude and shape. For example, the small amplitude events in the time-reversed traces (e.g., the energy encircled in green) correspond to the reflection energy that is reflected from the discontinuity at $Z = 800$ m in Figure 1. Note how the same reflected events (e.g., the energy encircled in red) are amplified by BG in the optimally computed signals.

Figures 3a and 3b show the P- and S-wave snapshots of the wavefield generated by the horizontal dislocation source in this experiment. More specifically, Figures 3a and 3b depict the portion of the divergence (radial component) and curl (transverse component) of the source displacement field enclosed within $W$ at the activation time of the double-couple point source. We can think of Figure 3a as the exact P-wave image and of Figure 3b as the exact S-wave image of the source. The radial and transverse components of the optimally reconstructed source displacement field $\psi(x, t)$ at the time of focus are shown in Figure 3a and Figure 3b. These are the P- and S-wave images of the source obtained by taking the divergence and curl, respectively, of $\psi(x, t)$. Finally, the P- and S-wave images of the source produced by TR are depicted in Figures 3e and 3f, respectively.

The BG images (second row of Figure 3) are superior to the TR images (third row) in many aspects. For example, note how the four lobes of the P-wave radiation pattern are resolved by BG in Figure 3c, whereas it is impossible to recognize them in the corresponding TR image in Figure 3e. The four-lobe S-wave radiation pattern is not resolved by either BG or TR. Nevertheless, compared to the TR image (3f), the BG image (3d) resembles the exact S-wave image (3b) more closely in both size and orientation.

The improvement of the resolution in the BG images, can be attributed to a more balanced illumination of the target. This is made possible by using the amplified reflected events in the optimized signals shown in Figure 2. When the optimized signals are propagated, a strong coherent burst of energy is create by these amplified events. This coherent energy travels in advance and part of it, after bouncing off the reflector at $Z = 800$ m, illuminates the target from underneath. Of course, BG did not create this energy out of nowhere. The energy is also present in the TR experiment, but it is much weaker. All that is done by BG was to detect this weak energy and amplify it in order to balance the target illumination. Effectively, this is equivalent of using the reflector at $Z = 800$ m as an elastic mirror in order to augment the illumination angles, which in the case of TR are limited to the small angle subtended by the first and the last receivers.

5 DISCUSSION

In the formulation of the elastic BG method in section 2 several assumptions were made. The most important of these assumptions is knowledge of the elastic medium (e.g. P- and S-wave velocities and density). This is because an accurate model for the medium is vital for computing the Green tensor for each receiver location and the Green tensors are needed for computing the $\Gamma$ matrix.
Knowledge of the medium is essential to any source imaging method. In general, using an inaccurate velocity model degrades the image quality or causes the image of the source to be created at a wrong time or location. BG focusing is not an exception in this respect. For instance, in the example shown in section 4, BG relies on amplifying the weak reflection energy in data in order to compensate for the incomplete acquisition. In this case, to effectively balance the illumination, BG requires an accurate velocity model that contains the reflectors that are associated with the reflection energy recorded in the data. Otherwise, BG will not be as effective as it is portrayed by the example in section 4 and, depending on the fidelity of the velocity model in predicting the reflections and its accuracy, the BG image of the source may be degraded and approach that of TR. In short, because TR works based on the crude approximation $\Gamma \approx I$, when an accurate model is available, TR is not as effective as BG. However, TR is less sensitive to inaccuracies in the model and is more robust than BG in tolerating errors in the model.

Apart from the velocity model, we also need an estimate of the source location such that we can be certain that the source is somewhere within a limited area $W$. In practice, such an estimate is usually available. For instance, in earthquake seismology, the approximate location of the source is commonly calculated using common source location techniques that work based on kinematic information in data (Stein and Wyssession, 2003). Similarly, in hydraulic fracturing monitoring surveys, geophysicist often have a good estimate of the size of the affected area around the injection well where microseismic energy is most likely to originate. The size of $W$ is an important factor in determining the effectiveness of BG as a source imaging method. This is because $W$ is used directly in the definition of the objective function $\Gamma$ upon which we based the formulation of the BG method. With a limited number of receivers, the smaller $W$ (the smaller the uncertainty in the location of the source) is, the easier it is to minimize the BG objective function. However, there is a lower limit on how small $W$ can be to obtain best focusing results. Based on our tests (not shown here) the optimum focusing is achieved when the size of $W$ is 2 to 3 times the dominant wavelength in the data.

The presence of noise in recorded data is an important matter that needs careful attention when applying BG to source imaging. In our numerical example of section 4, we did not include noise. Applying the BG method for source imaging depends on the validity of equation (12) which allows us to solve for optimal signals without explicit knowledge of the source parameters. However, in a real source imaging scenario, equation (12) does not hold exactly because recorded data always contain noise. Therefore, equation (12) must be replaced by

$$\mathbf{\Gamma}(\omega) \mathbf{a}(\omega) \approx e^{i\omega T}(d^*(\omega) + \mathbf{n}^*(\omega)), \quad (19)$$

where $\mathbf{n}$ denotes the noise vector and $d$ the theoretical noise-free data described by equation (2).

The stability of the solution $\mathbf{a}$ in equation (19) depends on the condition number of $\mathbf{\Gamma}(\omega)$, which itself depends on the configuration of the receivers, the properties of the medium, and the frequency $\omega$ for which $\mathbf{\Gamma}(\omega)$ is computed. If $\mathbf{\Gamma}(\omega)$ is ill-conditioned, then some regularization technique, e.g., truncated singular value decomposition (TSVD) can be used to find a stable solution to equation (19).

Computing $\mathbf{\Gamma}$ requires simulating the wave propagation to model the Green tensor for each receiver location. However, for a fixed configuration of receivers, $\mathbf{\Gamma}$ and its inverse have to be computed only once. (Recall that $\mathbf{\Gamma}$ on the left hand side of equation (12) depends only on the receiver geometry and the properties of the medium and the right hand side depends only on the recorded data.) Therefore, once $\mathbf{\Gamma}^{-1}$ is computed and stored, we can reuse it to solve equation (12) whenever a new source occurs. This means that for receivers with a fixed geometry, using BG to image multiple sources can be computationally very efficient.

### 6 CONCLUSION

Time-reversal (TR) focusing methods rely on the time-reversal invariance of the wave operator. When acquisition is incomplete, source imaging techniques based on time-reversal focusing are not optimal. In this paper, to overcome the limitations imposed by incomplete acquisition, we approach source imaging in elastic media as an optimization problem. The new approach, named Backus-Gilbert focusing (BG), provides a more general solution to elastic wave focusing than TR does.

To apply BG in source imaging, the medium must be known and also an estimate of the source position must be available. Apart from these two requirements, no other a priori information is needed by the method. Of course, since the method is formulated as an inverse problem that relies on the knowledge of the medium, it is more sensitive than TR to inaccuracies in the model. We showed with a numerical example simulating an elastic source imaging experiment with sparse receiver geometry, that BG can produce better resolved images of an unknown source than TR. For source imaging using receiver stations with a fixed geometry, the application of the method can be computationally efficient.

The presence of noise in data is an important issue that needs to be addressed by applying proper regularization. A high level of noise in data, depending on the condition number of the inverse problem, can compromise the effectiveness of BG.

One area of geophysics that is a good candidate for
application of BG is imaging earthquake sources. This is because of three reasons: The first reason is that a relatively reliable elastic model of the earth is available in global seismology which can be used by BG. The second reason is that the advantages of BG over TR are most pronounced in experiments with sparse (incomplete) acquisition geometry. The global network of seismometers does not have a uniform distribution on the Earth and therefore earthquakes that occur in certain areas cannot be recorded properly. And the third reason is that the data quality in terms of signal-to-noise ratio for the strong earthquakes recorded in global seismology are often very good.

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REFERENCES


APPENDIX A: OPTIMIZED IMAGING OF AN EXTENDED ELASTIC SOURCE

In section 2 we formulated the BG optimization for imaging a point source. Here, we show that this formulation is equally applicable for imaging any source (not just point source) with arbitrary spatio-temporal characteristics.

We can model a distributed source as a succession of point sources that occur at locations $\mathbf{x}$, and with moment tensor $M(\mathbf{x}, t)$ defined for $\mathbf{x} \in W$ and $t \in [0, T]$, and write

$$M(\mathbf{x}, t) = \int M(\mathbf{x}, t) \delta(\mathbf{x} - \mathbf{y}) \, d\mathbf{y}. \quad (A1)$$

The $n^{th}$ component of the data vector $d(t)$ associated with the extended source and recorded by the station at $\mathbf{x}'$ can be expressed as

$$d_n(t) = \int M_{pq}(\mathbf{x}', t) \frac{\partial M_{pq}(\mathbf{x}', t; \mathbf{y}, 0)}{\partial x_q} \, d\mathbf{y}. \quad (A2)$$

where $M_{pq}(\mathbf{x}, t)$ denotes the elements of the moment tensor $M$ of the point source at $\mathbf{x}$, and $G_{np}$ represents the elements of the elastodynamic Green tensor $G$. In the frequency domain, equation $\text{(A2)}$ becomes

$$d_n(\mathbf{\omega}) = \int M_{pq}(\mathbf{x}, \mathbf{\omega}) \frac{\partial G_{np}(\mathbf{x}; \mathbf{\omega}, 0)}{\partial x_q} \, d\mathbf{\omega}. \quad (A3)$$

The body force equivalent of the distributed source can be written as $[\text{Aki and Richards 2002}]$

$$f_n(\mathbf{x}, t) = -\int M_{pq}(\mathbf{x}; \mathbf{\omega}) \frac{\partial \delta(\mathbf{x} - \mathbf{y})}{\partial x_q} \, d\mathbf{y}. \quad (A4)$$

where $\delta$ represents the Dirac delta function.

With $\text{A3}$ and $\text{A4}$ defining the data and the body force equivalent for the distributed source, we can repeat the same steps we took in section 2 to formulate the
Optimal wave focusing in elastic media

optimization problem and obtain
\[ a'_p(\omega) \int_W \nabla^i \nabla_j G_{ij}^* dx = \int e^{i\omega T} M_{pq}^{*} \frac{\partial \nabla_{ij}^* (x', \omega)}{\partial x_\xi} d\xi, \]  
(A5)

which can be simplified using (A3) as
\[ a'_p(\omega) \int_W \nabla^i \nabla_j G_{ij}^* dx = e^{i\omega T} d^s \]  
(A6)

Equation (A6) represents a system of equations that can be concisely expressed as
\[ \Gamma(\omega) a(\omega) = e^{i\omega T} d^s \]  
(A7)

which has the exact form as equation 12 we obtained for a point source in section 2.

APPENDIX B: PARTICLE MOTION NEAR SOURCE

In section 2, we assumed that in the near-source region the particle displacements are proportional to the source equivalent force. Here, we justify this assumption by studying the behaviour of the particle displacements in the region near a unidirectional point force.

Consider an elastic medium with a point force \( f(t) \) applied at the origin. The displacement field \( u(x, t) \) due to this point force is (Aki and Richards, 2002)
\[ u_i(x, t) = \frac{1}{4\pi \rho} \left( 3\gamma_i \gamma_j - \delta_{ij} \right) \frac{1}{r^3} \int_{r/\alpha}^{r/\beta} \tau f_j(t - \tau) d\tau 
+ \frac{1}{4\pi \rho \alpha^2} \gamma_i \gamma_j \frac{1}{r} f_j(t - r/\alpha) 
- \frac{1}{4\pi \rho \beta^2} (\gamma_i \gamma_j - \delta_{ij}) \frac{1}{r} f_j(t - r/\beta), \]  
(B1)

where \( r = |x| \) is the distance from the origin, \( \gamma_i = x_i/r \) are the direction cosines, \( \delta \) is the Dirac delta function, \( \rho \) is the density, \( \alpha \) is the \( P \)-wave velocity, and \( \beta \) is the \( S \)-wave velocity with \( \alpha > \beta \). Using (B1) we can calculate the limit
\[ \lim_{r \to 0} 4\pi \rho r u_i(x, t) = \frac{1}{2} \left( \frac{1}{\alpha^2} + \frac{1}{\beta^2} \right) \delta_{ij} f_j(t) 
+ \frac{1}{2} \left( \frac{1}{\beta^2} - \frac{1}{\alpha^2} \right) \gamma_i \gamma_j f_j(t). \]  
(B2)

The expression on the right hand side of (B2) is called the Somigliana tensor (Aki and Richards, 2002). In vector form, equation (B2) is given by
\[ \lim_{r \to 0} 4\pi \rho r u(x, t) = \frac{1}{2} \left( \frac{1}{\alpha^2} + \frac{1}{\beta^2} \right) f(t) 
+ \frac{1}{2} \left( \frac{1}{\beta^2} - \frac{1}{\alpha^2} \right) \dot{r} \cdot f(t), \]  
(B3)

where \( \dot{r} = \gamma_i \).

When \( \dot{r} \parallel f \), expression (B3) reduces to
\[ \lim_{r \to 0} 4\pi \rho r u(x, t) = \frac{1}{2} \left( \frac{1}{\alpha^2} + \frac{1}{\beta^2} \right) f(t), \]  
(B4)

and when \( \dot{r} \perp f(t) \), expression (B3) reduces to
\[ \lim_{r \to 0} 4\pi \rho r u(x, t) = \frac{1}{2} \left( \frac{1}{\alpha^2} + \frac{1}{\beta^2} \right) f(t). \]  
(B5)

In both situations the displacement at the source location is parallel to the excitation. This result is confirmed by Wu and Ben-Menahem (1985) who show (Figure 2 of their paper) the motion of the elastodynamic field in the near-source region of a unidirectional force.

Optimal wave focusing in elastic media
Improved temporal focusing at source location using deconvolution

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ABSTRACT

Time reversal techniques are used in ocean acoustics, medical imaging and non-destructive evaluation to backpropagate recorded signals to the source of origin. We demonstrate experimentally a technique which improves the temporal focus achieved at the source location. The experiment consists of propagating a signal from a transducer within a concrete block to a single receiver on the surface, and then applying time reversal or deconvolution to focus the energy back at the source location. The results show that we are able to generate a focus in time at the correct location. The proposed method is simple and proven to be robust. Additionally, its costs are negligible due to deconvolution being a preprocessing step to the recorded data. The technique can be applied for detailed investigation of the source mechanisms (e.g. cracks) but also for monitoring purposes.

Key words: Time Reversal, Deconvolution, Experimental, Focusing

1 INTRODUCTION

Several methods are used to evaluate acoustic signal generated by events in media such as water, rocks, metals or concrete. Most of them are summarized as acoustic emission methods (AE), mainly travel time evaluation methods to localize the point of origin. Sophisticated methods have been developed in seismology to localize and characterize earthquakes. Time Reversal (TR) has been the focus of much research in acoustics due to its ability to compress the measured scattered waveforms back at the source point in both space and time Parvulescu and Clay (1965); Fink (1997); Anderson et al. (2008); Larmat et al. (2010). This has led to TR being applied in a wide variety of fields such as medicine, communication or nondestructive evaluation. However, continued work is being done to improve TR ability to focus energy. Some newly developed techniques use an array of input transducers, measure the wave field with an array near the desired focal spot, and then optimize the spatial and temporal focusing Tanter et al. (2000, 2001); Montaldo et al. (2004); Vignon et al. (2006); Roux and Fink (2000); Aubry et al. (2001); Jonsson et al. (2004). Other methods use an array of input transducers and optimize the temporal focusing at an output transducer Daniels and Heath (2005); Qiu et al. (2006); Blomgren et al. (2008); Zhou et al. (2006); Zhou and Qiu (2006). However, these techniques can require/benefit from large arrays while simultaneously losing spatial localization in order to enhance temporal focusing.

In this paper, we design and execute an evaluation experiment to compare conventional time reversal to an improved variant which uses deconvolution (DC). We explore the application of DC, which is a primitive though robust version of the inverse filter Tanter et al. (2000); Gallot et al. (2011), to calculate the optimal signal for backpropagation. The experiment consists of a concrete block which has a source embedded within. Instead of using a large array of receivers, the experiment uses only a single receiver to record the scattered waveform. TR or DC is then applied to the measured scattered waveform to calculate the TR and DC signal. The calculated signals are then backpropagated from a transducer on the surface of the block into the medium and recorded at the original source location transducer. By this experiment, we are able to explore and compare the capabilities of TR and DC to focus the measured waveforms at a point in both space and time. We show that DC significantly improves the temporal focus compared with TR.
2 THEORY: DECONVOLUTION

Time reversal (TR) is a process used to compressed the measured scattered waveforms at a point in both space and time to ideally a dirac delta function $\delta(t)$. It uses the recorded impulse response which can be represented by a Green function $G_{AB}$ between two points $A$ and $B$. TR then simply reverses the signal in time and propagates it back from the receiver location into the same medium. By doing so, one expects the energy to focus at the source location. The TR process can be represented by the following equation,

$$
\int_{-\infty}^{\infty} G_{AB}(\tau) G_{AB}(\tau - t) d\tau = \delta(t),
$$

(1)

where reciprocity has been used to replace the Green function $G_{BA}$ with $G_{AB}$. According to equation 1, the TR process, which is the autocorrelation of $G_{AB}(t)$, should equal a delta function. However, in practice, one cannot truly recreate a dirac delta function focus due to one or more conditions, necessary to satisfy Eq. 1, not being upheld. In order for it to work perfectly, one must record for infinite time, Green’s functions are assumed to contain flat, infinite bandwidth, and one must have full coverage of the wavefield. These requirements are not upheld during an experiment. This led us to explore the application of deconvolution.

Going back to Eq. 1, we can rewrite this (using a convolution notation, rather than the integral form) as

$$
F(t) = g(t) \ast R(t) \approx \delta(t),
$$

(2)

where $F(t)$ is the focal signal or source reconstruction, $R(t)$ is the recorded signal measured at the receiver location $B$ from the initial source propagation, and $g(t)$ is the signal necessary to be back propagated for focusing. We are able to go from Eq. 1 to Eq. 2 because we only investigate signals between the two points $A$ and $B$, and remove the green function notation to indicate we do not have infinite bandwidth. Thus, we remove some of the unrealistic conditions that Eq. 1 required. For a TR process, the signal for backpropagation is purely the time reversed recorded signal $g(t) = R(-t)$. Our goal is to calculate the optimal signal $g(t)$ such that the focal signal $F(t)$ approximately equals a dirac delta function $\delta(t)$.

Deconvolution equates to inverse filtering by transforming to the frequency domain, thus Eq. (2) becomes

$$
F(\omega) = g(\omega) R(\omega) \approx 1,
$$

(3)

Equation (3) is used to solve for $g(\omega)$,

$$
g(\omega) = \frac{1}{R(\omega)} \frac{R^*(\omega)}{|R(\omega)|^2},
$$

(4)

where $*$ denotes a complex conjugate operation. This expression gives a mathematical expression for $g(\omega)$.

3 EXPERIMENTAL SET UP

The purpose of our study is to explore the application of the deconvolution variant compared to conventional time reversal in focusing energy at a source location within a sample. In order to make such comparison, we created the following experimental set up. A 30 by 30 by 37 cm$^3$ concrete block had been casted from 72 kg
Quickcrete mix (No. 1101, max aggregate grain size ≤ 4 mm), about 60 l water and 5 kg additional gravel (5-15 mm grain size). Three ultrasonic piezo transducers (proprietary type Acsys S0807, center frequency 60 kHz, ES in figure 1) were positioned inside the formwork in various orientations before concreting. The block contains only minimal reinforcement, mainly to hold the embedded sensors in place while pouring the concrete. Broadband Acsys point contact sensors type S1803 (center frequency about 100 kHz, PT in figure 1) were used as external transducers. They are piezo based as well and feature a spring loaded 2 mm diameter ceramic tip for contact to the concrete. They are sensitive to acceleration vertical to the concrete surface. The transmitter signal is generated by custom made rectangular signal generator/amplifier (BAM US in figure 1) triggered by a TTL impulse issued by our data acquisition device (National Instruments model 6366). Signals received by the external sensors are 1 kHz low pass filtered and amplified by a Stanford Research low noise preamplifier (SR 566) before being digitized and recorded. The workflow and set up used for acquiring the data is shown in Fig. 1(a). For backpropagation, the setup is reversed. The BAM US device is removed. The transmitter signal generated is replaced by the digital/analog converter integrated in the data acquisition device, sending the computed, time reversed/deconvolved waveforms to the external sensor. The embedded sensor is used as receiver, again using the preamplifier before AD conversion and recording. This reversed set up is shown in Fig. 1 (b).

We have used a sampling frequency of 2 MHz and 20000 samples per trace (10 ms recording time). A 4000 sample (2 ms) pretrigger interval was set. Amplitude resolution is 16 bit. True zero time of the transmitter could be identified by electromagnetic crosstalk between transmitter and receiver cables, generating a small but easy to recognize impulse in the receiver data.

Note, that due to a high noise lab environment and a lack of a power amplifier for the backpropagated transmitter signal we had to apply an additional 2 kHz low-pass Butterworth filter on all data.

4 DISCUSSION

Our experiments started with propagating a defined 60 kHz source function from the embedded source towards the external receiver. Note the pretrigger time present in recorded signal shown in Figure 2. This pretrigger time was proven by Ulrich et al Ulrich et al. (2013) to be necessary if one desires to use DC for any application where high signal fidelity is important or where multiple impulses may be focused successively with little separation. Additionally, it was shown by Ulrich et al Ulrich et al. (2013) that the acausal portion of the signal is necessary in order to create a symmetric focus.

Once our wave field was recorded at the single receiver, TR or DC was applied to calculate the back propagating signals. Applying these two methods, one achieves the results shown in Figure 3 where (a) highlights the backpropagating signal calculated using TR while (b) shows the DC back propagation signal. These two signals shown in Fig. 3 differ significantly in one feature: TR has zero values past approximately 8 ms while DC’s backpropagation signal is nonzero for this time range. This is due to the acausal nature of the recorded signal.

These signals were then backpropagated from the receiver location into the same medium where the receiver on the surface of the concrete block now acted as a source. Propagating these calculated TR and DC signals back into the medium, one would expect a focus at the source location. Figure 4 shows the focuses recorded at the source location by using the embedded transducer as a receiver during the backpropagation. Figure 4 (a) shows the temporal focus recorded using the TR calculated signal while Figure 4 (b) represents the temporal focus achieved using the DC calculated signal. The temporal focus achieved using TR has significant side lobes present away from the time of focus. However, the tem-
This "virtual" source can then be used to characterize the medium. For example, after convolution (DC) processes is to generate a signal such that one gets an improved temporal focus at the source location. This was proven to hold true for an experimental set up having the source embedded within a concrete block and using only a single receiver. The method does not require any information about the medium. It only needs the recorded signal in order to generate the optimal signal for backpropagation. Due to the simplicity, low cost, and robust nature of this method, one could easily implement this technique into pre-existing workflows. This method may then be used to characterize the medium through use of "virtual" sources, repeated backpropagation or backpropagating signals to investigate the changes occurring within the medium. Finally, using DC, one can define the type of source function focus that will occur at the event location. This was shown to work in Ulrich et al. (2013) Therefore, one could determine the frequency of the focused wave field and allow different frequency focuses to occur. One then records the scattered wave field generated by the virtual sources consisting of different frequencies to characterize the medium. Therefore, by using DC, we could potentially improve the characterization of the medium compared with TR. In addition, the amplitude can be varied to study nonlinear effects.

In order to proof the robust nature of this method, we reran the experiment using a different location for the receiver (placing the receiver on a different side of the block). The purpose of this experiment was to not only test and make sure DC still had an improved temporal focus compared to TR but to also demonstrate our decision for \( \gamma \) which was the regularization constant. For this set up, DC's temporal focus was 79 % while TR had a temporal focus of 47 %. We would not expect to see the exact same temporal focusing numbers as before because we changed the direction of displacement we record and the distance between the source and receiver. We reran the experiment numerous times, recalculating the DC signal using different values for \( \gamma \), and recording the temporal focus at the source location. The experiment showed that the optimal value to be \( \gamma = 0.9 \). However, even for different \( \gamma \) values, one still achieves some form of a temporal focus until the \( \gamma \) values becomes too small where too much noise gets included into the focus.

The purpose of using time reversal (TR) and deconvolution (DC) processes is to generate a signal such that it will focus at the source location. One can use this feature of the methods for a range of applications in order to characterize the medium. For example, after the time of focus, the wave field will propagate away from the source location with the characteristic as if it were generated by a source mechanism at the focused event location. This is defined as a "virtual" source. This "virtual" source can then be used to for a wide variety of applications from multiple suppression, to medium characterization Wapenaar et al. (2012); Belhura and Snieder (2013); Mehta et al. (2008a); Snieder et al. (2006); Mehta et al. (2008b). Additionally, one can continuously monitor and backpropagate signals to investigate the changes occurring within the medium. Finally, using DC, one can define the type of source function focus that will occur at the event location. This was shown to work in Ulrich et al Ulrich et al. (2013) Therefore, one could determine the frequency of the focused wave field and allow different frequency focuses to occur. One then records the scattered wave field generated by the virtual sources consisting of different frequencies to characterize the medium. Therefore, by using DC, we could potentially improve the characterization of the medium compared with TR. In addition, the amplitude can be varied to study nonlinear effects.

5 CONCLUSION

We have introduced a simple though robust method for determining the optimal signal for backpropagation such that one gets an improved temporal focus at the source location. This was proven to hold true for an experimental set up having the source embedded within a concrete block and using only a single receiver. The method does not require any information about the medium. It only needs the recorded signal in order to generate the optimal signal for backpropagation. Due to the simplicity, low cost, and robust nature of this method, one could easily implement this technique into pre-existing workflows. This method may then be used to characterize the medium through use of "virtual" sources, repeated backpropagation or backpropagating with different source function focuses.

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Improved temporal focusing at source location using deconvolution


Focusing of elastic waves for microseismic imaging

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ABSTRACT

Microseismic events generate compressive waves and shear waves which can be recorded at receivers. We present a theory that shows how elastic P and S waves separately backpropagate to the original source location. These refocused P and S wavefields are free of singularities. We also demonstrate a technique that enhances the ability to image the spatial focus for each wave type using elastic waves. The improved spatial focus obtained is achieved in a velocity model for which the interface boundaries are approximate but where the mean slowness is correct. Deconvolution designs a signal to be rebroadcasted from the receivers, using only the waves recorded at each receiver, such that the wavefield has an optimal temporal focus at the source location. We demonstrate theoretically and numerically that improved temporal focusing of elastic waves leads to improved spatial focusing for each wave type. This proposed technique only involves a simple preprocessing step to the recorded data and its cost is hence negligible compared to the total cost of microseismic imaging.

Key words: Deconvolution, Signal Focusing, Image processing, Fractures and faults, Computational seismology, and Earthquake source observations

1 INTRODUCTION

Due to hydraulic fracturing becoming a common practice for unconventional gas and oil fields, there has been an increased interest into the study of microseismic events. Clusters of microseismic events delineate faults and the formation of fractures, and can indicate new or reactivating regions of failure. These microseismic events can be generated naturally or as a result of hydraulic stimulation (Duncan, 2005; Kendall et al., 2011). Therefore, the petroleum industry desires to develop more accurate ways of locating, and monitoring microseismic events to potentially improve their relationship to production and completion data (Foulger and Julian, 2012).

A common processing method to locate microseismic events or earthquakes is based on picking arrival times of the acoustic and shear waves. This process, however, is difficult to do accurately when significant noise is present in the data (Bose et al., 2009; Bancroft et al., 2010; Kummerow, 2010; Song et al., 2010; Hayles et al., 2011). An alternative approach, which requires less user interaction and allows for more accuracy, is using time reversal to image the focus of the microseismic events or earthquakes at the source location (McMechan et al., 1985; Larmat et al., 2006, 2010; Lu, 2008; Steiner et al., 2008; Lu et al., 2008; Artman et al., 2010). In this imaging approach, one uses time reversal to focus the recorded signal at the source location in both time and space. The advantage of time reversal is that it does not require picking of arrival times which is important when dealing with noisy data.

If one would time reverse the waves at every point in space, the wavefield will focus onto the original source location. If, however, the wavefield is sampled at only a limited number of locations, then it is not obvious that time reversal is the optimal way to focus energy on the original source. Much research has been carried out on focusing sparsely sampled wavefields (Parvulescu, 1961; Fink, 1997; Roux and Fink, 2000; Tanter et al., 2000, 2001; Aubry et al., 2001; Bertaix et al., 2004; Jonsson et al., 2004; Montaldo et al., 2004; Vignon et al., 2006; Larmat et al., 2010; Gallot et al., 2011). In this paper, we explore a simple extension to time reversal, based on deconvolution, as previously derived by Ulrich et al. (2013). We have shown earlier that deconvolution improves the locating of microseismic events in an acoustic medium (Douma et al., 2013). We now demonstrate deconvolution’s ability to improve the imaging of a microseismic event in an elastic medium. This method is a robust, though simplified, version of the inverse filter (Tanter et al., 2000, 2001; Gallot et al., 2011). It calcu-
lates a signal to be rebroadcast from the receiver such that the output at the focal location becomes an approximate delta function $\delta(t)$ and uses only the recorded signals at each receiver.

As with all imaging methods, reverse time imaging is unable to locate the microseismic event to a point location when the velocity model used for the backpropagation differs from the true velocity model, or when the aperture is limited; it causes the spatial image to defocus. In the numerical example used for this paper, the aperture used is not perfect, thus, the wavefield is not known at every point in both time and space. Additionally, we complicate our model by back-propagating our wavefields not through the correct velocity model but through a smoothed version of the velocity model.

In this paper, we first derive a relationship between the temporal focus and the incoming wave for an elastic medium. The theory is used to show that improved temporal focusing leads to improved spatial focusing for both P- and S-waves. We then show a numerical example in which a horizontal point force excites elastic waves.

2 THEORY

In this section, we show that improved temporal focusing leads to improved spatial focusing for both P- and S-waves. We first consider a homogeneous elastic medium where either P, SV, or SH waves are incident on a focal point. According to expression (8.13) of Aki and Richards (2002), the elastic wavefield can be expressed as

$$ u(r, \theta, \varphi) = \sum_{lm} \left( U_l(r) R_l^m(\theta, \varphi) + V_l(r) S_l^m(\theta, \varphi) + W_l(r) T_l^m(\theta, \varphi) \right). $$

(1)

The vector spherical harmonics in equation 1 are given by

$$ R_l^m(\theta, \varphi) = Y_{lm}(\theta, \varphi) \hat{r} \quad \text{(P - waves)}, $$

(2)

$$ S_l^m(\theta, \varphi) = \frac{1}{\sqrt{l(l+1)}} \left( \frac{\partial Y_{lm}(\theta, \varphi)}{\partial \theta} \hat{\theta} + \frac{1}{\sin \theta} \frac{\partial Y_{lm}(\theta, \varphi)}{\partial \varphi} \hat{\varphi} \right) \quad \text{(SV - waves)}, $$

(3)

$$ T_l^m(\theta, \varphi) = \frac{1}{\sqrt{l(l+1)}} \left( \frac{\partial Y_{lm}(\theta, \varphi)}{\partial \theta} \hat{\theta} - \frac{\partial Y_{lm}(\theta, \varphi)}{\partial \varphi} \hat{\varphi} \right) \quad \text{(SH - waves)}, $$

(4)

where $Y_{lm}(\theta, \varphi)$ denote the spherical harmonics. The radial functions $U_l(r)$, $V_l(r)$ and $W_l(r)$ are spherical Bessel functions or spherical Hankel functions which satisfy equation (8.6) of Aki and Richards (2002).

We consider the case of an incoming wave that, at a large distance from the focal point $r = 0$, is given by $f_i(t + r/c)/r$. We study the properties of this incoming wave at the focus for every angular degree $l$ separately. For a perfect aperture, the angular degree $l$ describes an explosive source when $l = 0$, a point force when $l = 1$, and a double-couple source when $l = 2$. Since there is no source at the location $r = 0$, the solution is finite and is therefore given by spherical Hankel functions $j_l$:

$$ U_l(r) \propto j_l(kr) \quad \text{where} \quad k = \omega/\alpha, $$

(5)

with $\alpha$ representing the P-wave velocity, and

$$ V_l(r) \quad \text{and} \quad W_l(r) \propto j_l(kr) \quad \text{with} \quad k = \omega/\beta, $$

(6)

where $\beta$ the S-wave velocity. Thus, the radial variation of the wavefield is proportional to $j_l(kr)$, with $k$ the wavenumber of the wave type under consideration. In the following, we study wavefields with radial dependence $j_l(kr)$ and denote the wavenumber as

$$ k = \omega/c, $$

(7)

where $c$ is the appropriate wave velocity ($\alpha$ or $\beta$ depending on the wave type). It is understood that the total wavefield follows by multiplying with the appropriate vector spherical harmonic as given in expressions (2)-(4). The radial dependence of the wavefield in the frequency domain, therefore, is given by

$$ U_l(\omega, r) = A_l(\omega) j_l(kr), $$

(8)

where $A_l(\omega)$ is a Fourier coefficient.

Using the Fourier convention $f(t) = \int F(\omega) e^{-i \omega t} d\omega$, the wavefield in the time domain is given by

$$ u_l(r, t) = \int A_l(\omega) j_l(kr) e^{-i \omega t} d\omega. $$

(9)

The waves impinging on the focus are determined by the incoming waves. For large radius $r$, the incoming waves can be written as

$$ u_{i}^{inc}(r, t) = \frac{f_i(t + r/c)}{r} \quad \text{as} \quad r \to \infty. $$

(10)
We show in the appendix that the total wavefield is given for all values of $r$ by,

$$u_i(r, t) = (-i)^l \frac{2}{c} \int (-\omega) F_l(\omega) j_l(k r) e^{-i\omega t} d\omega,$$  \hspace{1em} (11)

where $F_l(\omega)$ is the temporal Fourier transform of $f_l(t)$. 

Because our derivation involves repeated differentiations and integrations, we employ the following notation:

$$f_l^{(n)}(t) \equiv \frac{d^n f_l(t)}{dt^n}.$$  \hspace{1em} (12)

For negative values of $n$, this notation implies integrating $f_l(t)$ $n$-times. In the frequency domain, the notation (12) translates into

$$F_l^{(n)}(\omega) \equiv (-i\omega)^n F_l(\omega).$$  \hspace{1em} (13)

Next, we relate the total wavefield $u_i(r, t)$ to the spherical Bessel function of order 0 which makes it possible to evaluate the Fourier integral in equation (11) analytically. This derivation, found in the appendix, gives:

$$u_i(r, t) = e^{i\theta_0} \left( \frac{1}{r} \frac{d}{dr} \right)^l \left( \frac{f_l^{(-1)}(t + r/c) - f_l^{(-1)}(t - r/c)}{r} \right).$$  \hspace{1em} (14)

This expression shows an explicit relationship between the total wavefield $u_i(r, t)$ for all values of $r$, and the incident wave, $f_l(t + r/c)/r$ at a great distance $r \to \infty$. Note that using the notation of equations (12)-(13), the incoming wave is integrated $l$ times in the factor $f_l^{(-1)}(t \pm r/c)$, and that the differential operator $(1/r)(d/dr)$ acts $l$ times as well. The term $f_l^{(-1)}(t + r/c)$ is the wave that converges on the focal point, while $f_l^{(-1)}(t - r/c)$ gives the outgoing wave that radiates from the focus before the incoming wave has passed through that point. The incident wave $f_l(t + r/c)$ and the outgoing wave $f_l(t - r/c)$ have opposite sign because the focus is a caustic in two angular directions, hence the Maslow index increases by two, which corresponds to a sign change (Chapman, 2004).

We now demonstrate how we can use the expression for the wavefield near the focal point to demonstrate that improved temporal focusing leads to improved spatial focusing. The spatial focus $R_l(r)$ is defined as the wavefield at time $t = 0$. It follows from expression (14) that the spatial focus is given by:

$$R_l(r) \equiv u_i(r, t = 0)$$

$$= e^{i\theta_0} \left( \frac{1}{r} \frac{d}{dr} \right)^l \left( \frac{f_l^{(-1)}(r/c) - f_l^{(-1)}(-r/c)}{r} \right).$$  \hspace{1em} (15)

This expression gives the spatial focus in terms of the incoming wave.

One might be tempted to define the temporal focus as $u_I(r = 0, t)$. This field, according to expression (9), is proportional to $j_l(k r)$ in the frequency domain. The zero order Bessel function $j_0(k r)$ is nonzero for $r = 0$, but $j_l(k r = 0) = 0$ for $l \geq 1$ (Arfken and Weber, 2001). This means that for $l \geq 1$, the wavefield vanishes at the focal point. Physically, this is due to the fact that for $l \geq 1$, the focal point $(r = 0)$ is located at the intersection of nodal lines. Since $U_I(r = 0, t)$ vanishes at $r = 0$ for $l \geq 1$, this quantity is not a useful diagnostic of the temporal focus. To remedy this, we define the temporal focus instead as

$$T_l(t) \equiv \frac{d^l}{dr^l} u_i(r = 0, t).$$  \hspace{1em} (16)

As shown in the appendix, the $l$-th derivative of $j_l(k r)$ is finite and nonzero for $r = 0$. We derive in the appendix the following relation between the temporal focus and the incoming waves,

$$T_l(t) = \frac{2b_l}{c^{l+1}} f_l^{(l+1)}(t),$$  \hspace{1em} (17)

where $b_l$ is given by

$$b_l = \frac{2l^2(l)!}{(2l + 1)!}.$$  \hspace{1em} (18)

Specifically,

$$b_0 = 1, \hspace{1em} b_1 = 1/3, \hspace{1em} b_2 = 2/15.$$

According to equation (17), the temporal focus thus is proportional to the $(l + 1)$-th time derivative of the incoming wave. Of these $(l + 1)$ derivatives, given by equation (17), $l$ are due to the derivatives in definition (16). One time derivative is due to the fact that (14) contains the combination $(f_l(t + r/c) - f_l(t - r/c))/r$. Using a Taylor expansion and taking the limit $r \to 0$, gives

$$\lim_{r \to 0} \frac{f_l(t + r/c) - f_l(t - r/c)}{r} = \lim_{r \to 0} \frac{2(r/c) f_l'(t)}{r} = \frac{2}{c} f_l'(t).$$

This explains an additional time derivative in expression (17).

3 SPATIAL AND TEMPORAL FOCUS FOR EACH ANGULAR COMPONENT

The spatial and temporal focus defined by expressions (15) and (17) both relate to the incoming waves and can be combined to explicitly relate the spatial and temporal focus.
Equation (19) is the main result of the theory. Although the relation (19) between the spatial and temporal focus is complicated, it does show that good temporal focusing implies good spatial focusing. Good temporal focusing implies that \( T_1(t) \) is strongly peaked near \( t = 0 \), i.e. that \( T_1(t) \) only differs appreciably from zero for a small range of time values \(- t_f < t < t_f\), where \( t_f \) is the half width of the temporal focus. Expression (19) implies that the spatial focus differs appreciably from zero for values of \( r \) that satisfy \( 0 \leq r < ct_f \) (radius is always positive). A good temporal focus (small \( t_f \)) thus implies a good spatial focus.

The spatial and temporal focus, and their relationship defined by equations (15), (17), and (19) all depend on the order \( l \). In this section, we show the explicit forms of these expressions for the case \( l = 0, 1, \) and \( l = 2 \). These cases are relevant for an explosive source (\( l = 0 \)), point force (\( l = 1 \)) and double-couple source (\( l = 2 \)) if the aperture would be perfect. For \( l = 0 \), equations (15), (17), and (19) become respectively,

\[
R_0(r) = \left( f_0(r/c) - f_0(-r/c) \right) r, \tag{20}
\]

\[
T_0(t) = \frac{2}{c} f_0^{(1)}(t), \tag{21}
\]

\[
R_0(r) = \frac{c}{2} \left( T_0^{-1}(r/c) - T_0^{-1}(-r/c) \right), \tag{22}
\]

Equations (20), (21), and (22) are the same as shown in the previous derivation of Ulrich et al. (2013) which dealt with an explosive source in an acoustic medium where the temporal focus is the time derivative of the incoming wave. We now continue the derivation for \( R(r) \) due to a point force (\( l = 1 \)) and double-couple (\( l = 2 \)). For \( l = 1 \), equations (15), (17), and (19) become respectively,

\[
R_1(r) = \frac{1}{r} \left( f_1(r/c) + f_1(-r/c) \right) r, \tag{23}
\]

\[
C_2 \left( f_1^{(-1)}(r/c) - f_1^{(-1)}(-r/c) \right), \tag{24}
\]

\[
R_1(r) = \frac{4c^2}{3r} \left( T_1^{(-2)}(r/c) + T_1^{(-2)}(-r/c) \right) r, \tag{25}
\]

\[
T_1(t) = \frac{3}{2c} f_1^{(2)}(t), \tag{24}
\]

\[
\frac{4c^3}{3r^2} \left( T_1^{(-3)}(r/c) - T_1^{(-3)}(-r/c) \right). \tag{26}
\]

It may appear that equation (23) is singular at \( r = 0 \). Even though each of the two terms in this expression diverge as \( r \to 0 \), the singularities cancel. This can be verified by writing \( f_1(r/c) = a_0 + O(r) \). Integrating this once gives \( f_1^{(-1)}(r/c) = a_0(r/c) + O(r^2) \). Inserting this into equation (23) gives

\[
R_1(r) = \frac{1}{r} \left( 2a_0 + O(r) - \frac{c}{r^2} \left( 2a_0(r/c) + O(r^2) \right) \right). \tag{27}
\]

The terms proportional to \( a_0 \), which caused each of the individual terms in expression (23) to be singular, cancel out. The remainder of equation (26) is finite as \( r \to 0 \).

For \( l = 2 \), equations (15), (17), and (19) become respectively,

\[
R_2(r) = \frac{1}{r} \left( f_2(r/c) - f_2(-r/c) \right) r, \tag{28}
\]

\[
- \frac{3c}{r^2} \left( f_2^{(-1)}(r/c) + f_2^{(-1)}(-r/c) \right) \tag{29}
\]

\[
+ \frac{3c^2}{r^3} \left( f_2^{(-2)}(r/c) - f_2^{(-2)}(-r/c) \right). \tag{29}
\]

The wavefields computed are finite at the focal point \( r = 0 \) when one refocuses either P or S-waves. In contrast, when P and S-waves are excited by a point force, the P-wave component and the S-wave component behave as \( 1/r^3 \) as \( r \to 0 \), hence the P-wave and S-wave separately have a non-integrable singularity at \( r = 0 \), while their sum has an \( 1/r \) singularity Wu (1985), which is integrable. The refocused wavefields don’t display this behavior because these fields are source-free at \( r = 0 \), and therefore the wavefield is finite. Therefore, the P and S-waves can be refocused separately without causing singularities, and the treatment given here is applicable to P, SV, and SH waves separately.

The expressions above must be multiplied with the appropriate vector spherical harmonic (2)-(4) to obtain
the full focused wavefield. For each wave type, a different spherical harmonic must be multiplied to characterize the wavefield. Additionally, in equations (2)-(4), the vector spherical harmonics are summed over the angular order \( l \) and degree \( m \) which captures the imprint of the source properties on each wave type. Therefore, each wave has its own dependence on the angles and on space and time.

If one were to use one component of the motion, such as the \( x \)-component, the source properties for the different wave types are superposed on each other. Since for a fixed source mechanism the radiation pattern of P and S waves are different, the focused wavefields do not provide clear information about the source mechanism. In order to avoid mixing of P- and S-radiated waves, one must decompose the wavefield using the divergence and curl in order to investigate the focus of each wave type (P and S) separately.

If one does not have a perfect aperture, a blurred focus will occur, and the focus cannot be characterized by one angular degree \( l \) but by the superposition of different angular degrees \( l \). This is important to describe the spatial focus achieved in our numerical modeling.

4 NUMERICAL EXAMPLE

We illustrate the theory with a numerical example. We use the velocity model shown in the top panel of Figure 1 to propagate the source wavefield to the receivers. The model consists of horizontally continuous layers whose P-wave velocities range from approximately 4.7 km/s to 5.9 km/s and S-wave velocities range from approximately 2.3 km/s to 2.9 km/s. In practice, one does not know the true velocity model. For this reason, we used the smoothed velocity model, shown in the bottom panel of Figure 1, for the back-propagation. The velocity model is smoothed by using a two-dimensional triangle smoothing of the slowness with a smoothing radius of \( 0.185 \) km in the \( x \) and \( z \) directions (Fomel, 2007). This smoothed velocity model has the same mean slowness as the correct velocity model.

We use a horizontal point force located at \((x, z) = (0.51 \text{ km}, 2.68 \text{ km})\). The source is characterized by Ricker wavelet with dominant frequency of 100 Hz. There are 56 receivers distributed over 2 vertical boreholes in our model. The \( x \)-locations of the receiver boreholes are 0.74 km and 0.88 km respectively. The receivers range from a depth of 2.36 km to 2.86 km with a spacing of 18.5 m.

5 HORIZONTAL POINT FORCE

This section describes the numerical modeling that demonstrates that improved temporal focusing leads to improved spatial focusing for each wave type. We first model the wavefield due to a horizontal point force excitation at the source location. The horizontal and vertical displacements of the wavefield are then recorded at each receiver. Afterwards, we apply either time reversal or deconvolution to the recorded signals to generate the wavefields which are back-propagated.

We use the time reversed or deconvolved signals to excite waves that backpropagate through the smoothed velocity model, using the following forces acting at each of the receiver locations,

\[
\vec{F}_{\text{TimeReverse}} = (U_x(-t), U_z(-t)),
\]

\[
\vec{F}_{\text{Deconv}} = (t_1^{\text{Inverse}}(t), U_1^{\text{Inverse}}(t)).
\]

Here, \( \vec{F}_{\text{TimeReverse}} \) and \( \vec{F}_{\text{Deconv}} \) are the source functions for time reversal and deconvolution respectively, and \( U_x(t) \) and \( U_z(t) \) are the recorded signals. The inverse signal of a time series \( g(t) \) is defined as,

\[
g^{\text{Inverse}}(t) \ast g(t) = \delta(t).
\]

where \( \ast \) denotes convolution. In order to solve for \( g^{\text{Inverse}}(t) \) and avoid instability for \( g^{\text{Inverse}}(\omega) \) when \( g(\omega) = 0 \), we have apply a water level regularization. Thus,
Figure 2. Decomposed P wavefield at time of focus due to horizontal point force. Top panel is P-wave just after the horizontal point force is emitted. Middle panel indicates the result of injecting the time reversed signal back into the smoothed velocity model from the receiver locations. Bottom panel shows result of injecting the inverse signal calculated using deconvolution back into the smoothed velocity model from the receiver locations.

Figure 3. Decomposed S wavefield at time of focus due to horizontal point force. Top panel is S-wave just after the horizontal point force is emitted. Middle panel indicates the result of injecting the time reversed signal back into the smoothed velocity model from the receiver locations. Bottom panel shows result of injecting the inverse signal calculated using deconvolution back into the smoothed velocity model from the receiver locations.

\[
g_{\text{inverse}}(\omega) = \frac{1}{g(\omega)} \Rightarrow \frac{g^*(\omega)}{|g^*(\omega)|^2 + \epsilon}. \quad (33)
\]

The derivation and explanation of these two methods are discussed in more detail by Douma et al. (2013).

After backpropagation, the wavefield is decomposed into P and S components for a crucial reason. We demonstrated in the theory section that improved temporal focusing leads to improved spatial focusing for each wave type. We do not consider the focus for the vertical or horizontal displacements. Rather, we use the displacement components to calculate the P and S wavefields using divergence and curl, respectively. This allows us to retrieve the P and S waves that have backpropagated from the sources. For each wave type, our theory predicts that an improved temporal focusing leads to improved spatial focusing.

We first model the wavefield due to a horizontal point force excitation at the source location. The top panel of Figures 2 and 3 show the P and S wavefields’ radiation pattern just after the horizontal point force is emitted and represent a pure angular degree \( l = 1 \). In perfect source imaging, we would reconstruct these radiation patterns. However, our aperture is not perfect and we backpropagate through a smoother version of the velocity model. Thus, we do not expect to be able to reconstruct these radiation patterns perfectly.

In order to show that deconvolution generates an improved spatial focus, we first demonstrate that deconvolution enhances the temporal focus. Thus, we calculate the temporal focusing for the P and S component as a result of deconvolution (equation (31)) compared with time reversal (equation (30)). We defined the temporal focusing in equation (16) as the \( l \)-th derivative of the incoming wavefield. This is necessary because the
wavefield is zero at our source location due of nodal lines. In order to demonstrate improved temporal focusing for a horizontal point force, we take the derivative of the P wavefield in the $x$-direction and the derivative of the S-wavefield in the $z$-direction because these derivatives are the radial derivatives perpendicular to the nodal lines for each wave type. We change the direction of the derivative because the radiation pattern of the P-wave due to a horizontal point force is a dipole in the $x$-direction while the S-wave radiation has a dipole pattern oriented in the $z$-direction (Aki and Richards, 2002). This is visible in the top panel of Figures 2 and 3, which show the radiation patterns of the P and S wavefields just after the source has acted. We calculate the derivatives as defined above to show the temporal focus for the P and S wave at the source location. Comparing Figure 4(a) to 4(b), one can clearly note that deconvolution has significantly improved the temporal focusing compared to time reversal for the P-wave. In contrast, Figures 4(c) and (d) show that both time reversal and deconvolution produce a similar temporal focus for the S wave. Because improved temporal focusing implies better spatial focusing, see equation (26), one would expect to see an improved spatial focus improved for the P-wave using deconvolution compared to using time reversal. Additionally, we don’t expect the S-wave’s spatial focus to improve using deconvolution because the temporal focus did not improve.

After having demonstrated that deconvolution improved the temporal focusing for the P wave, we compare the spatial focus generated by deconvolution and time reversal for each wave type. The backpropagated wavefields at $t = 0$ for the two methods are shown in the middle and bottom panels of Figure 2 and 3. The middle panel of Figure 2 represents the spatial focus of the P wave using time reversal whereas the bottom panel shows the spatial focus of the P wave using time-reversal. Figure 2 shows that deconvolution drastically improves the spatial focus compared to deconvolution. Figure 3 does not show a clear improvement of spatial focusing between time reversal (middle panel) and deconvolution (bottom panel) for the S component. This was expected due to deconvolution and time reversal producing similar temporal focuses for the S-wave.

The aperture, over which we record the data that we back-propagate, is not perfect. This causes the spatial focuses, created using time reversal and deconvolution shown in Figures 2 and 3, to not be confined to one angular degree $l$ because the spatial focuses are blurred in the $z$-direction. A perfect spatial focus would consist of only the $l = 1$ component. Figure 5 (a)-(b) and (c)-(d) shows cross sections of the backpropagated wavefields in Figures 2 and 3 in the $x$ and $z$ directions, respectively, so that it is easier to assess the improvements and comparisons between the two methods. Note that the scales of the horizontal axis for Figure 5 (a)-(b) are different from Figure 5 (c)-(d). Figure 5(a) demonstrates that time reversal is not able to create a well defined dipole focus in the $x$-direction which represents the radiation pattern of a P wave due to a horizontal point force. Figure 5(b) shows that deconvolution is able to reconstruct the dipole radiation pattern of the P wave due to a horizontal point force. Figure 5(c) and (d) demonstrate that there seems to be no significant difference between time reversal (c) and deconvolution (d) to reconstruct the S wave’s focus.

Our numerical results have shown that deconvolution improves the temporal focusing for the P wave, we compare the spatial focus generated by deconvolution and time reversal for each wave type. The backpropagated wavefields at $t = 0$ for the two methods are shown in the middle and bottom panels of Figure 2 and 3. The middle panel of Figure 2 represents the spatial focus of the P wave using time reversal whereas the bottom panel shows the spatial focus of the P wave using time-reversal. Figure 2 shows that deconvolution drastically improves the spatial focus compared to deconvolution. Figure 3 does not show a clear improvement of spatial focusing between time reversal (middle panel) and deconvolution (bottom panel) for the S component. This was expected due to deconvolution and time reversal producing similar temporal focuses for the S-wave.
ution was able to improve the temporal focus for the P wave which led to an improved reconstruction of the P-wavefield's radiation pattern. However, deconvolution was not able to improve the temporal focus for the S-wave, due to a horizontal point force, which led to it also not improving the reconstruction of the S-wavefield's radiation pattern. This can be attributed to the fact that a nodal line for the S-wavefield's radiation pattern intersects the receiver array. Deconvolution will then apply a larger weight to the receivers near the nodal line in order to increase a weak recorded signal. This is unphysical because there is no information to be gained in these weak recorded waveforms near the nodal lines. These receivers are supposed to record no information about the source and should, therefore, not propagate any information back. However, it simultaneously demonstrates the robust nature of deconvolution. For the radiation pattern which has a nodal line intersecting the receiver array, deconvolution does not generate an inaccurate but rather a comparable reconstruction of the radiation pattern as time reversal.

We conclude that, for an elastic media without a perfect aperture and true velocity model, improved temporal focusing leads to improved spatial focusing. We have shown this both theoretically and numerically to be the case. Because deconvolution has the ability to improve the temporal focusing, one can improve the spatial focusing.

6 CONCLUSION

We have introduced deconvolution which improves the temporal focusing of microseismic events. We demonstrated theoretically and numerically that this improved temporal focusing leads to improved spatial focusing for each wave type in an elastic medium. This improved spatial focusing is beneficial for enhancing the focus of the elastic waves. The simplicity and robust nature of this method allows for a simple incorporation into existing reverse-time imaging methods. Additionally, the cost of deconvolution is minimal compared to running the finite difference modeling. Therefore, it can be added as a preprocessing step without significant additive cost.

7 ACKNOWLEDGMENT

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Seismic reflector imaging using internal multiples with Marchenko-type equations

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ABSTRACT
We present an imaging method that creates a map of reflection coefficients in correct one-way time with no contamination from internal multiples using purely a filtering approach. The filter is computed from the measured reflection response and does not require a background model. We demonstrate that the filter is a focusing wavefield that focuses inside a layered medium and removes all internal multiples between surface and the focus depth. The reflection response and the focusing wavefield can then be used for retrieving virtual vertical seismic profile data, thereby redatuming the source to the focus depth. Deconvolving the upgoing by the downgoing vertical seismic profile data redatums the receiver to the focus depth and gives the desired image. We then show that for oblique angles of incidence in horizontally layered media the image of the same quality as for 1D waves can be constructed. This step can be followed by a linear operation to determine velocity and density as a function of depth. Numerical simulations show the method can handle finite frequency bandwidth data and the effect of tunneling through thin layers.

Key words: Imaging, Multiples, Inverse Scattering

1 INTRODUCTION

From the late 1960's to the early 1980's much work has been done on one-dimensional exact inversion methods for scalar wave problems for applications in geophysics (Ware and Aki, 1969; Burridge, 1980; Coen, 1981). Most derivations use stretching and scaling of the wave equation to write it in a form resembling the Schrödinger equation for which the exact inversion method originally was developed (Agranovich and Marchenko, 1963; Lamb, 1980). The interest was briefly revived when Rose (2002) showed how 1D focusing can be achieved with the Marchenko equation and he used an iterative solution method to demonstrate it. Recently, Broginni et al. (2012) used the idea to retrieve a virtual Vertical Seismic Profile (VSP) with the virtual source inside the layered medium at any depth location and the receivers at the surface. The virtual VSP was retrieved from surface reflection data and the method was extended to 3D data-driven redatuming (Wapenaar et al., 2012). Wapenaar et al. (2013a) derived a new Newton-Marchenko scheme in 3D to image reflectors without creating ghosts images from internal multiples. In that sense such schemes are distinctly different from known methods that aim to remove internal multiples using the inverse scattering series (Zhang and Weglein, 2009; Weglein et al., 2012) or to predict and subtract internal multiples (Jakubowicz, 1998; ten Kroode, 2002) from the reflection data at the surface. Imaging schemes that use Marchenko type equations focus the wavefield inside a heterogeneous medium, use the internal multiples to construct correct image amplitudes, and do not create ghost reflectors because the internal multiples are handled correctly.

Direct inversion using reflection data measured at one side of the target in 1D is possible for infinite bandwidth data, which is not available in measured seismic data. Imaging can be achieved with a limited frequency bandwidth as available from seismic data with a penalty on the achievable resolution. Similar to the work of Wapenaar et al. (2013a) our aim is to form an image using only reflection data measured at one side of the target. By investigating how downgoing and upgoing waves in a layered medium can be focused at a chosen depth level, we derive an exact imaging scheme. To create an image of only primary reflection events
from measured data at the surface it is necessary to re-
datum the sources and receivers to the image location.
This should be done in a data driven way. Direction is
important and we keep upgoing and downgoing waves
separated.

In this paper we follow a different route than Wape-
naar et al. (2013a) by first finding out what is the in-
cident field that creates a focus just below the bot-
ttom interface of a layered medium and we call this the focus-
ing wavefield. We then find two relations between the fo-
cusing wavefield, the surface impulse reflection response
and VSP responses for a source at the focus depth in-
side a layered medium. From these relations we derive
an imaging scheme and show how the image can be used
to determine the medium properties and layer thickness
in case waves at several oblique angles of incidence are
used. With a numerical example we investigate the ef-
effects of finite bandwidth and of tunneling waves on the
performance of the imaging scheme. In the appendix
we give a mathematical derivation of the results with
the aid of the acoustic reciprocity theorems of the time-
convolution and time-correlation types.

2 FOCUSING A WAVEFIELD JUST
BELOW A LAYERED MEDIUM

This section is dedicated to finding expressions for the
focusing wavefield. Later we will investigate how we can
retrieve the focusing wavefield from the measured im-
pulse reflection response. For a normal incidence plane
wave we assume to have measured the acoustic pressure,
p(z, t), below the source, but just above the first re-
fractor, which we denote as the surface. We can describe
the acoustic pressure as the sum of downgoing, \( p^-(z, t) \),
and upgoing, \( p^+(z, t) \), waves as \( p(z, t) = p^-(z, t) + p^+(z, t) \),
where \( z \) is depth and \( t \) is time. We start with a medium
containing three layers and the two interfaces separating
them are located at depth levels \( z_0 \) and \( z_1 > z_0 \).
The densities and velocities in the three layers are de-
noted \( \rho_i, c_i \), with \( i = 0, 1, 2 \), respectively. The thickness of
the second layer is given by \( d_1 = z_1 - z_0 \) and the cor-
responding one-way traveltime is denoted \( t_1 = d_1/c_1 \).
For a downgoing pressure wave incident on an interface
at depth level \( z_1 \) the local reflection and transmission coefficients are
given by \( r_1, \tau_1^\pm \), and for an upgoing inci-
dent wave by \( -r_1, \tau_1^\mp \). The two-way transmission coef-
ficient is given by \( \tau_1^2 = \tau_1^\pm \tau_1^\mp \). First we send a normal
incidence plane downgoing acoustic pressure wave from
above and the upgoing field just above \( z_0 \) consists of
an infinite number of events. The first two are primary
reflections followed by a series of multiple reflections as
indicated in the top panel of Figure 1. We call the total
upgoing field the impulse reflection response \( R(z_0, t) \).
The position \( z_0 \) in the argument indicates that both
source and receiver are at the same depth level, hence
\( p^-(z_0, t) = R(z_0, t) \). In the third layer the downgoing
wavefield consists of a direct arrival followed by mul-
tiples. We call this the transmission response \( T^+(z_1, z_0, t) \),
measured at \( z_1 \) and generated by the source at \( z_0 \), hence
\( p^+(z_1, t) = T^+(z_1, z_0, t) \). We can write these upgoing
and downgoing pressure fields in the frequency domain,
with radial frequency \( \omega \), as

\[
\hat{p}^+(z_0, \omega) = 1, \quad \hat{p}^-(z_0, \omega) = \frac{r_0 + r_1 \exp(-2i\omega t_1)}{1 + r_0 r_1 \exp(-2i\omega t_1)},
\tag{1}
\]

\[
\hat{p}^+(z_1, \omega) = \frac{r_0^+ r_1^+ \exp(-i\omega t_1)}{1 + r_0 r_1 \exp(-2i\omega t_1)}, \quad \hat{p}^-(z_1, \omega) = 0,
\tag{2}
\]

where the diacritical hat denotes a quantity in frequency
domain and \( \hat{p}(z_0, \omega), \hat{p}(z_1, \omega) \) denote the acoustic pres-
sure just above \( z_0 \) and just below \( z_1 \). This notion is used
throughout the paper. By expanding the denominator
the infinite number of events are obtained that belong
to the reflection and transmission responses.

From the top panel of Figure 1 it is clear that if we
are able to eliminate the second downgoing event just
below the reflector at \( z_0 \) only one event reaches depth
level \( z_1 \). This would constitute a focused wavefield at \( z_1 \).
It is achieved by sending in a new downgoing wave with
amplitude \( r_0 r_1 \) that reaches depth level \( z_0 \) at \( t = t_1 \) as
indicated in the bottom panel of Figure 1. In the figure
all incident waves are time advanced by the one-way
traveltime \( t_1 \) of the second layer such that the focus
occurs at \( t = 0 \). Notice that by focusing the wavefield
at depth level \( z_1 \) all internal multiples have been elimi-
nated. Another interesting feature is that the reflection
response in the bottom panel of Figure 1 has only two
events and both have the correct local reflection ampi-
tudes of the two reflectors. To create a unit amplitude
focus the inverse of the transmission response has to be
sent in. We denote this downgoing focusing wavefield
\( f_1^+(z_0, z_1, t) \). The upgoing part is the corresponding re-
flection response and we denote it $f_{1}^{+}(z_0, z_1, t)$. The argument $z_1$ is inserted to indicate the depth level just below which the pressure wavefield is focused. These two wavefields together form the focusing wavefield. By looking at equations (1) and (2) we can see that at $z_0$ the focusing wavefield can be written as

$$
\hat{f}_{1}^{+}(z_0, z_1, \omega) = \frac{1}{\hat{T}^{+}(z_1, z_0, \omega)} = e^{i\omega t_1} + \frac{r_0 r_1 e^{-i\omega t_1}}{\tau_0^{+} \tau_1^{+}},
$$

(3)

$$
\hat{f}_{1}^{-}(z_0, z_1, \omega) = \frac{\hat{R}(z_0, \omega)}{\hat{T}^{+}(z_1, z_0, \omega)} = \frac{r_0 e^{i\omega t_1} + r_1 e^{-i\omega t_1}}{\tau_0^{+} \tau_1^{+}}.
$$

(4)

Because time-reversed solutions satisfy the same wave equation, we investigate the result of sending in the time-reversed reflection response $\hat{f}_{1}^{-}(z_0, z_1, -t)$. This is depicted in Figure 2 where the incident wavefield is the time-reversed reflection response of the lower panel of Figure 1 given by $r_2 \delta(t + t_1) + r_0 \delta(t - t_1)$. The corresponding reflection response is also shown in Figure 2. The first event in the reflection response, $r_0 r_1 \delta(t + t_1)$, is the time-reverse of the second incident wave of the focusing wavefield. The first term in the second reflection event, $\delta(t - t_1)$ is the time-reverse of the first incident wave of the focusing wavefield. The second term, $-\frac{r_0^2 \tau_1^2 \delta(t - t_1)}{\tau_0^+ \tau_1^+}$, can be interpreted as minus the direct arrival from an upgoing source wavefield generated just below $z_1$ and received at $z_0$. The source strength is equal to the product of the local transmission coefficients. The third term in the reflection event, $\frac{r_0^2 \tau_1^2 r_1 r_2 \delta(t - 3t_1)}{\tau_0^+ \tau_1^+}$, can be interpreted as the first multiple of the transmission response of the same source at $z_1$ and receiver at $z_0$. Because all later terms will just be multiples inside the second layer, we can conclude that sending in the time-reversed reflection response of the focusing wavefield, $\hat{f}_{1}^{+}(z_0, z_1, -t)$, results in a new reflection response that is equal to the time-reverse of the incident focusing wavefield, $\hat{f}_{1}^{+}(z_0, z_1, -t)$, minus the pressure field Green’s function, $G^{h^-}(z_0, z_1, t)$, that belongs to an upgoing source at $z_1$ and pressure field receiver at $z_0$. In the frequency domain we can express this as

$$
[\hat{f}_{1}^{+}(z_0, z_1, \omega)]^* = \hat{G}^{h^-}(z_0, z_1, \omega) = R(z_0, \omega)\hat{f}_{1}^{+}(z_0, z_1, \omega)^*,
$$

(5)

and equation (5) can be interpreted as a VSP type Green’s function expression in terms of the focusing wavefield and the impulse reflection response. The fact that this is only for the upgoing part of the source wavefield is indicated by the minus sign in the superscript with the Green’s function. The superscript $^p$ indicates that it is the pressure field. Notice that in the time domain the focusing wavefield and the Green’s function in the left-hand side of equation (5) are non-zero in mutually exclusive time windows except for the overlapping time instant, $t = t_1$, of the first arrival of the Green’s function and the last arrival of the time-reversed down-going focusing wavefield, see Figure 2.

The above focusing result can be used for any number of interfaces in the 1D model. In the frequency domain the reflection and transmission responses for any layered medium can be written in the fractional form used above and their denominators are always the same (Goupillaud, 1961). This means that the focusing wavefield for a medium with interfaces from $z_0$ to $z_i$ can be written as

$$
\hat{f}_{1}^{+}(z_0, z_i, \omega) = \hat{T}^{+}(z_i, z_0, \omega)^{-1},
$$

(6)

$$
\hat{f}_{1}^{-}(z_0, z_i, \omega) = \hat{R}(z_0, \omega)/\hat{T}^{+}(z_i, z_0, \omega),
$$

(7)

$$
\hat{f}_{1}^{-}(z_i, z_i, \omega) = 1,
$$

(8)

$$
\hat{f}_{1}^{-}(z_i, z_i, \omega) = 0.
$$

(9)

As an example we extend the model with a third interface, at $z_2$ with $z_2 > z_1$. Figure 3 shows seismic pressure reflection responses to a layered medium with the values for the velocities and densities of the first four layers in Table 1. In this model density contrasts are stronger than velocity contrasts. In the figure $z_0 = 75$ m, means that the source and receiver are 75 m above the first interface, $z_1 = 192$ m, and $z_2 = 291$ m. The interfaces are indicated in the plots by horizontal black lines. For all plots $t = 0$ is chosen such that it coincides with first arrival at $z_2$, which is the depth level where we want to focus the wavefield. The top left panel Figure 3 shows the acoustic pressure in this layered medium for

<table>
<thead>
<tr>
<th>Layer Number</th>
<th>Velocity (m/s)</th>
<th>Density (kg/m$^3$)</th>
<th>Thickness (m)</th>
</tr>
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<tr>
<td>1</td>
<td>1700</td>
<td>1430</td>
<td>∞</td>
</tr>
<tr>
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<td>117</td>
</tr>
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<td>3</td>
<td>2100</td>
<td>1750</td>
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</tr>
<tr>
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<td>1700</td>
<td>1430</td>
<td>85</td>
</tr>
<tr>
<td>5</td>
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<tr>
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<td>163</td>
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<td>10</td>
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<tr>
<td>11</td>
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</tr>
</tbody>
</table>
Figure 3. a) The impulse reflection response of a four layered medium where the layer boundaries inside the plot are indicated by three black horizontal lines labeled on the right side by $z_0, z_1, z_2$, b) the reflection response of a) but now with an extra incident wave that cancels the downgoing field at the first interface; c) the reflection response of b) but now with an extra incident wave that cancels the downgoing wave at the second interface; d) the reflection response of c) but now with an extra incident wave that cancels the downgoing wave at the first interface generated by the reflected wave from the bottom interface.

a single incident pressure wave. The top right panel of Figure 3 shows the response for an impulse followed by a second incident wave. In this case the upward traveling reflection from $z_1$ arrives at $z_0$ at the same moment the new incident waves arrives at $z_0$ and together they create only an upgoing wave. At this moment the wavefield is focused at depth level $z_1$ as in the first example, but because the focused wave continues to travel downward it creates new reflections from the interface at $z_2$ and those waves interact with all the interfaces. The bottom left panel of Figure 3 shows the same as in the top right panel, but now with a third incident wave that is taken such that it arrives at $z_1$ at the same moment that the upcoming reflection from $z_2$ arrives at $z_1$ and again such that no downgoing wave is created at $z_1$. This third incident wave creates an extra reflection at $z_0$, which has very small amplitude and is barely visible in the plot. The bottom right panel of Figure 3 shows the final result where the wavefield is focused at $z_2$ and a fourth incident wave is taken such that it arrives at $z_0$ at the moment that the upgoing reflected wave from $z_2$ arrives at $z_0$. The incident wave is again such that no downgoing wave at $z_0$ is created. Now the wavefield is focused at $z_2$ and no other events occur at that depth level. The
mathematical details of the waves shown in the bottom right panel of Figure 3 are given in the Appendix.

It can be seen from Figure 3 that to focus the wavefield at the bottom reflector a finite number of waves have to be sent in from the first layer. Comparing this result with the result from the previous example with two interfaces shows that to focus the wavefield at the bottom reflector, the number of waves we need to send into a layered medium is doubled for every reflector that is added. The corresponding reflection response has the same number of waves. If we replace \( z_1 \) by \( z_i \) in equation (5) the equation is still valid. We conclude that this is a valid equation for an arbitrarily layered model and a general derivation is given in the Appendix.

Now we know what the focusing wavefield looks like. The downgoing focusing wavefield incident on a layered medium with interfaces from \( z_0 \) to the focus depth \( z_i \) has \( 2^i \) waves and the reflection response has the same number of waves. At the level \( z_0 \) the focusing wavefield exists in the time domain between

\[
-t_d(z_i, z_0) \leq t \leq t_d(z_i, z_0), \quad t_d(z_i, z_0) = \sum_{n=1}^{i} d_n/c_0
\]

being the one-way traveltime across the layered medium. We also know that at the focus time the upgoing part at the receiver level contains the local reflection coefficient, \( r_i \), scaled by the product of local transmission coefficients. This will be useful for imaging if we are able to make three more steps. First we should be able to use this focusing wavefield to focus inside a layered medium. Second, we must find the way to determine this focusing field from the measured reflection data and as third step we must formulate an imaging scheme. Each step is carried out in the three subsequent sections.

3 FOCUSING THE WAVEFIELD INSIDE A LAYERED MEDIUM

We extend the layered model further by introducing an arbitrary number of reflectors below \( z_i \), but use \( z_i \) as focus depth. Figure 4 shows the situation with four downgoing arrows at \( z_0 \) indicating waves that are sent into the medium and four solid upgoing arrows indicating the reflection response. The wavefield focuses at \( z_i \) at \( t = 0 \), but because the focusing wavefield continues to travel down it will generate new up- and downgoing waves indicated by the dashed arrows in Figure 4. These waves arrive at \( z_0 \) after the last arrival of the focusing wavefield. They all start as a unit amplitude downgoing wave just below \( z_i \) as if there was a downgoing source just below \( z_i \) and hence all waves that are generated and recorded at \( z_0 \) correspond to part of a VSP measurement that is complementary to what we saw in the previous section. In the frequency domain this pressure Green’s function can be written as \( G^{p,+}(z_0, z_i, \omega) \) and it corresponds to the events indicated by the dashed lines in Figure 4, while the focusing wavefield corresponds to the solid lines. The impulse reflection response, \( \hat{R}(z_0, \omega) \), generated by a downgoing unit amplitude pressure wavefield and measured at \( z_0 \) in the whole layered medium is related to the focusing wavefield and the downgoing Green’s function by (see Appendix for general derivation)

\[
\hat{f}_1^\pm(z_0, z_i, \omega) + \hat{\mathcal{G}}^{p,+}(z_0, z_i, \omega) = \hat{R}(z_0, \omega)\hat{f}_1^\pm(z_0, z_i, \omega).
\]

Equation (10) is the wavefield retrieval equation for the wavefield generated by a downgoing source at \( z_i \) and received at \( z_0 \). It is obtained from the measured impulse reflection response and the functions \( \hat{f}_1^\pm \). This equation can be interpreted as follows. If the wavefield \( \hat{f}_1^\pm(z_0, z_i, t) \) is sent into the layered medium the reflection response at \( z_0 \) is given by \( \hat{R}(z_0, \omega) \ast \hat{f}_1^\pm(z_0, z_i, t) \) where the \( \ast \) denotes temporal convolution. Equation (10) says that this response is equal to \( \hat{f}_1^\pm(z_0, z_i, t) \) plus the Green’s function \( G^{p,+}(z_0, z_i, t) \) corresponding to a downgoing source at the focusing level and a pressure receiver at the original receiver level. This is illustrated in Figure 5 where the model consists of seven layers and the values for velocity, density, and layer thickness are taken from Table 1. The source and receiver are at 75 m above \( z_0 \). The depth levels of the interfaces are indicated by black lines in the figure. The focusing time is again \( t = 0 \) at the moment where the first arrival reaches the focus depth \( z_2 = 291 \) m. From the figure it can be seen that \( \hat{f}_1^\pm(z_0, z_i, t) \) and \( G^{p,+}(z_0, z_i, t) \) are well separated in time as indicated above the figure. Equation (5) is also valid when the focusing depth level is inside a layered medium,

\[
[\hat{f}_1^\pm(z_0, z_i, \omega)]^* - \hat{\mathcal{G}}^{p,-}(z_0, z_i, \omega) = \hat{R}(z_0, \omega)[\hat{f}_1^\pm(z_0, z_i, \omega)]^*.
\]
In equation (5) the Green’s function is the transmission response of a layered medium with a source below and a receiver above a layered medium. Equation (11) retrieves the wavefield \( \hat{G}^{\pm}(z_0, z_i, \omega) \) at receiver level \( z_0 \) that is generated by an upgoing plane wave at the source level \( z_i \) inside the layered medium, from the measured impulse reflection response \( \hat{R}(z_0, \omega) \) and the functions \( f_1^\pm \). This equation says that the convolution of the time-reversed upgoing focusing wavefield and the impulse reflection response is equal to the time-reversed downgoing focusing wavefield minus the Green’s function for an upgoing source just below \( z_i \) and the pressure receiver just above \( z_0 \). It is illustrated in Figure 6 for the same configuration and focus depth as used for Figure 5. In this figure events that belong to the focusing wavefield are indicated above the figure as \( f_1^\pm(z_0, z_i, -t) + f_{-1}^\pm(z_0, z_i, -t) \). In this wavefield there are no up-down reflections. The only up-down reflections occurring in the time window of the focusing wavefield come from the upgoing wavefield after reflection at the boundary at \( z_2 \) as indicated by the arrows. This is part of the Green’s function as indicated above the figure with \( G^{\pm}\). Because the upgoing part of the Green’s function is emitted at \( t = 0 \), which is the focusing time-instant for the focusing wavefield, they overlap along the direct travel path from the focus depth to the surface as indicated above the figure. For the rest the focusing wavefield and the Green’s functions are separated in time. In the next section we show that the focusing wavefield can be computed from the measured reflection data and equations (10) and (11). From these virtual VSP data and a subsurface image can be constructed as is shown in the next two sections. Knowing the up-and downgoing Green’s function is important, because they are related to the impulse reflection response at the focus level through

\[
\hat{G}^{q\pm}(z_i, z_0, \omega) = \hat{R}(z_i, \omega)\hat{G}^{q\pm}(z_i, z_0, \omega),
\]

where \( \hat{R}(z_i, \omega) \) denotes the impulse reflection response for the medium that is layered below \( z_i \) and homogeneous above that depth level. The Green’s functions are the reciprocal version of the ones in equations (10) and (11). Here the sign in the superscripts refers to the direction of the wavefield at the receiver level \( z_i \) and the superscript \( q \) refers to a monopole point source at the source level \( z_0 \). The local reflection coefficient, \( r_i \), can be extracted in the time domain from \( R(z_i, t) \) around \( t = 0 \).

4 COUPLED MARCHENKO-TYPE EQUATIONS AND GREEN’S FUNCTION RETRIEVAL

The time-domain functions \( f_1^\pm \) are not causal, while \( G^k \) and \( R \) are. This can be exploited to find \( f_1^\pm \) from the time-domain equivalents of equations (10)-(11). Two coupled Marchenko-type equations are ob-
tained in which \( G^\pm \) do not occur and from which \( f_1^\pm \) can be found given the measured impulse response \( R(z_0, t) \). Once \( f_1^\pm \) are found, time-domain versions of equations (10) and (11) can be used to compute the up- and downgoing wavefields generated by a downhole source and measured at the surface. Hence, for any depth \( z_i \) we can create a virtual VSP data set from the measured impulse reflection response (Newton, 1981; Broggiini and Snieder, 2012; Wapenaar et al., 2012).

Transforming equations (10) and (11) to the time domain results in
\[
G^{0,+}(z_0, z_i, t) = -f_1^+(z_0, z_i, t) + \int_{t'+=-t_d(z_i, z_0)}^{t} f_1^+(z_0, z_i, t') R(z_0, t-t') dt', \quad (13)
\]
\[
G^{0,-}(z_0, z_i, t) = f_1^-(z_0, z_i, t) - \int_{t'=t_d(z_i, z_0)}^{t} f_1^-(z_0, z_i, t') R(z_0, t-t') dt'. \quad (14)
\]

The integration intervals of equations (13) and (14) are finite because the two functions in the integrands are non-zero only in a subinterval. The impulse reflection response \( R(z_0, t-t') \) is causal and therefore zero-valued for \( t' > t \), putting the upper limits at \( t' = t \). In the previous section we have seen that \( f_1^\pm(z_0, z_i, t) = 0 \) for \(|t| > t_d(z_i, z_0)\), because all internal multiples are eliminated between the depth levels \( z_0 \) and \( z_i \) and this defines the lower integration limit of equations (13) and (14).

We can write the downgoing wavefield as
\[
f_1^+(z_0, z_i, t) = T_i^{-1} \delta(t + t_d(z_i, z_0)) + M^+(z_0, z_i, t), \quad (15)
\]
where in our 1D model the first arrival of the transmission response has amplitude \( T_i = \prod_{j=0}^{i} \tau_j \) and \( M^+(z_0, z_i, t) \) denotes the coda following the first arrival; \( M^+(z_0, z_i, t) = 0 \) for \( t < -t_d(z_i, z_0) \). Because both \( M^+(z_0, z_i, t) = 0 \) and \( f_1^+(z_0, z_i, t) = 0 \) for \(|t| > t_d(z_i, z_0)\) and \( G^{\pm} = 0 \) for \( t < t_d(z_i, z_0) \), we can write in the interval \(-t_d(z_i, z_0) < t < t_d(z_i, z_0)\) equations (13) and (14) as two coupled equations as
\[
f_1^-(z_0, z_i, t) = T_i^{-1} R(z_0, t + t_d(z_i, z_0)) + \int_{t'=t_d(z_i, z_0)}^{t} M^+(z_0, z_i, t') R(z_0, t-t') dt', \quad (16)
\]
\[
M^+(z_0, z_i, t) = \int_{t'=t_d(z_i, z_0)}^{t} f_1^-(z_0, z_i, t') R(z_0, t-t') dt', \quad (17)
\]
which are coupled Marchenko-type equations (Lamb, 1980) valid on the interval \(-t_d(z_i, z_0) < t < t_d(z_i, z_0)\). These can be solved for the functions \( M^+(z_0, z_i, t) \) and \( f_1^-(z_0, z_i, t) \) from the measured impulse reflection response \( R(z_0, t) \) when the amplitude of the first arrival of the transmission response, \( T_i \), is known. A straightforward way to solve equations (16) and (17) is to discretize them and solve the resulting matrix inversion problem.

We can also solve the coupled system with an iterative procedure and start by taking
\[
f_{i,0}^+(z_0, z_i, t) = T_i^{-1} R(z_0, t + t_d(z_i, z_0)), \quad (18)
\]
and for \( n \geq 0 \) evaluate the \( n^{th} \)-iteration as
\[
M_{i,n}^+(z_0, z_i, t) = \int_{t'=t_d(z_i, z_0)}^{t} f_{i,n}^+(z_0, z_i, t') R(z_0, t-t') dt', \quad (19)
\]
\[
f_{i,n+1}^-(z_0, z_i, t) = f_{i,0}^-(z_0, z_i, t) + \int_{t'=t_d(z_i, z_0)}^{t} M_{i,n}^+(z_0, z_i, t') R(z_0, t-t') dt', \quad (20)
\]
for \(-t_d(z_i, z_0) < t < t_d(z_i, z_0)\). This scheme always converges, because the underlying Neumann series expansion of the Green’s function integral equation converges unconditionally (Lamb, 1980). Solving equations (16) and (17) using an iterative scheme is not necessary, but will often prove computationally advantageous. Once the functions \( f_{i}^\pm(z_0, z_i, t) \) are found in their time-window, equations (13) and (14) can be used to compute the Green’s functions and thereby the virtual VSP is retrieved. The image can be constructed from the VSP Green’s functions using equation (12). In the next section we first find a more direct route to the image.

5 IMAGING

To use equations (16) and (17) for imaging, we observe that our initial estimate of the scaled delta function is always correct in arrival time, because in a 1D model it is half the two-way traveltime. But, the amplitude \( T_i \) is not known, nor the actual depth level \( z_i \). The 1D image can therefore be constructed as a time image when there is no additional information on the layered medium. We can scale the downgoing coda and the upgoing wavefield by the same factor \( T_i^{-1} \),
\[
M^+(z_0, z_i, t) = T_i^{-1} h^+(z_0, z_i, t), \quad (21)
\]
\[
f_1^-(z_0, z_i, t) = T_i^{-1} h^-(z_0, z_i, t). \quad (22)
\]
Using these definitions in equations (16) and (17) we find a scheme in which the unknown factor \( T_i \) is absent. The bottom panel of Figure 1 shows the solution to equations (16) and (17) with substitution of equations (21) and (22) for our example model with three layers. The example showed that sending a unit amplitude impulse in time at \(-t_d(z_1, z_0)\) and the coda of the scaled downgoing focusing wavefield leads to a reflected signal at \( t_d(z_1, z_0) \), with the desired local reflection coefficient, \( r_1 \), of the interface at depth level \( z_1 \). These scaled downgoing and upgoing wavefields can be recognized from equations (21) and (22) as \( h^+(z_0, z_1, t) \) and \( h^-(z_0, z_1, t) \). By evaluating the time-domain equivalent of equation (10) we find that \( h^-(z_0, z_1, t) = r_1 \delta(t - t_d(z_1, z_0)) \) for \( t_d(z_1, z_0) - \epsilon < t < t_d(z_1, z_0) + \epsilon \), where \( \epsilon \) is an arbitrarily small time instant. Hence, for an arbitrary
depth level \( z_i \) inside a layered medium the upgoing field that arrives at \( t = t_d(z_i, z_0) \) has an amplitude that is equal to the local reflection coefficient of depth level \( z_i \), and \( t_d(z_i, z_0) \) is the known one-way traveltime to depth level \( z_i \). We can therefore take the amplitude of the upgoing field at the focus time as the imaging condition to obtain an image \( I \) containing the local reflection coefficient at the one-way traveltime

\[
I(t_d(z_i, z_0)) = r_i = \int_{t' = t_d(z_i, z_0) - \epsilon}^{t_d(z_i, z_0) + \epsilon} h^-(z_0, z, t')dt'.
\]  

Equations (16)-(17) with substitution of equations (21) and (22) together form the imaging scheme and equation (23) is the imaging condition. This is the most direct route to imaging local primary reflection coefficients and simultaneously eliminating effects from internal multiples. We have not used any knowledge of the layered medium, but we have constructed an image containing the local reflection coefficient as a function of one-way traveltime. In case the acoustic impedance is known in the first layer the impedance could be found as a function of one-way traveltime. With one-dimensional waves at normal incidence there are no independent ways to determine a velocity model. This can be done by using several angles of incidence for plane waves, which is discussed in the next section.

6 PLANE WAVES AT OBLIQUE INCIDENCE

The above results for normal incidence plane waves relied on the fact that the local reflection coefficients are independent of frequency. To include oblique angles a spatial Fourier transformation can be performed on measured data yielding the impulse reflection response in the wave number frequency domain. Newton (1981) used the wave number domain where the wave numbers are just parameters, but then the reflection coefficients become frequency dependent. He therefore continued with the assumption that the wave numbers are small and can be neglected. Coen (1981) showed that in the intercept-time slowness domain slowness is just a parameter representative of the angle of incidence. He had to exclude waves that are evanescent between the depth levels \( z_0 \) and \( z_i \). In our case this assumption is not necessary. We assume that the waves propagate through the two boundaries at \( z_0 \) and at \( z_i \), but they can be evanescent at other depth levels. Only radial slowness, \( \rho \), matters and the local reflection coefficient becomes a function of slowness, denoted \( r_i(\rho) \). The depth level \( z_i \) is still unknown, but the intercept-time is the apparent two-way vertical traveltime as a function of angle and we need the one-way intercept time as the image time.

For depth level \( z_i \) we denote the one-way intercept time as \( t_d(z_i, z_0, p) \). If we define the vertical slowness in layer \( n \) as \( q_n = \sqrt{1/c_n^2 - p^2} \), the intercept time is given by \( t_d(z_i, z_0, p) = \sum_{n=1}^{i} q_n |z_n - z_{n-1}| \). We can write equations (16)-(17) with substitution of equations (21) and (22) as

\[
h^-(z_0, z_i, t, p) = R(z_0, t + t_d(z_i, z_0, p), p) \\
+ \int_{t' = -t_d(z_i, z_0, p)}^{t_d(z_i, z_0, p)} h^+(z_0, z_i, t', p)R(z_0, t - t', p)dt',
\]  

valid for \(-t_d(z_i, z_0, p) < t < t_d(z_i, z_0, p)\). This leads to the angle dependent imaging condition similar to equation (23),

\[
I(t_d(z_i, z_0, p), p) = r_i(p) = \int_{t' = t_d(z_i, z_0, p) - \epsilon}^{t_d(z_i, z_0, p) + \epsilon} h^-(z_0, z_i, t', p)dt',
\]  

The image \( I \) is an image in the intercept-time slowness, or \( \tau - \rho \), domain. For every slowness value the image contains the local reflection coefficients at the corresponding one-way intercept times that we can call image times. Therefore the image is an image gather that can be understood as a "prestack" time image, where "prestack" should be understood in the slowness-domain and each time is the correct one-way traveltime corresponding to a particular slowness value. Having slowness as a free parameter implies that the local reflection coefficients in the image can be converted to velocity and density values in each layer. This would constitute an inversion step and after the velocities are found, the image times can be converted to depth. Much work has been carried out on AVA inversion (e.g. Raz (1981)), but here we have two advantages. The first is that no multiples are part of the image and we can construct local reflection coefficients as a function of angle without needing any subsurface information. The second is that we have a non-recursive scheme and therefore no error accumulation with increasing imaging depth. Imaging followed by inversion assumes that the medium is horizontally layered and that the reflection response can be obtained from the data, which requires knowledge of the source signature, density and velocity in the source layer, and removal of free-surface related multiples (Verschuur et al., 1992; van Dedem and Verschuur, 2005).

6.1 Finite frequency bandwidth

The above algorithm is formulated with impulse response functions, which we don’t have in seismic data. For measured signals we need to deal with the finite frequency bandwidth of the data. Let the finite bandwidth be represented by a zero-phase filter function, or wavelet \( W(t) \), in the time domain, the functions \( \int_{-\infty}^{\infty} \) and the measured reflection response \( \bar{R} \) can be written as the
time convolution of the functions \( f_1^\pm \) and \( R \) as

\[
\tilde{f}_1^\pm(z_0, z, t, p) = \int_{t_0}^{t_0 + t_u} f_1^\pm(z_0, z, t', p)W(t - t')dt',
\]

where \( \tilde{f}_1^\pm(z_0, z, t, p) = 0 \) for \( |t| > t_d^\pm \), \( t_d^\pm = t_d(z_0, z, p) + t_u \), and \( t_u \) denotes the half time window of the wavelet, while \( R \neq 0 \) for \( t > -t_u \). Now a time window has to be chosen instead of a single time instant for the arrival time of the first arrival. We have to take into account the finite bandwidth of the functions and \( \bar{R} \) where \( \bar{R} \) denotes the time convolution of the functions \( f_1^\pm \) and \( R \) as

\[
\bar{R}(z_0, t, p) = \int_{t_0}^{t_0 + t_u} R(z_0, t', p)W(t - t')dt',
\]

where \( \bar{R} = 0 \) for \( t > -t_u \), and \( \bar{R} \neq 0 \) for \( t < -t_u \). Now a time window has to be chosen instead of a single time instant for the arrival time of the first arrival. We have to take into account the finite bandwidth of the functions and \( \bar{R} \) where \( \bar{R} \) denotes the time convolution of the functions \( f_1^\pm \) and \( R \) as

\[
\hat{R}(z_0, t, p) = \int_{t_0}^{t_0 + t_u} R(z_0, t', p)W(t - t')dt',
\]

For the examples we take a medium with 11 layers separated by 10 interfaces. The velocities, densities, and thicknesses of the layers are given in Table 1. The reflection data is computed in the frequency-slowness domain for 36 slowness values corresponding to for 36 angles of incidence from normal incidence to \( \alpha = 35^\circ \). The data are computed using a reflectivity code and are then transformed to the intercept-time slowness domain with source and receivers at the same height of 75 m above the top interface. This is equivalent to modeling a single shot gather in space-time and then transforming the data to the \( \tau - p \) domain. The source signature is a 40 Hz Ricker wavelet. The sixth layer is a high velocity thin layer. The seventh and eight layers have no velocity contrast and the reflection coefficient is determined by the medium that is layered above \( z_1 \) and homogeneous above \( z_2 \). We can therefore write them in a similar form as equation (12), given by

\[
\hat{G}^{-\alpha}(z_1, z_0, t, p) = \int_{t_0}^{t} R(z_i, t - t', p)\hat{G}^{\alpha}(z_1, z_0, t', p)dt'.
\]

From this equation \( R(z_i, t, p) \) is found by deconvolution and the image can be constructed. Another option is to pick a time above a reflection event, compute the reflection response and image the first reflector using inverse wavefield extrapolation. As a direct consequence of our development here we can state that a reflector at depth level \( z_i \) can be imaged using this scheme when the waves propagate through this depth level, while they are allowed to be evanescent between \( z_0 \) and \( z_1 \). It can be understood that signal to noise ratios will determine the accuracy of such images, but in principle our schemes can image interfaces below thin high-velocity layers through which the waves have tunneled. We show a numerical example later, but we can already understand from the above scheme that when \( \mathcal{T}_0 \) contains information on tunneling waves, it can be important to have an accurate time-function estimate of \( \mathcal{T}_0^{-1} \), because it is not updated. This can lead to small errors in \( M^+ \) and \( f_1^+ \) that are not eliminated by the deconvolution procedure for imaging.

### 7 Numerical Examples

For the examples we take a medium with 11 layers separated by 10 interfaces. The velocities, densities, and thicknesses of the layers are given in Table 1. The reflection data is computed in the frequency-slowness domain for 36 slowness values corresponding to for 36 angles of incidence from normal incidence to \( \alpha = 35^\circ \). The data are computed using a reflectivity code and are then transformed to the intercept-time slowness domain with source and receivers at the same height of 75 m above the top interface. This is equivalent to modeling a single shot gather in space-time and then transforming the data to the \( \tau - p \) domain. The source signature is a 40 Hz Ricker wavelet. The sixth layer is a high velocity thin layer. The seventh and eight layers have no velocity contrast and the reflection coefficient of that interface is independent of incidence angle. The eighth
and ninth layers have no density contrast. The incidence angle of 31.6° in the first layer becomes critical in the high-velocity thin layer, which amounts to a critical angle of 40.25° at the top of the thin layer. At normal incidence the fast thin layer is just over one fifth of the wavelength at 40 Hz. We solve equations (29) and (30) for each of the 36 angles of incidence using the iterative scheme similar to the one described in equations (19) and (20) in which we use $p = \sin(\alpha)/c_0$, $c_0$ being the velocity in the first layer. We use two different schemes solving equations (29) and (30). First we use the direct imaging method using the initial estimate of equation (18) and equations (21) and (22), all three extended as functions of $p$, in combination with imaging condition of equation (26). Secondly, we use the imaging by multidimensional deconvolution method, which requires an initial estimate given by equation (31). Once the $f^r_{i1}$ functions are determined the Green’s functions are computed using equations (13) and (14), both extended as functions of $p$, from which the reflection response is computed using equation (32) and the image is constructed by standard wavefield extrapolation and imaging. Angles reported in results and figures below refer to the incidence angle in the first layer.

7.1 Imaging directly using equation (26)

We solve equations (29) and (30), but the first term in the right-hand side of equation (29) is replaced by $R(z_0, t + t_d(z, z_0, p))$ and we use equations (21) and (22), extended as functions of $p$, in combination with the imaging condition of equation (26). The iterations stop when

$$\sqrt{\sum_{m} (\bar{f}_{1,n}(z_0, z_i, m\Delta t, p) - \bar{f}_{1,n-1}(z_0, z_i, m\Delta t, p))^2} \leq \sqrt{\sum_{m} (\bar{f}_{1,n}(z_0, z_i, m\Delta t, p))^2}/1000,$$

in which $n$ is the iteration number and the discrete time steps are given by $t = m\Delta t$, $\Delta t$ being the time step. For angles up to 25° the average number of iterations slowly rises from 8 at normal incidence to 11 at 25°, between 26° and 31° the number of iterations rises from 12 to 17, and the scheme needs 21, 24, 28, and 31 iterations for the last four angles where the first is just below and the last three are beyond the critical angle for the high-velocity thin layer. The data and the resulting image are shown in the intercept time-slowness domain where slowness has been converted to incidence angle in Figures 7 and 8. Figure 7 shows that it is hard to discriminate multiples from primary reflections from 0.5 s onward. In Figure 8 the expected model reflection coefficients are shown in black solid lines and the image amplitudes are shown in red dashed lines. From the figure it can be seen that all multiple energy has been eliminated while all primary reflections are imaged to their image times. The amplitudes and wave shapes of the first four reflectors are constructed almost exactly. For later reflectors two types of errors occur. For all these arrivals the phase changes due to thin layer effects, because multiples arrive within the time window of the Ricker wavelet. Tunneling effects of waves transmitted through the fast thin layer are visible in the last three traces.

To see the effect of the thin layer and of tunneling we zoom in and take a look at the image results of the first and last traces from these data. Figure 9 shows the normal incidence image as a function of image time. To
create this image only the solutions to equations (29) and (30) were computed, which requires no information other than the earth reflection response and the wavelet. We observe that the image inside the fast thin layer is quite accurate although the amplitude has a very small error, which can only be seen when the image is enlarged. The images of the last four interfaces show some phase changes in the wavelet and the maximum amplitude is not entirely correct. This is caused by the fact that in this scheme the initial estimate is a scaled and band limited delta-function, which ignores thin layer multiples overlapping with the first arrival. This is an effect of the finite bandwidth and is a resolution issue. Still the amplitudes of the last four reflectors are quite accurate and we would need to zoom in much further to make the errors visible. The obtained reflection coefficient of the eighth interface, e.g., has an amplitude error of just over 1% and zero timing error.

Figure 10 shows the image from the data with a plane wave at 35° incidence. For this angle of incidence the waves tunnel through the fast thin layer and all waves below this thin layer propagate again. As it can be seen from the amplitude mismatch at 0.2 s, inside the thin layer the imaging scheme does not give the correct amplitude, because waves that are evanescent at the imaging level are not properly treated by equation (25). For depth levels below the thin layer, the remaining four interfaces can still be imaged. Because tunneling waves show a phase shift due to the fact that the reflection coefficients of the top and bottom interfaces of the tunneling layer are complex, the image times and amplitudes are incorrect. If we look again at interface eight, the obtained reflection coefficient has an amplitude error of 3% and a timing error of 2.5 ms. This result can be improved by using a more accurate initial estimate and solve equations (29) and (30). Then we compute the up- and downgoing Green’s functions and create the image by deconvolution using equation (32). We show results of this approach below.

7.2 Imaging by MDD using equation (32)

To correctly image inside and below thin layers it can be necessary to include some multiples in the initial estimate of $T_d$ because this signal is not updated by the Marchenko-type scheme. Incorporating a good estimate of $T_d$ in the scheme also allows for imaging interfaces below layers through which the waves have tunneled. To show the improvement of incorporating multiples in thin layers and the effect of tunneling we take a close look at the eighth interface, which is located 776 m below the source. We select incidence angle dependent focus times at half the intercept times in the middle of the eighth layer. For this particular focus time we compute the first arrival of the transmission event within the time window of the Ricker wavelet around the focus time. When the model is unknown a smooth background model can be
constructed in the same way as it is normally done for migration. This background model can then be used to generate the initial estimate. With this initial estimate we solve equations (29) and (30), compute the directional Green’s functions of the angle dependent equivalents of equations (13) and (14). From those we construct the reflection response of the medium below the focus time with the aid of equation (32) and compute the time image by inverse wavefield extrapolation. We converted focus time to depth for display purposes. The result is given in Figure 11, which shows the angle dependent local reflection coefficient image as a function of angle versus depth for the two imaging schemes.

The red dashed lines are the image events obtained by starting with the correct first arrival followed by MDD and imaging. The blue dash-dotted lines are taken from the direct image of Figure 8 converted to depth. The black lines are obtained from modeling the expected primary event with the amplitude of the local reflection coefficient. Notice that the timing errors (displayed as depth errors) in the direct image result start to be visible around 25° where the wavelength at 40 Hz is almost 9 times the thickness of the thin layer. For all angles of incidence the thin layer effect has disappeared when the correct first arrival is used in combination with MDD. The reflection coefficient amplitude as a function of angle is shown in Figure 12 for the maximum amplitudes obtained from the time images that are displayed as depth images in Figure 11. The black dashed and dash-dotted lines give the normalized errors of the two numerical results shown in the dashed red and dash-dotted blue lines, respectively. The direct image, obtained with a scaled delta-function, is also very accurate for small angles of incidence. The error increases for angles larger than 20° due to neglecting the effect of the thin layer on the finite resolution result in solving the Marchenko-type equations. Still the image amplitude errors remain well below 5%, whereas the deconvolution image shows errors around 1%. For comparison the additional green solid line shows the amplitude obtained by standard one-way migration using the correct background velocity model. The large difference with the true values is caused by transmission and internal multiple effects.

8 DISCUSSION

The direct image is obtained by using a delta-function as initial estimate for the transmitted first arrival at the focus time. This has the advantage that the time-image can be formed without any knowledge of the subsurface. It has the disadvantage that thin layer effects and the effects of waves that tunnel above the image depth are not always properly accounted for. This can create small amplitude errors in the images of interfaces below thin layers, but it can create substantial time-errors when the waves have tunneled through a thin layer above the image depth. This is not a particular drawback of the method because no imaging scheme based on one-way propagation handles evanescent waves properly. In principle the constructed time image can be converted to depth by an inversion step that would compute the density and velocity from the obtained reflection coefficients as a function of incidence angle. In practice this inversion can be quite difficult, because it is a non-linear process. Once the velocity of each layer is known time to depth conversion can be performed to construct the final depth image.

The construction of an accurate estimate of the direct transmission event at the focus time can be necessary for obtaining an image that is better than the image obtained with a delta-function as first arrival. This can be done in the same way that is used in standard migration schemes by estimating a background model and use that to compute the first arrival of the transmission response. The importance of retrieving the Green’s functions is twofold. First the retrieved Green’s functions are the downward continued wavefields from the measurement surface to the interior. These are retrieved from the measured reflection response without any modeling. Second, since the theory gives the upgoing and downgoing Green’s function in the interior, one can immediately use these for imaging. This can be achieved by multidimensional deconvolution (van der Neut et al., 2011).

This is just a first step toward a new scheme for acoustic data imaging and possibly inversion. The present scheme is formulated in the intercept timeslowness domain, but it could have been formulated in space-time domain. The product of the reflection response and the upgoing and downgoing focusing functions as a function of slowness would become two-dimensional convolutions in space over the horizontal coordinates of the receiver plane. The time instant of the direct arrival would become a function of each point...
on the receiver plane to the focusing point in the subsurface. We would lose the advantage of being able to solve for the focusing functions per slowness value and for that reason we choose this option here.

Several open questions remain for further study, such as the effect of inaccuracies in the determined source wavelet, in the initial estimate, the effect of noise in the recorded data, and the effect of intrinsic losses in the earth. It is also important that this method can be generalized to 3D by combining the 3D method presented in Wapenaar et al. (2013a) and this scheme. This is not difficult in theory, but it will also present new challenges that are beyond the scope of the present study. We did see that the results are not exact when the image depth is at locations where the waves did not propagate and this will occur in a 3D setting in a more complicated way than in a 1D model. A second aspect is that a direct arrival time must be estimated for every receiver point on the surface to the focusing point, putting more conditions on the background model and complications may occur in strongly lateral heterogeneous models. This does not necessarily require more work than is done to build a background model for standard migration. Computing the total focusing wavefield from a similar Marchenko equation has been shown to work with computed 2D acoustic data (Wapenaar et al., 2013b; van der Neut et al., 2013).

9 CONCLUSIONS

We present an algorithm to compute the up- and downgoing parts of a focusing wavefield from the measured reflection response. This is done under the assumption of a plane wavefield and a 1D earth model. The focusing wavefield is then used together with the measured reflection response to compute the up- and downgoing VSP Green’s functions representations. These two relations can be derived from the reciprocity theorems of the time-convolution and time-correlation types. The focusing wavefield focuses at the depth where the VSP Green’s function has its virtual source. The up- and downgoing focusing wavefields are non-zero in a finite time window where the Green’s functions are zero. Therefore the focusing wavefields can be obtained from the reflection response by solving the resulting two coupled Marchenko-type equations.

We have shown that the downgoing focusing wavefield in the upper half space is the inverse of the transmission response. The upgoing focusing wavefield is the reflection response to the downgoing wavefield in a model that is the same as the earth from the surface to the focus depth, but is homogeneous below this focus depth. Once the two focusing wavefields are found from finite frequency bandwidth reflection data, a virtual VSP Green’s function can be computed. We have shown that a subsurface image free from effects of multiple reflections in the data can be found directly in the upgoing part of the focusing wavefield or from performing multidimensional deconvolution on the VSP Green’s functions.

The direct imaging method produces an image with accurate amplitudes, but small timing errors can occur for reflectors below a thin layer. In our example this occurred when the layer thickness is less than one-eighth of the dominant wave length or when waves have tunneled through the thin layer. In that case an improved initial estimate leads to a correct image using the MDD method.

APPENDIX A: APPENDIX A

WAVEFIELD FOCUSING AND GREEN’S FUNCTION REPRESENTATIONS

A1 Wavefields in a medium with three interfaces

Let us look at the expressions for the reflection response $R(z_0, t)$ for a source and receiver at $z_0$ and the transmission response $T^+(z_2, z_0, t)$ for a source at $z_0$ and a receiver at $z_2$. In the frequency domain the corresponding reflection $\hat{R}(z_0, \omega)$ and transmission $\hat{T}^+(z_2, z_0, \omega)$ responses generated by the unit amplitude plane wave are
given by (Goupillaud, 1961)

\[
\hat{R}(z_0, \omega) = \frac{r_0 + r_1 e^{-2i\omega t_1} + r_0 r_1 r_2 e^{-2i\omega t_2}}{1 + r_0 r_1 e^{-2i\omega t_1} + r_0 r_2 e^{-2i\omega (t_1 + t_2)} + r_1 r_2 e^{-2i\omega t_2}},
\]

(A1)

where we notice that the denominators are the same for the reflection and transmission responses. Comparing the events in the reflection response shown in Figure A1 with the numerator of the reflection response of equation (A1) we can see that they are the same. We conclude that sending in the inverse of the transmission response of equation (A2) is exactly the necessary wavefield that focuses to a unit amplitude at \(z_2\) and in the time domain at \(t = 0\). In the frequency domain the focusing wavefield is therefore given by

\[
\hat{f}_+^+(z_0, z_2, \omega) = \frac{1}{\hat{T}_+^+(z_0, z_2, \omega)} e^{i\omega(t_1 + t_2)}
\]

\[
\times \frac{1 + r_0 r_1 e^{-2i\omega t_1} + r_0 r_2 e^{-2i\omega (t_1 + t_2)} + r_1 r_2 e^{-2i\omega t_2}}{r_0 \tau_1 \tau_2^+}.
\]

(A3)

The four incident and four reflected wavefields are indicated in Figure A1.

A2 Derivation of the Green’s function representations

Equation (10) can be derived from the reciprocity theorem of the time-convolution type and equation (11) can be obtained from the reciprocity theorem of the time-correlation type. For more information about acoustic reciprocity theorems, see de Hoop (1995). In one-dimensional space and for a source free domain that is bounded by two depth levels, \(z_0\) and \(z_1\), they are given in the frequency domain by

\[
\hat{p}_A(z_0, \omega) \hat{v}_{z,B}(z_0, \omega) - \hat{p}_B(z_0, \omega) \hat{v}_{z,A}(z_0, \omega) = \hat{p}_A(z_1, \omega) \hat{v}_{z,B}(z_1, \omega) - \hat{p}_B(z_1, \omega) \hat{v}_{z,A}(z_1, \omega),
\]

\[\text{A5}\]

\[
\hat{p}_A(z_0, \omega) \hat{v}_{z,B}(z_0, \omega) + \hat{p}_B(z_0, \omega) \hat{v}_{z,A}(z_0, \omega) = \hat{p}_A^*(z_1, \omega) \hat{v}_{z,B}(z_1, \omega) + \hat{p}_B^*(z_1, \omega) \hat{v}_{z,A}(z_1, \omega),
\]

\[\text{A6}\]

where the subscripts \(A\) and \(B\) refer to two independent states and \(\hat{v}_z(z, \omega)\) denotes the particle velocity related to pressure as \(\hat{v}_z(z, \omega) = -i(\omega \rho(z))^{-1} \partial_z \hat{p}(z, \omega)\). We have assumed that no sources are present for \(z_0 \leq z \leq z_1\) and that the media between the two levels are identical in the two states. Equation (A5) is generally valid under these assumptions, whereas equation (A6) is a correct equation under the additional assumption that the medium between the two depth levels is dissipation free. The pressure is the sum of down- and upgoing wavefields as \(\hat{p}(z, \omega) = \hat{p}_+^*(z, \omega) + \hat{p}_-^*(z, \omega)\) and the particle velocity can be written as \(\hat{v}(z, \omega) = -(\omega \rho(z))^{-1} \partial_z \hat{p}(z, \omega)\). Substituting these decompositions in equations (A5) and (A6) and following a similar analysis as in Wapenaar and Berkholz (1989) gives

\[
\rho_0^{-1} \lim_{z \rightarrow z_0} (\hat{p}_A^*(z) \partial_z \hat{p}_B^*(z) + \partial_z \hat{p}_A^*(z_0) \partial_z \hat{p}_B^*(z)) = -\rho_0^{-1} \lim_{z \rightarrow z_1} (\hat{p}_B^*(z) \partial_z \hat{p}_A^*(z) + \partial_z \hat{p}_B^*(z_1) \partial_z \hat{p}_A^*(z)),
\]

\[\text{A7}\]

\[
\rho_0^{-1} \lim_{z \rightarrow z_0} ((\hat{p}_A^*(z))^{\ast} \partial_z \hat{p}_B^*(z) + (\hat{p}_B^*(z))^{\ast} \partial_z \hat{p}_A^*(z)) = -\rho_0^{-1} \lim_{z \rightarrow z_1} ((\hat{p}_B^*(z))^{\ast} \partial_z \hat{p}_A^*(z) + (\hat{p}_A^*(z))^{\ast} \partial_z \hat{p}_B^*(z))^{\ast},
\]

\[\text{A8}\]

where we have omitted \(\omega\) in the arguments for brevity. We use equations (A5) and (A6) to a configuration in which state \(A\) corresponds to the medium that is homogeneous below \(z_i\), while state \(B\) is the actual medium. In state \(A\) we use the functions \(\hat{f}_i^+(z, z_1, \omega)\) that focus just below depth level \(z_1\). Below the depth level \(z_1\) the focusing wavefield is a unit amplitude downgoing wave that can be written as \(\hat{f}_i^+(z, z_1, \omega) = \exp[-i\omega(z_1 - z)/c_{i1}]\) and because there is no upgoing wave we have \(\hat{f}_i^+(z, z_1, \omega) = 0\). We then find

\[
\hat{p}_A^+(z_0, \omega) = \hat{f}_i^+(z_0, z_1, \omega),
\]

\[\text{A9}\]

\[
\lim_{z \rightarrow z_1} \partial_z \hat{p}_A^+(z, \omega) = -i\omega/c_{i1}, \quad \lim_{z \rightarrow z_1} \partial_z \hat{p}_A^-(z, \omega) = 0.
\]

(A10)

In state \(B\) we take the actual configuration with a unit amplitude incident wave above the depth level \(z_0\) given by \(\hat{p}_B^+(z, \omega) = \exp[-i\omega(z - z_0)/c_0]\) and the reflection response is given by \(\hat{p}_B^-(z, \omega) = \hat{R}(z_0, \omega) \exp[i\omega(z - z_0)/c_0]\). We find

\[
\lim_{z \rightarrow z_0} \partial_z \hat{p}_B^+(z, \omega) = -i\omega/c_0,
\]

(A11)

\[
\lim_{z \rightarrow z_0} \partial_z \hat{p}_B^-(z, \omega) = i\omega \hat{R}(z_0, \omega)/c_0,
\]

(A12)

\[
\hat{p}_B^+(z_1, \omega) = \hat{G}^{\pm,q}(z_1, z_0, \omega),
\]

(A13)

where just below depth level \(z_1\) the wavefield is given by the up- and downgoing field Green’s functions. Notice that here the sign in the superscript of the Green’s function relates to upgoing and downgoing waves just below the level \(z_1\), which is the receiver level for this Green’s function. The superscript \(\hat{+}\) indicates the wavefield is generated by a monopole source just above \(z_0\). Substituting these choices in equation (A7) and (A8)
leads to
\[
\frac{Z_0}{Z_{i+1}} \hat{G}^{-\theta}(z_i, z_0, \omega) = \hat{R}(z_0, \omega) \hat{f}_1^+(z_0, z_1, \omega) - \hat{f}_1^-(z_0, z_1, \omega),
\]
\[
\frac{Z_0}{Z_{i+1}} \hat{G}^{+\theta}(z_i, z_0, \omega) = [\hat{f}_1^+(z_0, z_1, \omega)]^* - \hat{R}(z_0, \omega)[\hat{f}_1^-(z_0, z_1, \omega)]^*,
\]
where the impedance is given by \( Z_i = \rho_i c_i \). Equations (A14) and (A15) are the wavefield retrieval equations for the scaled up- and downgoing wavefields, respectively, at the receiver level \( z_i \) that is obtained from the measured reflection response and the functions \( \hat{f}_1^\pm \). Equation (A14) is equal to equation (10) and equation (A15) is equal to equation (11), because the Green’s functions satisfy the reciprocity relation
\[
Z_0 \hat{G}^{+\theta}(z_i, z_0, \omega) = Z_{i+1} \hat{G}^{-\theta}(z_0, z_1, \omega),
\]
where \( \hat{G}^{+\theta} \) are the Green’s functions of equations (10) and (11). The reason for the factors \( Z_0 \) and \( Z_{i+1} \) is that the Green’s functions and the focusing functions have been defined according to the transmission responses and the downgoing and upgoing pressure transmission responses satisfy
\[
Z_0 \hat{T}^\theta(z_i, z_0, \omega) = Z_{i+1} \hat{T}^{-\theta}(z_0, z_1, \omega).
\]
By adding equations (A14) and (A15) and using reciprocity of equation (A16) we find an expression for the Green’s function corresponding to the virtual VSP as
\[
\hat{G}(z_0, z_1, \omega) - [\hat{f}_2(z_0, z_1, \omega)]^* = \hat{R}(z_0, \omega) \hat{f}_2(z_1, z_0, \omega).
\]
Equation (A17) is the 1D equivalent of equation 9 in Wapenaar et al. (2013a) with \( \hat{f}_2 = \hat{f}_2^+ + \hat{f}_2^- = \hat{f}_1^+ - [\hat{f}_1^-]^* \). Without going into the details of this relation, the equation shows that if we send in the focusing wavefield \( \hat{f}_2(z_0, z_1, t) \) the response is equal to minus the time-reverse of this function \( \hat{f}_2(z_1, z_0, -t) \) and the VSP Green’s function.

Plane waves at oblique angles of incidence can be obtained by transforming the wavefields from space-time domain to the intercept-time slowness domain and the radial slowness \( p \) becomes a parameter. For each slowness value equation (A7) remains valid and therefore also equation (A14) remains valid. Equation (A8) and therefore equation (A15) remain valid under the condition that the waves propagate through the depth levels \( z_0 \) and \( z_1 \), while they can be evanescent at depth levels between \( z_0 \) and \( z_1 \).

From this result it would be a small step to deriving space-time relations for media with three-dimensional variations in velocity and density, but this is beyond the scope of the current paper. Examples of such schemes and results on data modeled for two-dimensional heterogeneous subsurface models can be found in Wapenaar et al. (2014) and Broggiini et al. (2014).

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Marchenko imaging

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ABSTRACT
Traditionally, the Marchenko equation forms a basis for 1-D inverse scattering problems. A 3-D extension of the Marchenko equation enables the retrieval of the Green’s response to a virtual source in the subsurface from reflection measurements at the Earth’s surface. This constitutes an important step beyond seismic interferometry. Whereas seismic interferometry requires a receiver at the position of the virtual source, for the Marchenko scheme it suffices to have sources and receivers at the surface only. The underlying assumptions are that the medium is lossless and that an estimate of the direct arrivals of the Green’s function is available. The Green’s function retrieved with the 3-D Marchenko scheme contains accurate internal multiples of the inhomogeneous subsurface. Using source-receiver reciprocity, the retrieved Green’s function can be interpreted as the response to sources at the surface, observed by a virtual receiver in the subsurface. By decomposing the 3-D Marchenko equation, the response at the virtual receiver can be decomposed into a downgoing and an upgoing field. By deconvolving the retrieved upgoing field with the downgoing field, a reflection response is obtained, with virtual sources and virtual receivers in the subsurface. This redatumed reflection response is free of spurious events related to internal multiples in the overburden. The redatumed reflection response forms the basis for obtaining an image of a target zone. An important feature is that spurious reflections in the target zone are suppressed, without the need to resolve first the reflection properties of the overburden.

Keywords: autofocusing, multiples, imaging

INTRODUCTION
The Marchenko equation has since long been used by mathematical physicists as a basis for 1-D inverse scattering theory (Marchenko, 1955; Lamb, 1980; Burridge, 1980; Ge, 1987; Chadan and Sabatier, 1989). It relates the reflection response, measured on one side of a lossless 1-D medium, to a field inside that medium, which, in turn, is related to the scattering potential in the medium. Inverse scattering methods derived from the Marchenko equation fully account for internal multiple scattering. Recently, Broggini and Snieder (2012) discussed an interesting link between the Marchenko equation and seismic interferometry. They showed that, by using the Marchenko equation, the Green’s function between an arbitrary virtual-source position inside the 1-D medium and a receiver at the surface can be retrieved from the reflection response measured at the surface of that medium. This constitutes an important step beyond seismic interferometry. To retrieve a Green’s function with seismic interferometry, the 1-D medium should be illuminated from both sides and a physical receiver should be present inside the medium, at the position of the virtual source. For example, the 1-D version of the virtual-source method proposed by Bakulin and Calvert (2006) requires a receiver in a borehole, illuminated from above and below Curtis et al. (2006); the fact that in practice the illumination occurs only from above implies that the retrieved Green’s function contains spurious multiples Snieder et al. (2006). In contrast to this, the scheme proposed by Broggini and Snieder (2012) requires no physical receivers inside the medium, and
illumination from one side suffices. Nevertheless, the retrieved Green’s function contains accurate internal multiple reflections. A restriction is that the virtual-source position is not defined in depth but in one-way traveltime. Hence, whereas the Green’s function is retrieved from the reflection response alone (i.e., without any knowledge about the medium), determining the position of the virtual source requires velocity information to perform time-depth conversion (a smooth velocity model usually suffices for this purpose).

We have recently extended the 1-D approach of Broginni and Snieder (2012) to three dimensions Wapenaar et al. (2012, 2014). To this end we derived a 3-D Marchenko equation, which relates the reflection response at the surface of a lossless 3-D inhomogeneous medium to a field inside that medium. This field is called the “focusing function”. It is obtained from the reflection response by solving the 3-D Marchenko equation via an iterative procedure. Subsequently, the Green’s function between a virtual source inside the medium and receivers at the surface is retrieved from the reflection response and the focusing function. Assuming the reflection response at the surface is well sampled, this recovered Green’s function properly contains the internal multiples of the 3-D inhomogeneous medium. As in the 1-D case, no physical receiver is required at the position of the virtual source. Apart from the reflection response at the surface, the method requires an estimate of the direct arrivals (including triplications in case of multipathing) between the virtual-source position and the receivers at the surface. This requirement replaces the need for time-depth conversion in the 1-D scheme.

In this paper we briefly review the 3-D Marchenko equation and the method for retrieving the Green’s function from the reflection response at the surface and an estimate of the direct arrivals. We interpret the retrieved Green’s function as the response to sources at the surface, observed by a virtual receiver in the subsurface (which, via reciprocity, is the same as the response to a virtual source in the subsurface, observed by receivers at the surface). Next we discuss decomposition of the Marchenko equation and show how this can be used to retrieve the downgoing and upgoing parts of the Green’s function at a virtual receiver in the subsurface, in response to sources at the surface. These retrieved downgoing and upgoing Green’s functions in the subsurface form the basis for redatuming and imaging. Unlike standard redatuming and imaging, which uses approximations of the downgoing and upgoing wave fields in the subsurface, our new method uses true downgoing and upgoing wave fields (i.e., Green’s functions, including accurate internal multiple reflections). This implies that the internal multiples contribute to imaging the reflectivity and that the obtained image will be free of spurious events related to the internal multiples. We illustrate this with some simple examples. In a companion paper Broginni et al. (2014) we discuss various aspects of this new imaging approach in more detail, including its sensitivity to errors in the estimated direct arrivals.

**GREEN’S FUNCTIONS AND FOCUSING FUNCTIONS**

We start by reviewing Green’s functions, focusing functions, and their mutual relations in a 3-D inhomogeneous medium (the details of the derivation are given in Appendix A). Consider the configuration depicted in Figure 1. It consists of a 3-D inhomogeneous lossless half-space below a transparent boundary ∂D₀ and a homogeneous half-space above that boundary. Spatial coordinates are denoted as $x = (x₁, x₂, x₃)$, with $x₃$ positive in the lower half-space and negative in the upper half-space. Boundary $\partial D₀$ is defined as $x₃ = x₃₀ = 0$. For convenience, coordinates at $\partial D₀$ are denoted as $x₀ = (x₁₀, x₃₀)$, with $x₁₀ = (x₁₀, x₂)$. Similarly, coordinates at an arbitrary depth level $\partial Dₐ$ are denoted as $xₐ = (x₁ₐ, x₃ₐ)$, where $x₃ₐ$ denotes the depth of $\partial Dₐ$.

**Green’s functions**

We define the Green’s function $G(x, x''ₐ, t)$ as the causal solution of the scalar wave equation in the actual inhomogeneous medium, with a source at $x''ₐ$, according to

$$\rho \nabla \cdot \left( \frac{1}{\rho} \nabla G \right) - \frac{\partial^2 G}{\partial t^2} = -\rho \delta(x - x''ₐ) \frac{\partial \delta(t)}{\partial t}. \quad (1)$$

Here $c = c(x)$ and $\rho = \rho(x)$ are the propagation velocity and mass density of the inhomogeneous medium and $t$ denotes time. Defined in this way, the Green’s function $G(x, x''ₐ, t)$ is the response (in terms of acoustic pressure) to an impulsive point source of volume injection rate at $x''ₐ$, observed at $x$ (de Hoop, 1995). In the following, $x''ₐ$ is chosen just above $\partial D₀$, see Figure 1. Hence, $x''ₐ = (x₁₀, x₃₀ - \epsilon)$, with $\epsilon \rightarrow 0$. The Green’s function is decomposed into downgoing and upgoing fields, which are mutually coupled by the inhomogeneities of the medium below $\partial D₀$. The downgoing and upgoing
components at observation point \( \mathbf{x} \) are represented by \( G^+(\mathbf{x}, \mathbf{x}_o^+, t) \) and \( G^-(\mathbf{x}, \mathbf{x}_o^-, t) \), respectively. Assuming the one-way wave fields are pressure-normalized, the two-way Green’s function is simply defined as the superposition of the downgoing and upgoing fields, according to
\[
G(\mathbf{x}, \mathbf{x}_o^+, t) = G^+(\mathbf{x}, \mathbf{x}_o^+, t) + G^-(\mathbf{x}, \mathbf{x}_o^-, t),
\]
for any \( \mathbf{x} \) at or below \( \partial \mathcal{D}_0 \).

The vertical derivative of the downgoing Green’s function at \( \partial \mathcal{D}_0 \) (just below the source point) is given by
\[
\partial_x G^+(\mathbf{x}, \mathbf{x}_o^+, t)|_{x_3=0} = -\frac{1}{2} \rho(\mathbf{x}_o^+) \delta(\mathbf{x}_H - \mathbf{x}_o^+) \frac{\partial \delta(t)}{\partial t},
\]
where \( \partial_x \) stands for \( \partial/\partial x_3 \). The vertical derivative of the upgoing Green’s function at \( \partial \mathcal{D}_0 \) is related to the pressure-normalized reflection response of the inhomogeneous medium below \( \partial \mathcal{D}_0 \), according to
\[
\partial_x G^-(\mathbf{x}, \mathbf{x}_o^-, t)|_{x_3=0} = \frac{1}{2} \rho(\mathbf{x}_o^-) \frac{\partial R^-(\mathbf{x}_o^-, \mathbf{x}_o^-, t)}{\partial t},
\]
(Appendix A, equation A-6). Here the superscript \( \cup \) denotes that \( R^-(\mathbf{x}_o^-, \mathbf{x}_o^-, t) \) is the reflection response to downgoing waves of the medium below depth level \( x_3=0 \). This is to be distinguished from \( R^-(\mathbf{x}_j', \mathbf{x}_j, t) \), introduced in the section “Imaging from below”, which is the reflection response to upgoing waves of the medium above depth level \( x_3 \).

### Focusing functions

Next we define so-called focusing functions in a reference configuration, see Figure 2 (Wapenaar et al., 2014; Slob et al., 2014). The reference medium is identical to the actual medium above depth level \( \partial \mathcal{D}_1 \), and reflection-free below this depth level. First consider the focusing function \( f_1(\mathbf{x}, \mathbf{x}_i^+, t) \) in Figure 2a. Here \( \mathbf{x}_i^+ = (x_H, x_i, t) \) denotes a focal point at lateral position \( x_H \) anywhere on \( \partial \mathcal{D}_1 \), whereas \( \mathbf{x} \) represents an observation point anywhere in the medium. Analogous to equation 2, the focusing function is written as the superposition of (mutually coupled) pressure-normalized downgoing and upgoing components at observation point \( \mathbf{x} \), according to
\[
f_1(\mathbf{x}, \mathbf{x}_i^+, t) = f_1^+(\mathbf{x}, \mathbf{x}_i^+, t) + f_1^-(\mathbf{x}, \mathbf{x}_i^+, t).
\]
The focusing function \( f_1(\mathbf{x}, \mathbf{x}_i^+, t) \) is defined such that it focuses at \( x_H = x_H^+ \) at depth level \( \partial \mathcal{D}_1 \) and continues as a diverging downgoing field \( f_1^+(\mathbf{x}, \mathbf{x}_i^+, t) \) into the reflection-free reference half-space \( x_3 \geq x_3^+ \). Formally, the focusing condition is defined as (analogous to equation 3)
\[
\partial_t f_1^+(\mathbf{x}, \mathbf{x}_i^+, t)|_{x_3=x_3^+} = -\frac{1}{2} \rho(\mathbf{x}_i^+) \delta(\mathbf{x}_H - \mathbf{x}_H^+) \frac{\partial \delta(t)}{\partial t}.
\]
Similarly, the focusing function \( f_2(\mathbf{x}, \mathbf{x}_o^-, t) \) in Figure 2b is defined as a superposition of downgoing and upgoing components, according to
\[
f_2(\mathbf{x}, \mathbf{x}_o^-, t) = f_2^+(\mathbf{x}, \mathbf{x}_o^-, t) + f_2^-(\mathbf{x}, \mathbf{x}_o^-, t).
\]
It focuses at \( x_H = x_H^- \) at depth level \( \partial \mathcal{D}_0 \) and continues as a diverging upgoing field \( f_2^+(\mathbf{x}, \mathbf{x}_o^-, t) \) into the homogeneous half-space \( x_3 \leq x_3^+ \). This condition is formally defined as
\[
\partial_t f_2^-(\mathbf{x}, \mathbf{x}_o^-, t)|_{x_3=x_3^+} = \frac{1}{2} \rho(\mathbf{x}_o^-) \delta(\mathbf{x}_H - \mathbf{x}_H^-) \frac{\partial \delta(t)}{\partial t}.
\]

### Example of a focusing function

To provide some insight in the properties of the focusing functions, here we present a somewhat intuitive preview of the focusing function \( f_1^+(\mathbf{x}, \mathbf{x}_i^+, t) \) (the derivation is discussed later). Figure 3a shows a 2-D inhomogeneous subsurface model (the different colors represent different propagation velocities). The red rays represent a wave field emitted from the surface \( \partial \mathcal{D}_0 \) into the subsurface, with the aim to focus at \( \mathbf{x}_i^+ \) at \( \partial \mathcal{D}_1 \) and at \( t = 0 \). The initial estimate of this focusing wave field at \( \partial \mathcal{D}_0 \), \( f_1^+(\mathbf{x}_0, \mathbf{x}_i^+, t) \), is obtained by time-reverting the
response at $\partial D_0$ to a point source at $x'$. It is shown in grey-level display in Figure 3b. We let this initial estimate propagate through the medium and evaluate its response at $\partial D_i$. Its vertical derivative at $\partial D_i$ (conform equation 6) is shown in Figure 3c. We observe a focus at the lateral position of the point source at $t = 0$ and a number of events at positive times. Note that the display in Figure 3c is clipped to enhance these events; as a consequence the artefacts of the focus around $t = 0$ are also enhanced. The events at positive times are a result of multiple reflections, see Figure 3d. These extra arrivals violate the focusing condition of equation 6, which states that $\partial_x f_i^1(x, x', t)$ evaluated at $\partial D_i$ should reveal a focus only.

Figure 4a shows a ray diagram (in red) of the actual focusing wave field $f_i^1(x, x', t)$. The blue rays represent the reflected field $f_i^1(x, x', t)$. The additional red rays reach the interfaces at the same time as the upgoing blue rays, in such a way that they compensate for the downward reflected red rays in Figure 3d. Figure 4b shows the actual focusing wave field at $\partial D_0$ and Figure 4c the vertical derivative of its response at $\partial D_i$. Note that in the latter figure there are no undesired events at positive times; only the focus remains. The same clipping factor has been applied as in Figure 3c, hence, the focusing artefacts are again enhanced. The inset in Figure 4c shows a cross-section of the focus at the central frequency (20 Hz). It is approximately a sinc-function, with the zero-crossings of the main lobe separated 125 m, which is close to $\lambda/\sin \alpha_{\max}$, where $\lambda$ is the wavelength (2300/20 = 115 m) and $\alpha_{\max}$ the maximum propagation angle (64 degrees). This confirms that the focus represents a spatially band-limited delta-function (resulting from the absence of evanescent waves and large angles of incidence).

We emphasize that, although in this example we propagated the focusing function through the exact subsurface model in order to evaluate its focusing properties, we do not need this exact model to estimate the focusing function. We show later that the focusing functions can be derived from the measured reflection response at the surface and an estimate of the direct arrivals between the focus position and the receivers at the surface. For the estimation of the direct arrivals, a smooth subsurface model suffices.
Relations between Green’s functions and focusing functions

At depth levels $\partial \mathbb{D}_0$ and $\partial \mathbb{D}_1$, the one-way focusing functions are mutually related, according to

$$f_1^+(x_0', x_1', t) = f_2^-(x_1', x_0', t)$$

and

$$-f_1^-(x_0', x_1', -t) = f_2^+(x_1', x_0', t),$$

respectively (Appendix A, equations A-9 and A-10). The one-way Green’s functions at depth level $\partial \mathbb{D}_1$ (Figure 1), the focusing functions $f_1^+$ at depth level $\partial \mathbb{D}_0$ (Figure 2a), and the reflection response at depth level $\partial \mathbb{D}_0$ are mutually related via

$$G^-(x_1', x_0''', t) = \int_{\partial \mathbb{D}_0} dx_0 \int_{-\infty}^{t} R^i(x_0'', x_0, t-t') f_1^+(x_0', x_1', t') dt' - f_1^-(x_0'', x_1', t)$$

and

$$G^+(x_1', x_0''', t) = -\int_{\partial \mathbb{D}_0} dx_0 \int_{-\infty}^{t} R^i(x_0'', x_0, t-t') f_1^+(x_0', x_1', -t') dt' + f_1^+(x_0'', x_1', -t).$$

see Appendix A, equations A-11 and A-12. The upper integration limit of the time integral ($t' = t$) follows from the causality of the reflection response. Adding the left- and right-hand sides of these expressions, using equations 2, 7, 9 and 10, we obtain

$$G(x_1', x_0''', t) = \int_{\partial \mathbb{D}_0} dx_0 \int_{-\infty}^{t} R^i(x_0'', x_0, t-t') f_2(x_1', x_0', t') dt' + f_2(x_1', x_0''', -t).$$

This equation relates the Green’s function $G(x_1', x_0''', t)$, with $x_1'$ in the subsurface, to the reflection response $R^i(x_0'', x_0, t)$, measured at the surface, and the focusing function $f_2(x_1', x_0''', t)$. In the next section we show that $f_2(x_1', x_0''', t)$ can be derived from the reflection response and the direct arrival $G_d(x_1', x_0''', t)$ of the Green’s function. Together with equation 13 this implies that the full Green’s function $G(x_1', x_0''', t)$ is retrieved from its direct arrival and the reflection response at the surface.

GREEN’S FUNCTION RETRIEVAL

Marchenko equation

Equation 13 is the starting point for deriving the 3-D Marchenko equation. Because of causality, the left-hand side is zero before the first arrival of the Green’s function. Hence

$$0 = \int_{\partial \mathbb{D}_0} dx_0 \int_{-\infty}^{t} R^i(x_0'', x_0, t-t') f_2(x_1', x_0', t') dt' + f_2(x_1', x_0''', -t), \quad \text{for } t < t_d(x_1', x_0''),$$

where $t_d(x_1', x_0'')$ denotes the traveltime of the first arrival of $G(x_1', x_0''', t)$ (in case of a triplicated wave, $t_d(x_1', x_0'')$ is the traveltime of the first onset). By convolving equation 14 with an arbitrary symmetric function $b(t)$, it follows that when $f_2$ is a solution, then so is the convolution of $f_2$ with $b(t)$. This means that this equation has no unique solution for $f_2$. To constrain the solution, we assume that $f_2(x_1', x_0''', t)$ can be written as a direct wave followed by a scattering coda, analogous to the 1-D case Lamb (1980). If the direct wave is known, then solving equation 14 reduces to solving the scattering coda. Before we define the direct wave of $f_2(x_1', x_0''', t)$, we first note that its upward part equals the inverse of the transmission response of the medium between $\partial \mathbb{D}_0$ and $\partial \mathbb{D}_1$, according to

$$f_2^-(x_1', x_0''', t) = T^{-1}_{\text{inv}}(x_1', x_0''', t),$$

see Appendix A, in particular equation A-14. Since $f_2^-(x_1', x_0''', t)$ is the field that is incident to the reference configuration (see Figure 2b), its direct wave is also the direct wave of $f_2(x_1', x_0''', t)$. Therefore we write

$$f_2(x_1', x_0''', t) = T^{-1}_{\text{inv}}(x_1', x_0''', t) + M(x_1', x_0''', t),$$

where $T^{-1}_{\text{inv}}(x_1', x_0''', t)$ is the direct arrival of the inverse of the transmission response (including possible triplications due to multipathing). Its arrival time is $-t_d(x_1', x_0'')$ (possible triplications in $T^{-1}_{\text{inv}}(x_1', x_0''', t)$ arrive before this time). $M(x_1', x_0''', t)$ is the scattering coda which follows the direct arrival, with

$$M(x_1', x_0''', t) = 0, \quad \text{for } t \leq -t_d(x_1', x_0'').$$
The transmission response $T(x'_i, x''_0, t)$ is related to a specific way to the Green’s function of the medium between $\partial D_0$ and $\partial D_1$ (see equations A-13 and A-15 in Appendix A). It is often sufficient to approximate the direct arrival of its inverse by

$$T^{-1}_{d}(x'_i, x''_0, t) \approx G_d(x'_i, x''_0, -t),$$

where $G_d(x'_i, x''_0, -t)$ is the time-reversal of the direct arrival of the Green’s function (including possible triplications). This approximation mainly implies that transmission losses at the interfaces are ignored Wapenaar and Berkhou (1989). In the following derivation we continue with $T^{-1}_{d}(x'_i, x''_0, t)$, but in the numerical examples we will approximate it by $G_d(x'_i, x''_0, -t)$.

By expressing $f_2$ as a direct wave, followed by a causal scattering coda (equations 16 and 17), we tacitly assumed that the first arriving wave is the direct wave. In the following we call this the “direct-wave assumption.” This assumption is not always fulfilled. For example, at large horizontal distances the first arriving wave may be a refracted wave. The conditions for the direct-wave assumption need further investigation. The following analysis is limited to situations for which this assumption is valid, for example, for finite horizontal distances in layered media with moderately curved interfaces.

Substituting equation 16 into 14, using the causality condition 17, yields the 3-D Marchenko equation

$$
\int_{\partial D_0} \frac{dx_0}{2\pi} \int_{-\infty}^{t_{d,0}(x'_i, x_0)} R^{ij}(x'_i, x_0, t - t')T^{-1}_{d}(x'_i, x''_0, t')dt' \\
+ \int_{\partial D_0} \frac{dx_0}{2\pi} \int_{t_{d,0}(x'_i, x_0)}^{t} R^{ij}(x'_i, x_0, t - t')M(x'_i, x_0, t')dt' + M(x'_i, x''_0, -t) = 0, \quad t < t_{d}(x'_i, x''_0),
$$

with $t_{d,0}(x'_i, x_0) = t_d(x'_i, x_0) - \epsilon$, where $\epsilon$ is a small positive constant (introduced so that the direct arrival is included in the first integral). The Marchenko equation is solved for $M(x'_i, x''_0, t)$ by the following iterative scheme

$$M_k(x'_i, x''_0, -t) = M_0(x'_i, x''_0, -t) - \int_{\partial D_0} \frac{dx_0}{2\pi} \int_{-t_{d,0}(x'_i, x_0)}^{t} R^{ij}(x'_i, x_0, t - t')M_{k-1}(x'_i, x_0, t')dt',
$$

with

$$M_0(x'_i, x''_0, -t) = - \int_{\partial D_0} \frac{dx_0}{2\pi} \int_{-\infty}^{t_{d,0}(x'_i, x_0)} R^{ij}(x'_i, x_0, t - t')T^{-1}_{d}(x'_i, x''_0, t')dt',
$$

for $t < t_d(x'_i, x''_0)$, whereas $M_k(x'_i, x''_0, -t) = 0$ for $t \geq t_d(x'_i, x''_0)$. Hence, assuming the reflection response at the surface and the direct arrival of the inverse transmission response are known, the $k$th iteration of the coda, $M_k(x'_i, x''_0, t)$, follows from equations 20 and 21. Using equation 16, the successive iterations for the focusing function $f_2(x'_i, x''_0, t)$ are then given by

$$f_{2,k}(x'_i, x''_0, t) = T^{-1}_{d}(x'_i, x''_0, t) + M_{k-1}(x'_i, x''_0, t),
$$

with $M_{-1}(x'_i, x''_0, t) = 0$. Assuming the scheme converges, the Green’s function $G(x'_i, x''_0, t)$ is subsequently obtained by substituting the converged solution $f_2(x'_i, x''_0, t)$ into equation 13. A numerical example of this scheme, for a situation with a triplicated direct wave, is discussed in Wapenaar et al. (2014). Here we continue with decomposition before we discuss a numerical example.

**Decomposition of the Marchenko equation**

The Marchenko equation, discussed in the previous section, forms the basis for retrieving the two-way Green’s function $G(x'_i, x''_0, t) = G^+(x'_i, x''_0, t) + G^-(x'_i, x''_0, t)$ from the reflection response $R^{ij}(x''_0, x_0, t)$ at the surface and the direct arrival of the inverse transmission response $T^{-1}_{d}(x'_i, x''_0, t)$. For redatuming and reflection imaging, the one-way Green’s functions $G^-(x'_i, x''_0, t)$ and $G^+(x'_i, x''_0, t)$ are needed separately. Different approaches to decompose the two-way Green’s function into one-way Green’s functions are possible (Wapenaar et al., 2012; Slob et al., 2014). We follow the approach of Slob et al. (2014), modified for the 3-D situation. This approach uses the one-way Green’s function representations 11 and 12.

Because of causality, the left-hand sides of equations 11 and 12 are zero before the first arrival of the Green’s function. Hence

$$0 = \int_{\partial D_0} \frac{dx_0}{2\pi} \int_{-\infty}^{t_{d,0}(x'_i, x_0)} R^{ij}(x''_0, x_0, t - t')f_+(x_0, x'_i, t')dt' - f_-(x''_0, x'_i, t) \quad \text{for} \quad t < t_d(x'_i, x''_0),
$$

$$0 = \int_{\partial D_0} \frac{dx_0}{2\pi} \int_{-\infty}^{t_{d,0}'} R^{ij}(x''_0, x_0, t - t')f_+(x_0, x'_i, t')dt' + f_-(x''_0, x'_i, t) - f_+(x''_0, x'_i, t_d(x'_i, x''_0)) \quad \text{for} \quad t < t_d(x'_i, x''_0),
$$

for $\epsilon > 0$.
and

\[ 0 = \int_{\partial \Omega} \int_{-\infty}^{t} R^-(x''_0, x_0, t - t') f^{+}_1(x_0, x'_i, t) \, dt' - f^{+}_1(x''_0, x'_i, -t) \quad \text{for} \quad t < t_d(x'_i, x''_0). \quad (24) \]

Analogous to equations 16 and 17, \( f^{+}_1(x''_0, x'_i, t) \) is written as

\[ f^{+}_1(x''_0, x'_i, t) = T_{\alpha}^{\text{inv}}(x'_i, x''_0, t) + M^+(x''_0, x'_i, t), \quad (25) \]

where \( M^+(x''_0, x'_i, t) \) is a causal coda, with

\[ M^+(x''_0, x'_i, t) = 0, \quad \text{for} \quad t \leq -t_d(x'_i, x''_0). \quad (26) \]

For \( f^{-}_1(x''_0, x'_i, t) \) we obtain from equation (10) and the causality of \( f_2(x'_i, x''_0, t) \),

\[ f^{-}_1(x''_0, x'_i, t) = 0, \quad \text{for} \quad t \geq t_d(x'_i, x''_0). \quad (27) \]

As in the previous section, the following analysis only applies to situations for which the “direct-wave assumption” (equations 25 – 27) holds true.

Substituting equation 25 into equations 23 and 24, using the causality conditions 26 and 27, gives

\[ \int_{\partial \Omega} \int_{-\infty}^{-t_d(x'_i, x''_0)} R^+(x''_0, x_0, t - t') T_{\alpha}^{\text{inv}}(x'_i, x_0, t') \, dt' \]

\[ + \int_{\partial \Omega} \int_{-t_d(x'_i, x''_0)}^{t} R^+(x''_0, x_0, t - t') M^+(x_0, x'_i, t') \, dt' - f^{-}_1(x''_0, x'_i, t) = 0, \quad \text{for} \quad t < t_d(x'_i, x''_0) \]

and

\[ \int_{\partial \Omega} \int_{-t_d(x'_i, x''_0)}^{t} R^+(x''_0, x_0, t - t') f^{-}_1(x_0, x'_i, -t') \, dt' - M^+(x''_0, x'_i, -t) = 0, \quad \text{for} \quad t < t_d(x'_i, x''_0). \quad (29) \]

This coupled system of Marchenko equations is solved by the following iterative scheme

\[ M^+_k(x''_0, x'_i, -t) = \int_{\partial \Omega} \int_{-t_d(x'_i, x''_0)}^{t} R^+(x''_0, x_0, t - t') f^{+}_1(x_0, x'_i, -t') \, dt', \quad (30) \]

\[ f^{+}_{1,k+1}(x''_0, x'_i, t) = f^{+}_{1,0}(x''_0, x'_i, t) + \int_{\partial \Omega} \int_{-t_d(x'_i, x''_0)}^{t} R^+(x''_0, x_0, t - t') M^+_k(x_0, x'_i, t') \, dt', \quad (31) \]

with

\[ f^{+}_{1,0}(x''_0, x'_i, t) = \int_{\partial \Omega} \int_{-\infty}^{-t_d(x'_i, x''_0)} R^+(x''_0, x_0, t - t') T_{\alpha}^{\text{inv}}(x'_i, x_0, t') \, dt', \quad (32) \]

for \( t < t_d(x'_i, x''_0) \), whereas \( M^+_k(x''_0, x'_i, -t) = f^{+}_{1,k+1}(x''_0, x'_i, t) = 0 \) for \( t \geq t_d(x'_i, x''_0) \). Note that \( f^{+}_{1,0}(x''_0, x'_i, t) \), defined in equation 32, is identical to \(-M_0(x'_i, x''_0, -t)\), defined in equation 21. In Appendix B it is shown that \( M^+_k(x''_0, x'_i, t) \) and \( f^{+}_{1,k}(x''_0, x'_i, t) \) can alternatively be obtained from the odd and even terms of an expansion of the coda \( M(x'_i, x''_0, t) \) of the focusing function \( f_2(x'_i, x''_0, t) \).

Using equation 25, the iterations of the one-way focusing function \( f^{+}_1(x''_0, x'_i, t) \) can be written as

\[ f^{+}_{1,k}(x''_0, x'_i, t) = T_{\alpha}^{\text{inv}}(x'_i, x''_0, t) + M^+_k(x''_0, x'_i, t), \quad (33) \]

with \( M^+_1(x''_0, x'_i, t) = 0 \). Assuming the scheme converges, the decomposed Green’s functions \( G^-(x'_i, x''_0, t) \) and \( G^+(x'_i, x''_0, t) \) are obtained by substituting the converged solutions \( f^{+}_{1,k}(x''_0, x'_i, t) \) and \( f^{-}_1(x''_0, x'_i, t) \) into equations 11 and 12.
Figure 5. 2-D inhomogeneous subsurface model. (a) Propagation velocity model, with some rays of the reflection response. (b) Smoothed version of the velocity model, with some rays of the direct arrival of the Green’s function. (c) Reflection response $R^i(x'_0, x_0, t)$, for fixed $x_0$ and variable $x'_0$. (d) Direct arrival of the Green’s function, $G_d(x', x_0, t)$, for fixed $x'_0$ and variable $x_0$.

Numerical example of Green’s function retrieval

We illustrate the iterative solution of the one-way focusing functions, and the subsequent retrieval of the decomposed Green’s functions, with a numerical example in a 2-D inhomogeneous subsurface model (Figure 5). The propagation velocities in the different layers are represented by different colors in Figure 5a. The reflector package below $\partial D_i$ (i.e., below $x_{3,i} = 1100$ m) represents a target zone on which we will zoom in later. A smoothed version of the propagation velocity model is shown in Figure 5b. The numerically modeled reflection response $R^i(x'_0, x_0, t)$, convolved with a Ricker wavelet with a central frequency of 20 Hz, is shown in grey-level display in Figure 5c. The source position is fixed at $x_0 = (0, 0)$ and the receiver position $x'_0 = (x'_1, 0)$ is variable, with $x'_1$ ranging from $-2250$ m to $2250$ m, with an inter-receiver distance of 10 m. The upper half-space is homogeneous, so the reflection response contains no surface-related multiples. The response in Figure 5c is displayed with a small time-dependent gain of $\exp(0.1 \ast t)$ to emphasize the internal multiples. In total 451 reflection responses $R^i(x'_0, x_0, t)$ have been modeled, for source positions $x_0 = (x_1, 0)$, with $x_1$ also ranging from $-2250$ m to $2250$ m, with a source interval of 10 m. Figure 5d shows the direct arrival of the Green’s function, $G_d(x', x_0, t)$, for variable source position $x_0$ (same range as above) and fixed receiver position $x'_0 = (0, 1100)$ m. This Green’s function has been modeled in a smoothed version of the model (Figure 5b), to acknowledge the fact that in practice no precise information about the interfaces and medium parameters is available. The reflection response in Figure 5c and
the direct arrival in Figure 5d are input to the iterative scheme.

The iterative procedure starts with \( f_{1,0}^{+}(x_0, x'_1, t) = T_d^{inv}(x'_1, x_0, t) \approx G_d(x'_1, x_0, -t) \). This approximation mainly implies that we ignore transmission losses at the interfaces. This will affect the retrieved Green’s functions \( G^{-}(x'_1, x'_0, t) \) and \( G^{+}(x'_1, x'_0, t) \) in approximately the same way. In the section “Marchenko redatuming” we will deconvolve \( G^{+}(x'_1, x'_0, t) \) with \( G^{-}(x'_1, x'_0, t) \). This deconvolution largely compensates for the mentioned approximation.

The zeroth order term of the focusing function, \( f_{1,0}^{+}(x_0, x'_1, t) \), is shown in Figure 6a, for a fixed focal point \( x'_1 = (0, 1100\, m) \) and variable \( x_0 = (x_1, 0) \). Its reflection response, \( f_{1,0}^{-}(x'_0, x'_1, t) \), is shown in Figure 6b. The functions \( f_{1,k}^{+}(x_0, x'_1, t) \) and \( f_{1,k}^{-}(x'_0, x'_1, t) \), for \( k = 1 \) and \( k = 5 \) are shown in Figures 6c,d,e,f. Note that new events are generated during iterations \( k = 0 \) and \( k = 1 \). During the higher order iterations, mainly the amplitudes of these events are modified. The results for \( k = 1 \) (Figures 6c,d) are already very close to those for \( k = 5 \) (Figures 6e,f).

The solutions after five iterations, shown in Figures 6e and 6f, are taken as the final estimates of \( f_{1,5}^{+}(x_0, x'_1, t) \) and \( f_{1,5}^{-}(x'_0, x'_1, t) \), respectively. These are used in equations 11 and 12 to retrieve the decomposed Green’s functions \( G^{-}(x'_1, x'_0, t) \) and \( G^{+}(x'_1, x'_0, t) \). These are shown in Figures 7a and 7b, respectively, for a fixed virtual-receiver position \( x'_1 = (0, 1100\, m) \) and variable source position \( x'_0 \). They are displayed with a time-dependent gain of \( \exp(1.0 + t) \) to emphasize the internal multiples. These decomposed Green’s functions will be used in the section “Marchenko redatuming”.

Figure 8 shows the sum of the results of Figure 7, displayed as seismic traces (only every 15th trace is shown). The black solid traces represent the retrieved Green’s function, whereas the grey dashed traces represent the directly modeled Green’s function, as a
Physical interpretation

We present a physical interpretation of the iterative scheme leading to the one-way focusing functions \( f^+_1(x_0, x'_1, t) \) and \( f^+_2(x''_0, x'_1, t) \), and of equations 11 and 12 for obtaining the decomposed Green’s functions \( G^- (x'_1, x''_0, t) \) and \( G^+ (x'_1, x''_0, t) \).

Figure 9 shows a physical interpretation of the first few iterations of the one-way focusing functions \( f^+_{1,k}(x_0, x'_1, t) \) and \( f^+_{2,k}(x''_0, x'_1, t) \). Recall that the focusing functions are defined in a reference configuration, which is equal to the actual medium above depth level \( \partial \Omega_d \) and reflection-free below this depth level (see also Figure 2). Therefore in Figure 9 the layers below \( \partial \Omega_d \) are absent. The red rays in Figure 9a schematically represent the zeroth order term \( f^+_{1,0}(x_0, x'_1, t) = T^\text{in}^\text{p}(x'_1, x_0, t) \), which was shown in grey-level display in Figure 6a (only two of 451 rays are shown). The rays start at \( x_0 = (x_1, 0) \) at the surface (for all \( x_1 \)-values), and cross each other at the focal point \( x'_1 \) at \( t = 0 \). During downward propagation towards the focal point, reflection occurs at the interfaces. This is expressed by equation 32. The blue rays in Figure 9a represent \( f^+_{1,0}(x''_0, x'_1, t) \). Although the integrand in equation 32 contains the reflection response of the actual medium, the result is evaluated only for \( t < t_d(x'_1, x''_0) \) and therefore reflections from below \( \partial \Omega_d \) do not contribute. The blue rays correspond to the main events in \( f^+_{1,0}(x''_0, x'_1, t) \) in Figure 6b (for simplicity not all reflections are shown in Figure 9a). Next, equation 30 is evaluated for \( k = 0 \). Hence, the reflection response of the medium is applied to the time-reversed version of \( f^+_{1,0}(x_0, x'_1, t) \), which results in the time-reversal of \( M^+_{1,0}(x''_0, x'_1, t) \). This is illustrated in Figure 9b. Here the downgoing red rays are the reversals of the outgoing blue rays in Figure 9a. They represent \( f^+_{1,0}(x_0, x'_1, -t) \), which is a downgoing field because of the time-reversal. The upgoing blue rays in Figure 9b represent \( M^+_{1,0}(x''_0, x'_1, -t) \), which is an upgoing field because of the time-reversal. Equation 30 is evaluated only for \( t < t_d(x'_1, x''_0) \), hence, only the rays corresponding to arrivals for \( t < t_d(x'_1, x''_0) \) are shown in Figure 9b. Next, \( f^+_{1,1}(x_0, x'_1, t) \) is constructed via \( f^+_{1,1}(x_0, x'_1, t) = T^\text{in}^\text{p}(x'_1, x_0, t) + M^+_{1,0}(x_0, x'_1, t) \). This is represented by the red rays in Figure 9c, consisting of the red rays in Figure 9a and the reversals of the blue rays in Figure 9b. Together, these red rays correspond to the main events in Figure 6c (the dashed parts of the red rays are very weak events which vanish during the iterative process). Next, equation 31 is evaluated for \( k = 0 \). This accomplishes that the reflection response to \( M^+_{1,0}(x_0, x'_1, t) \) is added to \( f^+_{1,0}(x''_0, x'_1, t) \), thus giving \( f^+_{1,1}(x''_0, x'_1, t) \). This is represented by the blue rays in Figure 9c, which correspond to the main events in Figure 6d.

The interpretation of higher order iterations goes along similar lines. The ray diagram of the converged solution is shown in Figure 4a, in which the red and blue rays correspond to the main events in Figures 6e and 6f, respectively. Note a peculiar property of the ray diagram in Figure 4a. Instead of the upgoing blue rays giving rise to downward reflected rays at the interfaces, downgoing red rays are launched from the surface, which meet the upgoing blue rays at the interfaces, in such a way that the downward reflected rays below the interfaces are annihilated. As a result, no rays other than the primary downgoing red rays reach the depth level \( \partial \Omega_d \). This is in agreement with the definition of the focusing function, which should focus only at \( x'_1 \) on \( \partial \Omega_d \) at \( t = 0 \) (equation 6).

Note that, although for the interpretation of \( f^+_{1,0}(x''_0, x'_1, t) \) we used the subsurface model to explain the different events, we did not use this model to obtain \( f^+_{1,0}(x''_0, x'_1, t) \) in Figure 6. The scheme discussed in the previous sections only uses the reflection response \( R^+(x''_0, x_0, t) \) (Figure 5c) and an estimate of the direct arrival of the Green’s function, \( G_d(x'_1, x''_0, t) \) (Figure 5d). All events appearing in \( f^+_{1,0}(x''_0, x'_1, t) \), except the time-reversed direct arrival, are retrieved entirely from the reflection response \( R^+(x''_0, x_0, t) \).

Next we interpret equations 11 and 12, which are rewritten as

\[
\frac{t < t_d(x'_1, x''_0)}{G^{p,+}(x''_0, x'_1, t) + f^+_{1,0}(x''_0, x'_1, t)} = \int_{\partial \Omega_d} dx_0 \int_{-\infty}^{t_d(x'_1, x''_0)} R^+(x''_0, x_0, t - t') f^+_1(x_0, x'_1, t') dt'
\]

and

\[
\frac{t \geq t_d(x'_1, x''_0)}{G^{p,-}(x''_0, x'_1, t) - f^+_{1,0}(x''_0, x'_1, -t)} = -\int_{\partial \Omega_d} dx_0 \int_{-\infty}^{t_d(x'_1, x''_0)} R^+(x''_0, x_0, t - t') f^+_1(x_0, x'_1, -t') dt'
\]

respectively. Note that we applied source-receiver reciprocity to the decomposed Green’s functions on the left-hand sides. We temporarily use two superscripts at the Green’s functions. The first superscript, \( p \), refers to the observed quantity (acoustic pressure) at the observation point \( x''_0 \); the second superscript (+ or −) refers to the propagation direction at the virtual source point \( x'_1 \). Figures 10a and 10b represent the total responses on the left-hand sides of equations 34 and 35, respectively. The traveltime curve \( t = t_d(x'_1, x''_0) \) is represented by the dashed lines in these figures.

The right-hand side of equation 34 describes the reflection response of the actual medium to \( f^+_1(x_0, x'_1, t) \), for an unbounded time interval. The downgoing red rays...
in Figure 11a represent the incident field \( f_{\gamma}^\dagger(x_0, x'_i, t) \), which focuses at \( x'_i \), similar as in Figure 4a. The upgoing blue rays represent again the response \( f_{\gamma}^\dagger(x'_0, x'_i, t) \), with \( t < t_d(x'_i, x'_0) \). Beyond the focal point the downgoing rays diverge and, after reflection, arrive at the surface. This is illustrated by the green rays in Figure 11a, which represent the Green’s function \( G^{\text{p}+}(x'_0, x'_i, t) \), \( t > t_d(x'_i, x'_0) \), with its virtual source at the focal point \( x'_i \). The left-hand side of equation 34 represents the superposition of \( f_{\gamma}^\dagger(x'_0, x'_i, t) \) and \( G^{\text{p}+}(x'_0, x'_i, t) \), see Figure 10a. By subtracting \( f_{\gamma}^\dagger(x'_0, x'_i, t) \) we obtain \( G^{\text{p}+}(x'_0, x'_i, t) \), or, after applying source-receiver reciprocity and returning to the single-superscript notation, \( G^{-}(x'_i, x'_0, t) \), see Figure 11b. This ray diagram represents the retrieved upgoing part of the Green’s function in Figure 7a.

The interpretation of equation 35 is slightly more complicated. The right-hand side describes the reflection response of the actual medium to \( -f_{\gamma}^\dagger(x_0, x'_i, -t) \), which, because of the time-reversal, is a downgoing field. It is represented by the downgoing red rays in Figure 12a, which are the reversals of the upgoing blue rays in Figure 11a. The upgoing blue rays in Figure 12a show the response \( -f_{\gamma}^\dagger(x'_0, x'_i, -t) \), \( t \leq t_d(x'_i, x'_0) \). The green-blue dashed ray represents the final arrival of \( -f_{\gamma}^\dagger(x'_0, x'_i, -t) \) (i.e., the time-reversed first arrival of \( -f_{\gamma}^\dagger(x'_0, x'_i, t) \)). Its path is shared with that of the direct arrival of the Green’s function \( G^{\text{p}-}(x'_0, x'_i, t) \). The downgoing red rays cross the paths of the direct wave and give rise to the multiple reflections of the Green’s function \( G^{\text{p}-}(x'_0, x'_i, t) \), represented by the solid green rays in Figure 12a. Note that all rays associated with \( G^{\text{p}-}(x'_0, x'_i, t) \) apparently originate from a virtual source at the focal point \( x'_i \). The left-hand side of equation 35 represents the superposition of \( -f_{\gamma}^\dagger(x'_0, x'_i, -t) \) and \( G^{\text{p}-}(x'_0, x'_i, t) \), see Figure 10b. By subtracting \( -f_{\gamma}^\dagger(x'_0, x'_i, -t) \) we obtain \( G^{\text{p}-}(x'_0, x'_i, t) \), or, after applying source-receiver reciprocity and returning to the single-superscript notation, \( G^{\dagger}(x'_i, x'_0, t) \), see
MARCHENKO REDATUMING

“Redatuming” refers to the process of virtually moving sources and receivers from a specific datum plane (typically the acquisition surface) to a new datum plane inside the medium, closer to a target zone. In classical redatuming (Berryhill, 1979, 1984; Kinneging et al., 1989), one-way primary operators are used to redatum the reflection response from $\partial D_0$ to $\partial D_i$. These operators are based on a background model of the medium between $\partial D_0$ and $\partial D_i$. These operators do not account for internal multiple reflections, hence, a classically redatumed response contains spurious reflections (i.e., reflections that do not belong to the reflection response at $\partial D_i$).

Redatuming can be seen as a two-step process: starting with sources and receivers at the surface $\partial D_0$, in the first step the receivers are moved from $\partial D_0$ to $\partial D_i$, followed by moving the sources to $\partial D_i$ in the second step. In some cases, acquisition is carried out with sources at the surface $\partial D_0$ and receivers at a datum plane $\partial D_i$ inside the medium. For example, receivers may be located in a horizontal borehole. For this situation, redatuming is a one-step process, bringing only the sources from $\partial D_0$ to $\partial D_i$ (Bakulin and Calvert, 2006). No information about the medium is needed because the receivers at $\partial D_i$ “measure” the required downward extrapolation operators. This data-driven approach to redatuming is called “interferometric redatuming” (Schuster and Zhou, 2006). Because the operators for interferometric redatuming are measured, they account, in principle, for internal multiple reflections. This property is exploited by a method called “interferometric redatuming by multidimensional deconvolution (MDD)”, which indeed leads to a ghost-free reflection response at $\partial D_i$ (van der Neut et al., 2011).

In most situations, however, both sources and receivers are situated at the surface. The iterative Marchenko scheme discussed in the foregoing sections is applied to the reflection response $R^-(x_i, x_0', t)$ at the surface $\partial D_0$, yielding the Green’s functions $G^-(x_i', x_0', t)$ and $G^+(x_i', x_0', t)$, i.e., responses to sources at the surface $\partial D_0$, observed by virtual receivers at datum plane $\partial D_i$ inside the medium. This can be seen as the first step of a two-step redatuming process. This form of receiver redatuming requires an estimate of the direct arrival of the inverse transmission response, $T_d^{inv}(x_i', x_0')$. This is actually the same as the one-way primary operator used in classical redatuming. The direct arrivals can, for example, be obtained by modeling in a background model between $\partial D_0$ and $\partial D_i$ or, when $\partial D_i$ coincides with an actual reflector, they can be derived from the reflection response at the surface via the common-focal-point (CFP) method (Berkhout, 1997; Berkhout and Verschuur, 2001). Note that, unlike the first step in classical redatuming, our iterative form of receiver redatuming accounts for internal multiple reflections. Once the receivers have been redatumed to $\partial D_i$, the sources can be redatumed to $\partial D_i$ in a similar way as in interferometric redatuming by MDD. We call the combination of these two steps “Marchenko redatuming.” In the remainder of this section, the source redatuming step is discussed in more detail.

The Green’s functions $G^-(x_i', x_0', t)$ and $G^+(x_i', x_0', t)$ are mutually related via the reflection response of the medium below $\partial D_i$, according to

$$G^-(x_i, x_0', t) = \int_{\partial D_i} \int_{-\infty}^{\infty} R^{\perp}(x_i, x_i', t') G^+(x_i', x_0', t-t') dt'$$

(Wapenaar et al., 2000; Amundsen, 2001). This expression states that the downgoing field $G^+$ at $\partial D_i$, convolved with the reflection response $R^{\perp}$ at $\partial D_i$ and integrated along all source positions of this reflection response at $\partial D_i$, gives the upgoing field $G^-$ at $\partial D_i$. Note that the fields $G^-$ and $G^+$ are defined in the actual medium (Figure 1), whereas the reflection response $R^{\perp}$ is defined in a reference medium that is identical to the actual medium below $\partial D_i$ and reflection-free above this depth level.

For each $x_0'$, equation 36 is an integral equation for
the reflection response $R^i(x_i, x'_i, t)$. Figure 13a shows
ray diagrams of $G^-(x_i, x'_0, t)$ and $G^+(x'_i, x''_0, t)$ for fixed
$x''_0$ and variable $x_i$ and $x'_i$. $R^i(x_i, x'_i, t)$ can be resolved
from equation 38 by MDD when the Green’s functions
are available for many source positions $x'_0$. To this end,
both sides of equation 38 are first correlated with the
downgoing Green’s function and integrated over $x'_0 \in \partial\Omega_0$.
This yields
\[ C(x_i, x''_0, t) = \int_{\partial\Omega_0} dx'_0 \int_0^\infty R^i(x_i, x'_i, t)\Gamma(x'_i, x''_0, t - t')dt', \]
with the correlation function $C(x_i, x''_0, t)$ and
pointspread function $\Gamma(x'_i, x''_0, t)$ defined as
\[ C(x_i, x''_0, t) = \int_{\partial\Omega_0} dx'_0 \int_0^\infty G^-(x_i, x'_0, t)G^+(x''_0, x'_0, t')dt', \]
and
\[ \Gamma(x'_i, x''_0, t) = \int_{\partial\Omega_0} dx'_0 \int_0^\infty G^+(x'_i, x''_0, t)G^+(x''_0, x'_0, t')dt''. \]
respectively. MDD involves inverting equation 37. This
gives the redatumed reflection response $R^i(x_i, x'_i, t)$,
with virtual sources ($x'_i$) and virtual receivers ($x_i$) at
$\partial\Omega_i$. For the practical aspects of this inversion,
van der Neut et al. (2011). As mentioned before, this
deconvolution process largely compensates for the am-
plitude errors in $G^-(x_i, x'_0, t)$ and $G^+(x'_i, x''_0, t)$, related
to approximating the direct arrival of the inverse trans-
mission response by the time-reversal of the direct ar-
rival of the Green’s function (equation 18).

We illustrate this source redatuming procedure by applying
MDD to the retrieved Green’s functions
$G^-(x_i, x'_0, t)$ and $G^+(x'_i, x''_0, t)$ of Figure 7. To this end
we repeat the receiver redatuming procedure that led
to Figure 7 for a range of $x'_0$-values along $\partial\Omega_i$. Next, we
evaluate the correlation and point-spread functions of
equations 38 and 39, respectively, and invert equation
37. The result, $R^i(x_i, x'_i, t)$, is shown in Figure 13b, for
fixed $x'_0 = (0, 1100$ m) and variable $x_i = (x_1, x_3, t =
1100$ m), with $x_1$ ranging from $-2000$ m to $2000$ m (i.e.,
the top of the target zone). Figure 13c shows a directly
modeled shot record, with its source at $x'_i$. Note that
the retrieved response (Figure 13b) clearly exhibits the
reflections of the layers in the target zone, with hardly
any contamination by internal multiples of the overbur-

Summarizing, we have presented Marchenko reda-
tuming of the reflection response from the surface $\partial\Omega_0$, to
$\partial\Omega_i$ inside the medium, as a two-step process. Step
one, employing the iterative Marchenko scheme, moves
the receivers from $\partial\Omega_0$ to $\partial\Omega_i$ (as illustrated with
the numerical example in Figures 7a and 7b). Step two, in-
terferometric redatuming by MDD, moves the sources
from $\partial\Omega_0$ to $\partial\Omega_i$ (as illustrated in Figure 13b). This
two-step redatuming process requires the same input as
classical redatuming (i.e., the reflection response at the
surface $\partial\Omega_0$ and an estimate of the direct arrivals be-
tween $\partial\Omega_0$ and $\partial\Omega_i$), but, unlike classical redatuming, it
uses the internal multiples in the original reflection re-
sponse $R^i(x'_0, x_0, t)$ to obtain the redatumed reflection response $R^i(x_i, x'_i, t)$ at $\partial\Omega_i$, free of spurious reflections
related to internal multiples in the overburden.

Until now it was assumed that $R^i(x'_0, x_0, t)$ is
measured in a configuration with a homogeneous upper
half-space above $\partial\Omega_0$ (Figure 1). In many cases the
acquisition surface $\partial\Omega_0$ is a free surface, meaning
that $R^i(x'_0, x_0, t)$ is contaminated with surface-related
multiples. In those cases, Marchenko redatuming should
be preceded by surface-related multiple elimination
(SRME) (Verschuur et al., 1992; van Groenestijn and
Verschuur, 2009). The output of SRME is the reflection
response $R^i(x''_0, x_0, t)$ in the configuration of Figure 1.

**MARCHENKO IMAGING**

**Imaging from above**

The redatumed reflection response $R^i(x_i, x'_i, t)$ can
be used in different ways for reflection imaging. The
most straightforward approach is to extract the zero-
offset $(x'_i = x_i)$, zero-time $(t = 0)$ component, i.e.,
$R^i(x_i, x_i, 0)$, repeat the process for all $x_i$ at a well-
sampled range of depth levels $x_3$, in the region of inter-
est, and thus construct an image of the zero-offset reflec-
tivity $R^i(x, x, 0)$. We call this Marchenko imaging.
Because $R^i(x_i, x'_i, t)$ is the correctly redatumed reflec-
tion response, the image thus obtained will be free of
spurious reflections Wapenaar et al. (2012). Note that
this method can start and end at any two depth levels,
leading to a ghost-free image of the target zone between
these depth levels, without the need to derive a detailed
model of the medium between the acquisition surface
and the target zone; as in standard imaging, a smooth
subsurface model which explains the direct arrivals suf-

Instead of selecting only the zero-offset, zero-time
component $R^i(x_i, x_i, 0)$, the full reflection response
$R^i(x_i, x'_i, t)$ can be used to derive the angle-dependent
reflectivity at each image point (de Bruin et al., 1990;
Sava and Fomel, 2003). This information can be used as
input for amplitude-versus-angle inversion to determine
the detailed medium parameters in the target zone.

Applying Marchenko redatuming to many depth
levels in a target zone is a costly process. Alternatively,
a relatively small number of sparse depth levels can be
selected to which the reflection response is redatumed.
Subsequently, standard imaging can be applied to the
redatumed reflection responses $R^i(x_i, x'_i, t)$ to image
the regions between the selected depth levels. Assuming
the depth levels are selected with care (for example
between the major reflecting boundaries), this will still
Figure 14. (a) Marchenko image of the target zone, obtained from the redatumed response $R^c(x_i, x'_i, t)$ at the top of the target zone, for $x'_i = (x'_1 = 0, x_{3,i} = 1100 \text{ m})$. Similar responses are computed for a range of $x'_1$ values, between $-2000$ m and $2000$ m. We use these redatumed responses as input for standard imaging in the target zone. The result is shown in Figure 14a, which clearly shows the reflectors of the target zone. For comparison, Figure 14b shows the same target zone, selected from a standard image obtained from the reflection response at the surface, $R^c(x''_0, x_0, t)$ (in both figures, amplitudes above 25% of the maximum amplitude have been clipped; no depth-dependent gain has been applied). Whereas the latter result is contaminated by spurious reflections, caused by internal multiples in the overburden, our image result in Figure 14a is free of these contaminations. Note that processes like full waveform inversion Virieux and Operto (2009) and full wavefield migration Berkhout and Verschuur (2011) might lead to results similar as the one shown in Figure 14a. A detailed comparison with these methods is beyond the scope of this paper. We only mention one important difference. Methods like full waveform inversion/migration are recursive methods, meaning that deep layers can only be imaged after the shallower structures have been resolved. In contrast, Marchenko imaging is a non-recursive method: the image in Figure 14a has been obtained without using information about the overburden, other than the smooth model of Figure 5b.

As in standard imaging, the exclusion of evanescent waves in the focusing functions implies a limitation to the maximum obtainable spatial resolution Berkhout and van Wulfften Palthe (1979). In principle this resolution limit can be overcome by taking evanescent waves into account (Williams and Maynard, 1980; Lerosey et al., 2007; Fink and Tanter, 2010; Cao et al., 2012). It remains to be investigated to what extent evanescent waves can be used to improve the resolution of Marchenko imaging.

Imaging from below

The decomposed Marchenko scheme not only derives the one-way Green’s functions from the reflection response at the surface, it also yields the one-way focusing functions $f^+_1(x'_1, x'_j, t)$ and $f^-_1(x'_i, x_j, t)$ and, via equations 9 and 10, the focusing functions $f_2^+(x'_j, x''_0, t)$ and $f_2^-(x_j, x''_0, t)$. Whereas the one-way Green’s functions are mutually related via the reflection response to downgoing waves of the medium below $\partial D_i$ (see equation 36), the one-way focusing functions $f^+_2(x'_j, x''_0, t)$ and $f^-_2(x_j, x''_0, t)$ (Figure 2b) are related via the reflection response to upgoing waves of the medium above...
Marchenko imaging

\( \partial \mathcal{D}_j \), according to

\[
\delta^+_j(x_j, x''_j, t) = \int_{\partial \mathcal{D}_j} \delta x_j \int_{-\infty}^{\infty} \mathcal{R}^c(x_j, x'_j, t') \delta f^{-}_j(x'_j, x''_j, t - t') \partial \mathcal{D}_j
\]

(see Appendix A, equation A-17). For each \( x''_j \), equation 40 is an integral equation for the reflection response \( \mathcal{R}^c(x_j, x'_j, t) \). Figure 15a shows ray diagrams of the focusing functions \( \delta^+_j(x_j, x''_j, t) \) and \( \delta^+_j(x_j, x''_j, t) \), for fixed \( x''_j \) and variable \( x_j \) and \( x'_j \) at \( \partial \mathcal{D}_j \), with \( x_{3,j} = 1600 \) m. Figures 15b and 15c show numerical examples of these functions, obtained from the reflection response \( \mathcal{R}^c(x''_j, x_0, t) \) of Figure 5c by applying five iterations of the Marchenko scheme. The variable \( x'_j \) is defined as \( x'_j = (x'_1, x_{3,j} = 1600 \) m), with \( x'_1 \) ranging from \(-2000 \) m to \( 2000 \) m. \( \mathcal{R}^c(x_j, x'_j, t) \) can be resolved from equation 40 by MDD, when the focusing functions are available for many focal points \( x''_j \). The procedure is the same as for resolving \( \mathcal{R}^c(x_j, t) \) from equation 36. Once \( \mathcal{R}^c(x_j, x'_j, t) \) has been obtained for all \( x_j \) at the bottom of the target zone, it can be used to image reflectors at and above \( \partial \mathcal{D}_j \) from below, in a similar way as \( \mathcal{R}^c(x_j, x'_j, t) \) is used in the previous section to image reflectors at and below \( \partial \mathcal{D}_j \) from above. An image thus obtained is shown in Figure 15d, which represents the zero-offset reflectivity \( \mathcal{R}^c(x_j, t) \) of the target zone, free of spurious reflections related to internal multiples in the overburden (amplitudes above 25% of the maximum amplitude have been clipped). Note the polarity change with respect to the image in Figure 14a.

The two imaging methods (from above and from below) use the same information, so the resulting images of \( \mathcal{R}^c(x_j, x'_j, t) \) and \( \mathcal{R}^c(x_j, x''_j, t) \) are not independent. Subtracting the two images (accounting for their opposite signs) will therefore not lead to a significant improvement of the signal-to-noise ratio. Hence, for linear imaging applications, the two methods are equivalent (i.e., not complementary). However, for nonlinear applications it can be useful to use the redatumed data \( \mathcal{R}^c(x_j, x'_j, t) \) and \( \mathcal{R}^c(x_j, x''_j, t) \) at two different depth levels \( \partial \mathcal{D}_i \) and \( \partial \mathcal{D}_j \) (with \( x_{3,i} > x_{3,j} \)). For example, these responses can be used as input for a new nonlinear imaging method, as introduced by Fleury and Vasconcelos (2012), Vasconcelos (2013) and Rayani and Curtis (2013), or for full waveform inversion (Virieux and Operto, 2009), to resolve the parameters of the target zone between these depth levels.

Van der Neut et al. (2013) discuss an alternative method to combine imaging from above with imaging from below. The first step involves Marchenko redatuming to two depth levels \( \partial \mathcal{D}_i \) and \( \partial \mathcal{D}_j \), yielding the Green’s functions \( G^+_{ij}(x'_j, x''_j, t) \) and \( G^+_{jk}(x'_j, x''_j, t) \). These Green’s functions are then used to simultaneously resolve the reflection responses to downgoing and upgoing waves of the target zone between \( \partial \mathcal{D}_i \) and \( \partial \mathcal{D}_j \). Note that, since \( G^+_{ij}(x'_j, x''_j, t) \) contains reflections from objects below the target zone, this method uses essentially different information to image the target zone from below.

**COMPARISON WITH SOURCE-RECEIVER INTERFEROMETRY**

Because Marchenko imaging involves redatuming of sources as well as receivers, it has an interesting relationship with source-receiver interferometry, as proposed by Curtis and Halliday (2010) and Halliday and Curtis (2010). We analyze this relation for the Marchenko scheme for imaging from below. We start this analysis by replacing \( f_{2}^{-} \) and \( f_{2}^{+} \) in equation 40 by their initial estimates \( f_{2,0}^{-} \) and \( f_{2,0}^{+} \), respectively. According to equations 9, 10, 18, 32 and 33 we have

\[
f_{2,0}^{+}(x_j, x''_j, t) = - \int_{\partial \mathcal{D}_0} \mathcal{R}^c(x''_j, x_0, -t) \ast G_4(x_0, x_j, t) \partial x_0,
\]

Figure 15. (a) Ray diagram of the focusing functions \( f_{2}^{-}(x'_j, x''_j, t) \) and \( f_{2}^{+}(x'_j, x''_j, t) \). (b) The retrieved focusing function \( f_{2}^{-}(x'_j, x''_j, t) \), for fixed \( x''_j \) and variable \( x'_j \). (c) The retrieved focusing function \( f_{2}^{+}(x'_j, x''_j, t) \), for fixed \( x''_j \) and variable \( x'_j \). (d) Marchenko image of the target area above \( \partial \mathcal{D}_j \), obtained by resolving \( \mathcal{R}^c(x'_j, x''_j, t) \) from the focusing functions, followed by standard imaging.
and
\[ f_{2,0}(x', x''_0, t) = G_d(x''_0, x', t), \tag{42} \]
respectively. Here the asterisk * denotes temporal convolution; for simplicity we ignored the integration limit in the convolutional integral in equation 32. Resolving \( R^+(x_j', x''_j, t) \) from equation 40 involves multidimensional deconvolution of \( f^{+2}_{2,0} \) and \( f^{-2}_{2,0} \). If we approximate this MDD by crosscorrelation of the initial estimates \( f^{+2}_{2,0} \) and \( f^{-2}_{2,0} \), we obtain
\[ R^+(x_j', x''_j, t) \approx C(x_j', x''_j, t) \]
\[ = \int_{\partial D_0} \int_{\partial D_j} G_0(x''_0, x'_j, t) * f^{+2}_{2,0}(x'_j, x''_0, -t) dx''_0 \]
\[ = -\left[ \int_{\partial D_0} \int_{\partial D_j} G_0(x''_0, x'_j, t) * R^-(x'_j, x''_0, -t) * \right. \]
\[ \left. G_d(x''_0, x_j', t) dx'_0 dx''_0 \right]. \]

The double integral on the right-hand side is a form of source-receiver interferometry. In particular, it represents the retrieval of the reflection response from below, as proposed by Poliannikov (2011) and Poliannikov et al. (2012). In their approach, the Green’s functions represent responses of real sources in the subsurface. Figure 16 shows the configuration for the case of a single reflector. The indicated rays are the ones for which the traveltime of the integrand is stationary. According to equation 43, the two Green’s functions and the time-reversed reflection response are convolved with each other, hence, the traveltimes along the solid rays in Figure 16 are added to each other and the traveltime along the dashed-dotted ray is subtracted from it. The resulting traveltime is that of the reflection response from below, indicated by the dashed ray.

This method works well for primary reflections, as is shown with numerical examples by Poliannikov (2011) and Poliannikov et al. (2012). However, equation 43 breaks down for multiple reflections. In contrast, the reflection response from below, obtained by inverting equation 40 with \( f^{+2}_{2,0} \) and \( f^{-2}_{2,0} \) obtained with the iterative Marchenko scheme, holds for primary as well as multiple reflections, as has been demonstrated with the numerical example in Figure 15.

CONCLUSIONS

We have shown that a decomposed version of the 3-D Marchenko equation leads to an iterative scheme that enables the retrieval of the decomposed Green’s functions \( G^-(x_i', x''_i, t) \) and \( G^+(x_i', x''_0, t) \) from reflection measurements at the acquisition surface \( \partial D_0 \). Here \( x''_0 \) denotes the position of the Green’s source at the surface, whereas \( x'_i \) represents an arbitrary virtual-receiver position in the subsurface. Hence, the decomposed iterative Marchenko scheme accomplishes receiver redatuming. Subsequently, source redatuming is achieved by deconvolving (in a multidimensional sense) \( G^-(x_i, x''_0, t) \) with \( G^+(x_i', x''_0, t) \). This results in the redatumed response \( R^+(x_i, x''_0, t) \), with virtual sources \( (x'_i) \) and virtual receivers \( (x_i) \) at depth level \( \partial D_i \) in the subsurface. This two-step redatuming procedure requires the same input as standard redatuming, namely the reflection response at the surface and an estimate of the direct arrivals between the surface and the redatuming level. However, whereas standard redatuming ignores the effects of internal multiples in the overburden, the redatuming procedure based on the Marchenko scheme and MDD properly accounts for internal multiples.

The redatumed response can be used in various ways for seismic imaging. We propose that these methods go together under the name “Marchenko imaging”. In its most general form, angle-dependent reflectivity information is retrieved at any image position. More simple approaches lead to images of the zero-offset reflectivity, \( R^+(x, x, 0) \), either by repeating the redatuming procedure for many depth levels \( \partial D_i \) and selecting the zero-offset, zero-time component, or by applying standard imaging to the redatumed response in a target zone below a fixed depth level \( \partial D_i \). In all cases, internal multiples contribute to the imaging process and the obtained image is free of spurious events related to these internal multiples. The non-recursive character of Marchenko imaging implies that accumulation of errors is avoided. An important feature is that spurious reflections in a target zone are suppressed (as illustrated in Figure 14), without the need to resolve first the reflection properties of the overburden.

Marchenko imaging also has limitations. The conditions for the validity of the direct-wave assumption need further investigation. Moreover, the medium is assumed to be lossless. For media with small losses it will probably suffice to apply loss corrections to the data prior to Marchenko imaging. However, for media with significant dissipation, more research is required. Losses due to mode conversion are also not covered by the current method. As in standard imaging, evanescent waves are excluded, which limits the spatial resolution. Sampling
issues, both for 2-D and 3-D data, also need further investigation. In its current form our method requires that the reflection response is well sampled at the surface. Particularly for 3-D data this assumption is never fulfilled, hence, our method needs to be modified before it can be applied to realistic 3-D data.

So far we have only considered scalar waves. Representations 11 and 12 can straightforwardly be generalized for vectorial waves, but the derivation of the Marchenko equation for vectorial waves needs further investigation (particularly for elastodynamic waves, with different propagation velocities for P- and S-waves, for which the causality arguments become more complex).

Several other approaches exist that deal with internal multiples (Weglein et al., 1997, 2011; Berkhout and Verschuur, 1997, 2011; Jakubowicz, 1998; Ten Kroode, 2002; Brookes, 2011). Also full wave form inversion deals to a certain extent with internal multiples Virieux and Operto (2009). It remains to be investigated how Marchenko imaging relates to these and other methods.

Despite the open questions, we believe that this work opens a new view on how to use information in internal multiple reflections to improve seismic imaging and monitoring of the Earth’s subsurface at different scales.

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APPENDIX : APPENDIX A: DERIVATION OF RELATIONS BETWEEN GREEN’S FUNCTIONS, FOCUSING FUNCTIONS, AND REFLECTION AND TRANSMISSION RESPONSES

We derive several relations between Green’s functions, focusing functions, and reflection and transmission responses, using reciprocity theorems in the space-frequency domain.

We define the temporal Fourier transform of a space- and time-dependent function \( p(x, t) \) as

\[
p(x, \omega) = \int_{-\infty}^{\infty} p(x, t) \exp(-j\omega t) dt,
\]

where \( \omega \) is the angular frequency and \( j \) the imaginary unit \((j = \sqrt{-1})\). To keep the notation simple, the same symbol is used for time- and frequency-domain functions (here \( p \)). In the space-frequency domain, we will make use of two reciprocity theorems for one-way wave fields at depth levels \( \partial\Omega_0 \) and \( \partial\Omega_1 \), enclosing a lossless medium (see Figures 1 and 2). Following a similar derivation as in Appendix B of Wapenaar and Berkhout (1989), we find the following reciprocity theorems for pressure-normalized one-way wave fields

\[
- \int_{\partial\Omega_0} \rho^{-1} \{ p_A^+(\partial_3 p_B) + p_A^-(\partial_3 p_B^+) \} d\mathbf{x}_0 = \int_{\partial\Omega_1} \rho^{-1} \{ (\partial_3 p_A^+) p_B + (\partial_3 p_A^-) p_B^+ \} d\mathbf{x}_1,
\]

(A - 1)

and

\[
- \int_{\partial\Omega_0} \rho^{-1} \{ (p_A^+)^* (\partial_3 p_B) + (p_A^-)^* (\partial_3 p_B^+) \} d\mathbf{x}_0 = \int_{\partial\Omega_1} \rho^{-1} \{ (\partial_3 p_A^+)^* p_B^* + (\partial_3 p_A^-)^* p_B^* \} d\mathbf{x}_1,
\]

(A - 3)

respectively. The asterisk denotes complex conjugation. The subscripts \( A \) and \( B \) refer to two independent acoustic states. The underlying assumptions are that the domain \( \Omega \) between \( \partial\Omega_0 \) and \( \partial\Omega_1 \) is source-free, and the medium parameters in this domain are the same in both states. Furthermore, in equation A-3 the evanescent wave fields at \( \partial\Omega_0 \) and \( \partial\Omega_1 \) are neglected.

In the frequency domain, equation 3 reads

\[
\partial_3 G^+(x, x_0, \omega) |_{x_2 = x_3 = 0} = -\frac{1}{2} j \omega \rho(x_0^0) \delta(x_0^1 - x_0^0).
\]

(A - 4)

To derive equation 4, substitute \( p_B^+(x, \omega) = G^+(x, x_0, \omega) \) into equation A-2. Moreover, replace \( \partial\Omega_0 \) by \( \partial\Omega_m \), where \( \partial\Omega_m \) is a boundary which lies below all inhomogeneities, so that \( p_A^+(x_m, \omega) = p_B^+(x_m, \omega) = 0 \). This gives

\[
p_A^+(x_0, \omega) = \int_{\partial\Omega_m} R^+(x_0^0, x_0, \omega) p_A^+(x_0, \omega) d\mathbf{x}_0,
\]

(A - 5)
with

\[ R^T(x_0', x_0, \omega) = \frac{\partial_t G^- (x, x_0', \omega)|_{x_3 = x_3, t}}{\frac{1}{2} j \omega p(x_0)} \]  

(A - 6)

Transforming this back to the time domain gives equation 4.

In the frequency domain, the focusing conditions 6 and 8 read

\[ \partial_t f^+_1 (x, x_i', \omega)|_{x_3 = x_3, t} = -\frac{1}{2} j \omega p(x_i') \delta(x_H - x_H) \]  

(A - 7)

and

\[ \partial_t f^-_1 (x, x'_0, \omega)|_{x_3 = x_3, t} = \frac{1}{2} j \omega p(x'_0) \delta(x_H - x_H) \]  

(A - 8)

respectively. Substituting \( p^+_A(x, \omega) = f^+_1 (x, x_i', \omega), \ p^-_B(x, \omega) = f^-_1 (x, x'_0, \omega) \) and \( \partial_t p^-_A(x, \omega)|_{x_3 = x_3, t} = \partial_t p^-_B(x, \omega)|_{x_3 = x_3, t} = 0 \) into equations A-2 and A-3, using equations A-7 and A-8, gives

\[ f^+_1 (x_0', x_i', \omega) = f^-_2 (x_i', x'_0, \omega) \]  

(A - 9)

and

\[ -\{f^+_1 (x_0', x_i', \omega)\}^* = f^-_2 (x_i', x'_0, \omega), \]  

(A - 10)

respectively. Transforming these expressions back to the time domain yields equations 9 and 10, respectively.

To derive equations 11 and 12, consider the configurations in Figures 1 and 2. The Green’s functions (state B) are defined in the actual medium (Figure 1), whereas the focusing functions (state A) are defined in a reference configuration (Figure 2). Between depth levels \( \partial D_0 \) and \( \partial D_1 \), the medium is source-free and the medium parameters are the same in both states, hence, the conditions for reciprocity theorems A-2 and A-3 are satisfied. Substituting \( p^+_A(x, \omega) = f^+_1 (x, x_i', \omega), \ p^-_B(x, \omega) = G^+(x, x'_0, \omega) \) and \( \partial_t p^-_A(x, \omega)|_{x_3 = x_3, t} \) into equations A-2 and A-3, using equations A-4, A-6, A-7 and A-8, yields

\[ \int_{\partial D_0} R^T(x_0', x_0, \omega) f^+_1 (x_0, x_i', \omega) d\mathbf{x}_0 - f^-_1 (x_0', x_i', \omega) = G^- (x_i', x'_0, \omega) \]  

(A - 11)

and

\[ -\int_{\partial D_0} R^T(x_0', x_0, \omega) \{f^+_1 (x_0, x_i', \omega)\}^* d\mathbf{x}_0 + \{f^+_1 (x_0', x_i', \omega)\}^* = G^+ (x_i', x'_0, \omega), \]  

(A - 12)

respectively. The inverse Fourier transforms of these expressions yield equations 11 and 12, respectively.

For the derivation of equation 15, consider Figure 17. Here \( T(x_i, x'_0, t) \) represents the transmission response of the reference configuration, with \( x'_0 \) just above \( \partial D_0 \). Analogous to equation A-6, the Fourier transform of the transmission response is related to the Green’s function of the reference configuration, according to

\[ T(x_i, x'_0, \omega) = \frac{\partial_t G^+ (x_i, x'_0, \omega)|_{x_3 = x_3, t}}{\frac{1}{2} j \omega p(x'_0)} \]  

(A - 13)

Substituting \( p^+_A(x_0, \omega) = \delta(x_H - x'_0), \ p^+_A(x, \omega) = T(x_i, x'_0, \omega), \ p^-_B(x, \omega) = 0, \ p^-_A(x, \omega) = f^-_2 (x_i, x_0', \omega) \) and \( \partial_t p^-_B(x, \omega)|_{x_3 = x_3, t} \) into equation A-2, using equation A-8, gives

\[ \delta(x'_H - x'_0) = \int_{\partial D_1} T(x_i, x'_0, \omega) f^-_2 (x_i, x'_0, \omega) d\mathbf{x}_i, \]  

(A - 14)

where the modified transmission response \( T(x_i, x'_0, \omega) \) is given by

\[ T(x_i, x'_0, \omega) = \frac{\partial_t T(x_i, x'_0, \omega)|_{x_3 = x_3, t}}{-\frac{1}{2} j \omega p(x_i)}. \]  

(A - 15)

Hence, \( f^-_2 (x_i, x'_0, \omega) \) is the inverse of the modified transmission response \( T(x_i, x'_0, \omega) \) in the sense of equation A-14. In the time domain this is formulated by equation 15.

For the derivation of equation 40, we consider the reference configuration of Figure 2b for both states A and B (but with subscripts i replaced by j, and a prime added to x). For state A we choose a Green’s function in the reference configuration, according to \( p^+_A(x', \omega) = G^+(x', x_j, \omega) \). Assuming the source at \( x_j \) is situated just below \( \partial D_j \), we have, analogous to equation A-4,

\[ \partial_j G^- (x', x_j, \omega)|_{x_3 = x_3, j} = \frac{1}{2} j \omega p(x_j) \delta(x_H - x_H) \]  

(A - 16)
Marchenko imaging

Appendix B: Relation between the Different Marchenko Equations

Marchenko equation 19 for the coda $M(x'_i, x''_0, t)$ of the two-way focusing function $f_2(x'_i, x''_0, t)$ is solved by the iterative scheme of equations 20 and 21. The expression for the $k$th iteration can alternatively be written as a series expansion, according to

$$M_k(x'_i, x''_0, t) = \sum_{l=0}^{k} N_l(x'_i, x''_0, t),$$  \hspace{1cm} \text{(B - 1)}

where

$$N_l(x'_i, x''_0, -t) = -\int_{\partial D_0} dx_0 \int_{-t}^{t} R^l(x''_0, x'_0, t - t') N_{l-1}(x'_0, x'_0, t') dt',$$  \hspace{1cm} \text{(B - 2)}

for $t < t_d(x'_i, x''_0)$, whereas $N_l(x'_i, x''_0, -t) = 0$ for $t \geq t_d(x'_i, x''_0)$. The expansion starts with

$$N_0(x'_i, x''_0, -t) = M_0(x'_i, x''_0, -t).$$  \hspace{1cm} \text{(B - 3)}

The coupled system of Marchenko equations (equations 28 and 29) for $M^+(x''_0, x'_i, t)$ (the coda of the one-way focusing function $f_1^+(x''_0, x'_i, t)$) and $f_1^+(x''_0, x'_i, t)$ is solved by the iterative scheme of equations 30 – 32. The expressions for the $m$th iteration can also be rewritten as series expansions, according to

$$M_m^+(x''_0, x'_i, t) = \sum_{l=0}^{m} N_{2l+1}(x'_i, x''_0, t)$$  \hspace{1cm} \text{(B - 4)}

and

$$-f_{1,m}^-(x''_0, x'_i, -t) = \sum_{l=0}^{m} N_{2l+1}(x'_i, x''_0, t),$$  \hspace{1cm} \text{(B - 5)}

respectively, and $M_{-1}^+(x''_0, x'_i, t) = 0$. Hence, $M_m^+(x''_0, x'_i, t)$ and $-f_{1,m}^-(x''_0, x'_i, -t)$ can be simply obtained by selecting the odd and even terms, respectively, of expansion B-1 of the two-way coda $M_k(x'_i, x''_0, t)$ (to be more specific, $M_m^+$ consists of the odd terms of the expansion of $M_{k=2m+1}$, whereas the time-reversal of $-f_{1,m}^-$ consists of the even terms of the expansion of $M_{k=2m}$).
REFERENCES


Autofocusing imaging: Imaging with primaries, internal multiples and free-surface multiples

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ABSTRACT
Recent work on autofocusing with the Marchenko equation has shown how the Green’s function for a virtual source in the subsurface can be obtained from reflection data. The response to the virtual source is the Green’s function from the location of the virtual source to the surface. The Green’s function is retrieved using only the reflection response of the medium and an estimate of the first arrival at the surface from the virtual source. Current techniques, however, only include primaries and internal multiples. Therefore, all surface-related multiples must be removed from the reflection response prior to Green’s function retrieval. Here, we extend the Marchenko equation to retrieve the Green’s function that includes primaries, internal multiples, and free-surface multiples. In other words, we retrieve the Green’s function in the presence of a free surface. We use the associated Green’s function for imaging the subsurface. The information needed for the retrieval are the reflection response at the surface and an estimate of the first arrival at the surface from the virtual source. The reflection response, in this case, includes the free-surface multiples; this makes it possible to include these multiples in the imaging operator and it obviates the need for surface-related multiple elimination.

1 INTRODUCTION
To focus a wavefield at a point in a medium only requires surface reflection data and an estimate of the first arriving wave at the surface from a point source at the focusing location (Broggini et al., 2012; Broggini and Snieder, 2012; Wapenaar et al., 2013a). Unlike in seismic interferometry (Bakulin and Calvert, 2006), no receivers are required at the desired focusing location, i.e. the virtual source location. Significantly, the detailed medium parameters need not be known to focus the wavefield. However the travel-time of the direct-arrival of the virtual source to the surface is required. To obtain this travel time, one only needs a macro-model of the velocity.

The focusing scheme of Broggini et al. (2012), Broggini and Snieder (2012), and Wapenaar et al. (2013a) is an extension of the algorithm of Rose (2002a,b) who shows an iterative scheme that solves the Marchenko equation for wavefield focusing in one dimension. The focused events in the wavefield for the virtual source consist of primaries and internal multiples (Wapenaar et al., 2013a) but not free-surface multiples. Importantly, Rose (2002a,b) derived the focusing method (autofocusing) for single-sided illumination with sources and receivers on one side of the medium, similar to current geophysical acquisition methods.

We summarize our work in Figure 1. In this paper, any variable with a subscript 0 (e.g. \( R_0 \)) signifies that no free-surface is present. As shown in Figure 1 (solid up-going arrow), for the algorithm of Broggini et al. (2012), one must remove the free-surface multiples from the reflection response of the medium to retrieve the Green’s function by autofocusing. The removal of the free-surface multiples can be achieved by Surface Related Multiple Elimination (SRME) (Verschuur et al., 1992).

Wapenaar et al. (2011a) illustrate imaging with the Green’s function in 1D and also discuss how to image in multi-dimensions (2D and 3D). Similarly, Behura et al. (2012) introduce an imaging algorithm based on the auto-focusing scheme that images not only primaries but also internal multiples, thereby reducing imaging artifacts. Broggini et al. (2014) extend the work of Behura et al. (2012) by using multidimensional deconvolu-
our work (equation (1))

\[ G(x'_i, x_0, \omega) = G_0(x'_i, x_0, \omega) \]

\[ - \int_{\partial D_0} G_0(x'_i, x, \omega) R(x, x_0, \omega) dx, \]  

where \( \partial D_0 \) is the acquisition surface, \( x_0 \) and \( x'_i \) are spatial positions along \( \partial D_0 \) and \( \partial D_i \) (arbitrary depth level), and \( R \) is the reflection response for a down-going incident wavefield. Equation 1 is shown in Figure 1 by the solid down-going arrow. This method of retrieving \( G \) from \( G_0 \) follows from Equation 22 of Wapenaar et al. (2004) which relates the transmission operators for media with and without the free surface. In our case, we replace the transmission operators with the corresponding Green’s functions, \( G \) or \( G_0 \), since the Green’s function is the total transmitted wavefield from the focusing point to the surface that includes multiples. Note that this approach, \( R \rightarrow R_0 \rightarrow G_0 \rightarrow G \), follows the tortuous path shown in Figure 1. We can, however, retrieve the Green’s function in the presence of the free surface directly from the measured reflection data \( R \rightarrow G \) (Figure 1, black arrow). This avoids the more elaborate path \( R \rightarrow R_0 \rightarrow G_0 \rightarrow G \) so that SRME and \( G_0 \) are not required.

We generalize the formulation of Wapenaar et al. (2013a) to include free-surface multiples (\( R \rightarrow G \)); the detailed mathematics of this retrieval is documented in the appendix. The reflections from the free surface are included in the focusing scheme similar to the treatment by Wapenaar et al. (2004) of free-surface multiples.

We define our spatial vector field by its horizontal coordinates and depth coordinates, for instance \( x_0 = (x_H, x_z, 0) \), where \( x_H \) are the horizontal coordinates at a depth \( x_z, 0 \). We define a solution for the waves that focus at a point in a medium, called the focusing functions \( f_1 \) and \( f_2 \). The \( f_1 \) function involves waves that focus at \( x_i \) at a defined depth level \( \partial D_i \) for incoming and outgoing waves at the acquisition surface \( (\partial D_0) \) at \( x_0 \) (Figure 2). The function \( f_2 \) is somewhat the opposite of \( f_1 \) as it is a solution for waves that focus just above \( \partial D_0 \) at \( x'_i \) for incoming and outgoing waves at \( \partial D_i \) (Figure 3). The focusing functions exist in a reference medium that has the same material properties as the actual inhomogeneous medium between \( \partial D_0 \) and \( \partial D_i \) and that is homogeneous above \( \partial D_0 \) and reflection-free below \( \partial D_i \) (Slob et al., 2014). Therefore, the boundary conditions on \( \partial D_0 \) and \( \partial D_i \) in the reference medium, where the focusing function exists, are reflection free. Note that this boundary condition need not be the same as the actual medium. The focusing functions can be separated into

2 THEORY

The theory of focusing the wavefield without a free surface, i.e. retrieving the Green’s function \( G_0 \), is covered by Rose (2002a), Broggini et al. (2012), and Wapenaar et al. (2013a). As summarized in Figure 1, we have to remove the free-surface multiples from the reflection response \( R \) (by SRME) to get \( R_0 \) and then compute \( G_0 \), the Green’s function in the absence of the free surface.

We can retrieve \( G \) (the Green’s function in the presence of the free surface) from \( G_0 \) in the frequency domain with the expression

Figure 1. Overview of the methods to focus the wavefield using an iterative approach. \( R \) denotes reflected waves recorded at the surface in the presence of a free surface, and \( R_0 \) is the reflected waves for a medium without a free surface. \( G \) is the Green’s function at the surface for a virtual source located at a point in the medium in the presence of a free surface and \( G_0 \) is the Green’s function in the absence of a free surface. The two dashed arrows indicate separate iterative schemes.

Elimination (MDD) as the imaging condition in place of conventional cross-correlation or deconvolution, which further reduces the artifacts. In other words, Broggini et al. (2014) retrieve the Green’s function from the acquisition surface to any point in the medium. This Green’s function is essentially an imaging or downward continuation operator. Since this Green’s function includes both primaries and internal multiples, we expect improved subsurface images compared to using primaries alone.

In this paper, we modify the earlier focusing algorithms (Rose, 2002a; Broggini et al., 2012; Wapenaar et al., 2013a) to focus not only primaries and internal multiples but also the free-surface multiples; this is labeled Our Work in Figure 1. We achieve such focusing using reflected waves in the presence of a free surface and an estimate of the first arrival from the focus location to the surface. Notably, our proposed auto-focusing scheme obviates the need for SRME (Figure 1).

The free surface is the strongest reflector in the system; therefore, in general, the free-surface multiples are stronger than internal multiples. In addition, free-surface multiples can be used to provide better illumination, higher fold, and better vertical resolution of the subsurface (Schuster et al., 2003; Jiang et al., 2007; Muijs et al., 2007a,b). For these reasons, by retrieving the Green’s function which includes primaries and all multiples (including free-surface multiples) and using the imaging condition proposed by Behura et al. (2012), we expect better imaging of the subsurface.
Autofocusing imaging: Imaging with primaries, internal multiples and free-surface multiples

In this paper the superscript (+) refer to down-going waves and (−) to up-going waves.

The focusing functions are the same regardless of the surface boundary condition of the actual medium as these functions reside only in the reference medium. The focusing function \(f_2\) consists of the direct arriving wave and the coda \(M\) consisting of the scattered waves that result when the direct arriving wave transmits through the medium to the focus point.

The second focusing function is related to the Green’s function \(G_0\) of the actual inhomogeneous medium without a free surface by (Wapenaar et al., 2013a):

\[
G_0(x_1',x_0',\omega) = f_2(x_1',x_0',\omega)^* + \int_{\partial D_0} f_2(x_1',x_0',\omega) R_0(x_0,x_0',\omega) dx_0.
\]

As shown in the appendix, the Green’s function in the presence of the free surface is given by

\[
G(x_1',x_0'',\omega) = f_2(x_1',x_0'',\omega)^* + \int_{\partial D_0} f_2(x_1',x_0'',\omega) R(x_0,x_0'',\omega) dx_0 + r \int_{\partial D_0} f_2(x_1',x_0'',\omega)^* R(x_0,x_0'',\omega) dx_0.
\]  

where * represents the complex conjugate, and \(r = -1\) is the reflection coefficient of the free surface. The reflection response \(R\) is flux-normalized so that the one-way reciprocity equations (Wapenaar and Grimbergen, 1996) hold. Note the up-going Green’s function (\(G^-\)) in the actual inhomogeneous medium at \(\partial D_0\) is the reflection response \(R\) for a downward radiating source at \(\partial D_0\).

Equation 5 differs from equation 4 in two ways. First, the last term on the right hand side of equation 5 accounts for the waves that are reflected off the free surface. Second, equation 5 contains the reflection coefficient \(R\) for a medium with a free surface, while expression 4 contains the reflection coefficients \(R_0\) for a medium without a free surface.

Similar to our treatment of the focusing function \(f_2\), we can define another focusing function \(g_2\) such that

\[
g_2(x,x_0'',\omega) = f_2^*(x,x_0'',\omega) - f_2(x,x_0'',\omega).
\]

Analogously, we can define a difference Green’s function \(\tilde{G}\) that is related to \(g_2\) similar to expression 5 by

\[
\tilde{G}(x_1',x_0'',\omega) = g_2(x_1',x_0'',\omega)^* + \int_{\partial D_0} g_2(x_1',x_0'',\omega) R(x_0,x_0'',\omega) dx_0 + r \int_{\partial D_0} g_2(x_1',x_0'',\omega)^* R(x_0,x_0'',\omega) dx_0.
\]

We call \(\tilde{G}\) the difference Green’s function since

\[
\tilde{G}(x_1',x_0'',\omega) = G^*(x_1',x_0'',\omega) - G^-(x_1',x_0'',\omega).
\]

To yield the up-going Green’s function, we subtract equations 5 and 8:

\[
G^-(x_1',x_0'',\omega) = \frac{1}{2}[G(x_1',x_0',\omega) - \tilde{G}(x_1',x_0'',\omega)].
\]
Similarly, we obtain the down-going Green’s function by adding equations 5 and 8:

\[ G^-(x'_i, x_0'', t) = \int_{\partial D_i} dx_i \int_{-\infty}^{\infty} G^+(x'_i, x_0'', t - t') R_0^i(x_i, x'_i, t') dt'. \]  

These up- and down-going (\(G^+\) and \(G^-\)) Green’s functions at the focal point are used for imaging and include primaries and all multiples. The up- and down-going Green’s functions have been used for imaging the subsurface, (Behura et al., 2012; Broggiini et al., 2014, 2012; Wapenaar et al., 2011a). However, their Green’s function only contains primaries and internal multiples. In this paper, the up- and down-going Green’s function also includes free-surface multiples.

The use of up- and down-going wavefield for imaging is not a new principle. Claerbout (1971), Wapenaar et al. (2000) and Amundsen (2001) have shown that one can get the reflection coefficient below an arbitrary depth level once the up- and down-going wavefields are available. The governing equation for imaging with these up- and down-going waves is

\[ G^-(x'_i, x_0'', t) \]

\[ = \int_{\partial D_i} dx_i \int_{-\infty}^{\infty} G^+(x'_i, x_0'', t - t') R_0^i(x_i, x'_i, t') dt', \]

where \(\partial D_i\) is an arbitrary depth level, \(R_0^i\) is the reflection response below \(\partial D_i\). In addition, \(R_0^i\) at \(\partial D_i\) is reflection-free above this depth level. We can think of \(R_0^i\) as the reflection response from a truncated medium; where the truncated medium is the same as the true medium below \(\partial D_i\) and reflection free above. Equation 11 states that we can recover \(G^-\) from the convolution of \(G^+\) with \(R_0^i\) and integrate along all source positions \(x'_i\) of \(R_0^i\).

We solve for \(R_0^i\) by multidimensional deconvolution (Wapenaar et al., 2008, 2011b) as the time integral is a convolution. The subsurface image is given by taking the zero lag of \(R_0^i\), i.e. \(t = 0\) at each depth level in the model, (for each \(\partial D_i\)), this is called the multidimensional imaging condition. Alternatively, once we obtain \(R_0^i\) at an arbitrary \(\partial D_i\) we can also apply a standard imaging procedure, for instance, downward continuation, to image below \(\partial D_i\). This is because \(R_0^i\) is the reflection response of the truncated medium below \(\partial D_i\) for sources and receivers at \(\partial D_i\).

**3 NUMERICAL EXAMPLES**

We consider a 1D model that has a high impedance layer generic to salt models as shown in Figure 4. A Receiver at the surface records the reflected waves. To retrieve the Green’s function in 1D, one needs the travel time of the first arriving wave from the virtual source to the surface. In 2D or 3D media, a smooth version of the slowness (1/velocity) can be used to get an estimate of the direct arriving wave from the virtual source to the surface. The direct arriving wave can be obtained using finite-difference modeling of the waveforms.

We obtain the focusing function \(f_2\), Figure 5, by setting the left-hand side of equation 5 to zero and evaluating this expression for a time earlier than the first arriving wave. (Details are given in equations A8 and A7 of the Appendix). The focusing function is substituted in equation 5 to retrieve the Green’s function located at 2.75 km to the surface, Figure 6. This Green’s function \(G\), arbitrarily scaled to its maximum amplitude (Figure 6), is the response at the surface \(\partial D_1\) to the virtual source (located at 2.75 km [dot in Figure 4]).

We also model the Green’s function using finite dif-
Autofocusing imaging: Imaging with primaries, internal multiples and free-surface multiples

Figure 6. Retrieved Green’s function (normalized by maximum amplitude), $G$, from a depth of 2.75 km to the surface (white). The model Green’s function is displayed (in black) in the background.

Figure 7. Impact of the free surface. White line: the Green’s function in a medium without a free surface $G_0$, and black line is the difference between $G$ and $G_0$, therefore the blue line shows events that are caused by the presence of the free surface.

Figure 8. Autofocusing imaging of Figure 4 in yellow, with the true reflectivity (in black) in the background.

Figure 9. Simple velocity model where the dot indicates the position of one virtual source at depth 1000 m.

Figure 10. Relative amplitude shown in Figure 8. In 1D, the autofocus image is the deconvolution of the up- and down-going Green’s function at each image point for $t = 0$. There are some anomalous amplitudes in the autofocus image (especially around 200 m) but they are small compared to the actual reflectors’ amplitude.

The next example is taken from Weglein and Dragoset (2007), where the second primary event cancels with the free surface multiple. I demonstrate with this numerical example the retrieval of the Green’s function (as well as its associated up- and down-going Green’s function) at depth 1000m for the model shown in Figure 9. The associated reflected waves at the acquisition level, shown in Figure 10, are recorded 5m below the free surface. Figure 11 illustrates some of the events that are present in the reflection response. As is shown...
in Figures 10 and 11, the second primary event $P_2$ is canceled by the free-surface multiple $F_1$ at 1.0 s, and the other events (internal multiples and free surface multiples) interfere destructively with each other at later times.

One needs to remove the multiples before implementing standard imaging; however, using SRME with adaptive subtraction for this removal will be problematic as the primary and the free surface multiples completely cancels each other at 1.0 s, and even at later travel times the multiples interfere destructively with the primaries. Hence it will be difficult to image the second reflector by SRME followed by standard imaging.

The Green’s function for a source at depth 1000m is shown in Figure 12. The corresponding up- and down-going Green’s function at depth 1000 m is illustrated in Figures 13 and 14, respectively. The computed travel times for this simple model of the up- and down-going Green’s function Figure 15 corresponds with the events in the retrieved up- and down-going Green’s functions; hence verifying our decomposition of the Green’s function into its associated up- and down-going wavefields.
Figure 14. Down-going Green’s function with virtual source at depth 1000 m and recording at the surface.

Figure 15. Sketch of some events that are present in the Green’s function from the virtual source (red dot) at 1000m and recorded at the surface.

Figure 16. Image of the velocity model after autofocusing imaging with the reflectivity overlain (in black).

4 DISCUSSION/CONCLUSION

The advantage of our autofocusing scheme is that we include the free-surface multiples in our Green’s function, hence we don’t need to remove the free-surface multiples in our reflection response. For our 1D numerical examples, we can isolate the events that are caused by the presence of the free surface (black line in Figure 7) by computing the difference of $G$ and $G_0$. These arrivals that are caused by the free surface (black line in Figure 7) have higher amplitude and greater waveform complexity compared to the events for the response without a free surface $G_0$ (white line in 7). This supports our conclusion that using the additional events and energy that free-surface reflections provide can benefit imaging of the subsurface, i.e. imaging using $G$ rather than with $G_0$.

In our autofocusing scheme we require that the wavelet be removed from the reflection response (which can be done by deconvolution). In addition, we assume that the reflection response is due to a down-going source. However in the marine case the source is generally placed a few meters below the surface, therefore there is not only a down-going component of the source, but also an up-going component. In such situations, we consider the source wavelet to also include the up-going component of the source. Consequently, the wavelet with which we deconvolve the reflection response at the surface is no longer a monopole source but a dipole source.

In summary, we extended the retrieval of the Green’s function to include the presence of a free surface. This function includes primaries, internal multiples, and now free-surface multiples. Significantly, our proposed method does not require any surface-related multiple removal of the reflection response. In addition,
we need an estimate of the first arrival at the surface from the virtual source in the subsurface. To obtain the first arrival, we only need a macro model of the velocity, but the small scale details of the velocity and density need not be known.

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APPENDIX A: GREEN’S FUNCTION RETRIEVAL IN THE PRESENCE OF THE FREE SURFACE

We discuss our adaptation of the method of Wapenaar et al. (2013b) to account for the free-surface reflections. The focusing functions are the wavefields that focus, in both time and space, at a point in the medium (Figure 2 and 3). These focusing functions exist in the reference medium, which is homogeneous above the depth level \( \partial D_0 \) and reflection-free below \( \partial D_0 \).

The reciprocity theorems for one-way (up-going and down-going) wavefields are derived by Wapenaar and Grimbergen (1996). We use the convolution-type and correlation-type reciprocity theorems given in Wapenaar and Grimbergen (1996) to find relationships between our up- and down-going wavefields. As discussed by Wapenaar et al. (2013a), we obtain a relationship for the focusing functions \( f_1 \) and \( f_2 \) by using their respective up- and down-going waves at each depth level \((\partial D_0 \text{ and } \partial D_i)\) with the convolution reciprocity theorem

\[
f_1^+(x_0', x_1, \omega) = f_2^- (x_1, x_0', \omega)
\]  

(A1)

and cross-correlation reciprocity theorem

\[
-f_1^- (x_0'', x_1', \omega)^* = f_2^+ (x_1', x_0'', \omega).
\]  

(A2)

Our actual inhomogeneous model with a free surface above \( \partial D_0 \) is shown in Figure A1. As opposed to the model in Wapenaar et al. (2013a) which does not have a free surface, we consider the reflections from the free surface for a down-going source similar to the work of Wapenaar et al. (2004). In Figure A1, we describe the wavefield in its up-and down-going components. The downward propagating component of the wavefield (Green’s function) at \( \partial D_0 \) is \( G^+ (x_0, x_0'', \omega) = \delta(x_H - x''_H) + rR(x_0, x_0'', \omega) \), which includes (in the right hand side) the downward-going impulsive source and the reflection from the free surface. The down-going source \( \delta(x_H - x''_H) \) is a 2-dimensional Dirac delta where \( x_H \) is the lateral position of the focal point.

Note \( G^+ \) is the component of the Green’s function that is propagating downwards at \( x_0 \) for a downward radiating source at \( x_0'' \). In the case without the free surface, there are no reflections from the free surface, hence \( G_0^+ (x_0, x_0'', \omega) = \delta(x_H - x''_H) \) because \( r = 0 \). The upward-going propagating part of the Green’s function \( G^- \) above \( \partial D_0 \) is the reflection response \( R(x_0, x_0'', \omega) \). We consider the up- and down-going components of the Green’s function just below \( \partial D_0 \). The down-going component is \( G^+(x_1, x_0'', \omega) \) while the up-going component is \( G^- (x_1, x_0'', \omega) (G^+ \text{ and } G^- \) respectively, Figure A1). We use the convolution and correlation reciprocity theorems to find relationships for the one-way wavefields of \( f_1 \) shown in Figure 2 and the one-way wavefields of the Green’s function in the actual medium shown in Figure A1:

\[
G^-(x_1', x_0'', \omega) = \int_{\partial D_0} \left[ f_1^+(x_0, x_1', \omega) R(x_0, x_0'', \omega) - r f_1^+ (x_0, x_1', \omega) R(x_0, x_0'', \omega) \right] dx - f_1^+ (x_0', x_1', \omega)
\]  

(A3)

and

\[
G^+(x_1', x_0'', \omega) = -\int_{\partial D_0} \left[ f_1^- (x_0, x_1', \omega)^* R(x_0, x_0'', \omega) - r f_1^+ (x_0, x_1', \omega)^* R(x_0, x_0'', \omega) \right] dx + f_1^+ (x_0', x_1', \omega)^*.
\]  

(A4)

Equations A3 and A4 are similar to the relation for the up- and down-going Green’s function in Wapenaar et al. (2013a), however equations A3 and A4 also considers the reflected waves from the free surface. These free-surface reflections are the expressions in equations A3 and A4 that are multiplied by \( r \). The two-way Green’s function is obtained by adding equations A3 and A4 as well as using equations 2, 3, A1, and A2:

\[
G(x_1', x_0'', \omega) = f_2^+(x_1', x_0'', \omega)^* + \int_{\partial D_0} f_2^+(x_1, x_0, \omega) R(x_0, x_0'', \omega) dx_0 + r \int_{\partial D_0} f_2^+(x_1, x_0, \omega)^* R(x_0, x_0'', \omega) dx_0.
\]  

(A5)
We consider equation A5, in time, for the interval $t < t_d(x'_i, x''_0)$, where $t_d$ is the travel time for the first arrival of $G$. No waves arrive before $t_d(x'_i, x''_0)$ since $t_d$ is the time for the first arriving event. Therefore, $G(x'_i, x''_0, \omega)$ vanishes for $t < t_d(x'_i, x''_0)$ and as a result

$$0 = f_2(x'_i, x''_0, -t) + \int_{\partial D_0} d\mathbf{x}_0 \int_{-\infty}^{t} f_2(x'_i, x_0, t') R(x_0, x''_0, t-t') dt' + r \int_{\partial D_0} d\mathbf{x}_0 \int_{-t}^{\infty} f_2(x'_i, x_0, t') R(x_0, x''_0, t+t') dt'.$$

We use the same ansatz for $f_2$ as Wapenaar et al. (2013a) because we are using the same reference medium, i.e. the model where the focusing functions exist. The ansatz is given by

$$f_2(x_i, x_0, t) = [T_d(x_i, x''_0, t)]^{inv} + M(x_i, x''_0, t),$$

where $[T_d(x_i, x''_0, t)]^{inv}$, defined as the inverse of the direct arrival of the transmission response, is the first arriving event of $f_2(x_i, x''_0, t)$ and $M(x_i, x''_0, t)$ is the scattering coda of $f_2$ following the first arrival as shown by Wapenaar et al. (2013a). The substitution of expression A7 in equation A6 yields

$$0 = M(x'_i, x''_0, -t) + \int_{\partial D_0} d\mathbf{x}_0 \int_{-\infty}^{-t_d(x'_i, x''_0)} [T_d(x'_i, x_0, t)]^{inv} R(x_0, x''_0, t-t') dt' + \int_{\partial D_0} d\mathbf{x}_0 \int_{-t_d(x'_i, x''_0)}^{t} M(x'_i, x_0, t') R(x_0, x''_0, t-t') dt'$$

$$+ r \int_{\partial D_0} d\mathbf{x}_0 \int_{-t_d(x'_i, x''_0)}^{\infty} M(x'_i, x_0, t') R(x_0, x''_0, t+t') dt'$$

for $t < t_d(x'_i, x''_0)$ with $t_d(x'_i, x''_0) = t_d(x'_i, x''_0) - \epsilon$ where $\epsilon$ is a small positive constant to include the direct arrival in the integral. Equation A8 is a Fredholm integral of the second kind and can be solved iteratively as follows:

$$M_k(x'_i, x''_0, -t) = M_0(x'_i, x''_0, -t) - \int_{\partial D_0} d\mathbf{x}_0 \int_{-t_d(x'_i, x''_0)}^{\infty} M_{k-1}(x'_i, x_0, t') R(x_0, x''_0, t-t') dt'$$

$$- r \int_{\partial D_0} d\mathbf{x}_0 \int_{-t_d(x'_i, x''_0)}^{\infty} M_{k-1}(x'_i, x_0, t') R(x_0, x''_0, t+t') dt',$$

where

$$M_0(x'_i, x''_0, -t) = - \int_{\partial D_0} d\mathbf{x}_0 \int_{-\infty}^{-t_d(x'_i, x''_0)} [T_d(x'_i, x_0, t)]^{inv} R(x_0, x''_0, t-t') dt'.$$
for $t < t_d(x', x'').$ In contrast to the algorithm of Wapenaar et al. (2013a), we use $R$ instead of $R_0$ and we also include the reflection from the free surface (last term in equation A10). We substitute the coda $M$ into equation A7 to yield the focusing function $f_2$. The $f_2$ solution is then used in equation A5 to obtain the two-way Green’s function (equation 5).

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ABSTRACT
Locating hydrocarbon reservoirs has become more challenging with smaller, deeper or shallower targets in complicated environments. Controlled-source electromagnetics (CSEM) is one geophysical method used by industry to find and derisk reservoirs in marine settings. The diffusive nature of CSEM fields means the signal from the target is only a small part of the total field. To reduce the impact of the complicated settings and improve the detecting capabilities of CSEM, we apply synthetic aperture to CSEM data. Synthetic aperture virtually increases the length and width of the CSEM source by combining the responses from multiple individual sources. Applying a weight to each source steers or focuses the synthetic aperture source array in the inline and crossline directions. We introduce an optimization method to find the optimal weights for synthetic aperture arrays that adapts to the information in the CSEM data. To demonstrate the benefits of weighted synthetic aperture, we apply a 2D synthetic aperture array and a crossline only synthetic aperture array to noisy, simulated electromagnetic fields. Both synthetic aperture arrays reduce the noise and increase the anomaly from the reservoir. The crossline only synthetic aperture array also preserves the structure of the model.

Key words: synthetic aperture, CSEM, steering, focusing, optimization

1 INTRODUCTION
Controlled-source electromagnetics (CSEM) is a geophysical method used primarily for finding oil reservoirs in marine settings (Constable and Slnka, 2007; Edwards, 2005; Constable, 2010). CSEM was first proposed in academic research and was implemented in industry over a decade ago (see Constable and Slnka, 2007; Edwards, 2005; Constable, 2010 for history and overview). Industry now uses the method widely to derisk and discover offshore reservoirs (Constable and Slnka, 2007). The method involves towing an electric dipole source over receivers placed on the ocean floor, which record the electric and magnetic fields. The dipole source, operating at low frequencies (typically around 0.1-1 Hz), emits a signal which travels down through the conductive subsurface creating diffusive fields (Constable and Slnka, 2007). The diffusive fields decay quickly which means that the signal from the reservoir is only a small part of the total field. The difficulty of identifying the signal from the reservoir is exacerbated in complicated environments. Finding and derisking reservoirs with CSEM has become more challenging because CSEM is applied to targets that are shallower, deeper, smaller, and in more complex settings. We apply synthetic aperture to reduce the impact of these issues and improve the detecting capabilities of CSEM. Researchers in the radar field first developed synthetic aperture and now many different fields, including medical imaging and geophysics, apply the technique (Van Veen and Buckley, 1988; Jensen et al., 2006). Synthetic aperture utilizes the information from multiple individual sources to create a source array with a longer aperture. Fan et al. (2010) first applied synthetic aperture to CSEM fields using sources from a single towline to create a source array several kilometers long. The use of synthetic aperture has expanded to include sources from multiple towlines which allows for the creation of a 2D source array. We give a weight to each source in the synthetic aperture source array to maximize the signal from the reservoir. The weighting is analogous to beamforming with synthetic aperture radar and allows us to steer or focus the energy in the inline, crossline or both directions. In this paper, we first review the application of weighted synthetic aperture to CSEM. Then we introduce a method to find the optimal weighting parameters for a synthetic
aperture source array. Finally, we present two examples of applying the optimal weighted synthetic aperture to synthetic electromagnetic fields with noise added.

2 WEIGHTED SYNTHETIC APERTURE

We review the theory and history of weighted synthetic aperture and present a new weighting formulation for applications to CSEM. Fan et al. (2010) first applied synthetic aperture to CSEM fields; however, the technique was developed earlier for radar (Barber, 1985). Currently, many fields use the technique, including radar, sonar, medical imaging, to increase resolution or detectability (Van Veen and Buckley, 1988; Barber, 1985; Jensen et al., 2006). The technique virtually increases the length of the aperture of a source by summing responses from multiple individual sources. To create a beam to steer or focus, one weights the sources in the synthetic aperture source array; there are numerous algorithms from the radar field to create a beam (Van Veen and Buckley, 1988). Synthetic aperture was only recently applied to CSEM fields because it was thought diffusive fields do not have a direction of propagation (Mandelis, 2000). Løseth et al. (2006) demonstrated that electromagnetic fields can be described by both wave and diffusion equations. A solution to the 3D scalar diffusion equation is a plane wave at single frequency with a defined direction of propagation (Fan et al., 2006). Synthetic aperture was only recently applied to CSEM fields because it was thought diffusive fields do not have a direction of propagation (Mandelis, 2000). Løseth et al. (2006) demonstrated that electromagnetic fields can be described by both wave and diffusion equations. A solution to the 3D scalar diffusion equation is a plane wave at single frequency with a defined direction of propagation (Fan et al., 2006). Synthetic aperture was only recently applied to CSEM fields because it was thought diffusive fields do not have a direction of propagation (Mandelis, 2000). Løseth et al. (2006) demonstrated that electromagnetic fields can be described by both wave and diffusion equations. A solution to the 3D scalar diffusion equation is a plane wave at single frequency with a defined direction of propagation (Fan et al., 2006).

The equation for synthetic aperture in the frequency domain is given by

\[ S(r) = \sum_j a_j F_j(r), \]  

where \( a_j \) is a complex weighting term and \( F_j(r) \) is the response of any component of the electric or magnetic field for each source \( j \) at the location \( r \) and single frequency \( \omega \). For CSEM, implementing synthetic aperture is a post-acquisition step and does not require any changes to the acquisition design. Because the synthetic aperture source array is composed of multiple individual sources, the direction of radiation can be steered or focused by weighting the individual sources. Fan et al. (2011, 2012) demonstrate steering and focusing with CSEM fields with a single towline. Generalizing to using sources from several towlines, one can choose the weights to steer the source array in the inline direction (along the towline), the crossline direction (perpendicular to the towline), or in both directions. Previously, we used exponential weighting where we chose a single value for the amplitude term and a steering angle for the phase shift (Fan et al., 2011, 2012; Knaak et al., 2013). This type of weighting is analogous to a fixed beamformer for radar where the weighting is independent of the signal (Van Veen and Buckley, 1988). The phase shift and amplitude terms for exponential weighting are linear in the spatial coordinates, which essentially forces the source array to radiate a plane wave. This type of weighting is not ideal for every situation. For example, a 2D source array centered over a reservoir would be more effective with weights that focus the energy towards the center. To achieve a less restrictive formulation, we define the weight as a single complex number for each source. The new weighting creates an adaptive, weighted synthetic aperture source array where the weight is allowed to take on any value. With this formulation, the number of weights corresponds to the number of sources in the synthetic aperture array. Previously, we tested different combinations of phase shifts and amplitude terms to find the best steering parameters, with the range of steering angles and amplitudes set by what seemed reasonable based on the geometry. Testing combinations is impractical given the large number of weights in the new formulation. Also with a 2D source array, the functional form of the weights is not easily known. Focusing may be optimal for some source locations while steering works better for others. To determine the optimal weights for a 2D source array, a new solving method is needed. In the next section, we introduce an optimization method used to find the optimal weights for the synthetic aperture source array.

3 OPTIMIZING THE WEIGHTS FOR SYNTHETIC APERTURE

To ensure that weighted synthetic aperture highlights the reservoir optimally for every source array location, we use optimization to solve for the weights used to steer the synthetic aperture source. The goal of applying synthetic aperture to CSEM data is to increase the detectability of the reservoir. We measure the detectability as the magnitude of the difference between the pay field and the wet field. The pay field is the electromagnetic field recorded from a CSEM survey or the fields generated from a model including a reservoir. The wet field is the background field without the reservoir. To implement this method with real data, one needs an estimate of the response without the reservoir or the response from a nearby location without a reservoir. We apply weighted synthetic aperture to both the pay and wet fields, as in equation 1, to determine the increase in the signal from the reservoir. The equations for the weighted synthetic aperture pay field and wet field are given below:

\[ S_p(r) = \sum_j a_j F_p^j(r) \]  

\[ S_w(r) = \sum_j a_j F_w^j(r), \]  

where \( F_p^j(r) \) and \( F_w^j(r) \) are any component of the electric or magnetic field at receiver \( r \) from the pay field and wet field respectively and \( a_j \) is the weight for the
source \( j \). One way to measure the anomaly from the reservoir is to take the difference between the pay and wet fields. The difference gives the contribution from the secondary field created by the presence of the reservoir. This is the measure we use in the optimization scheme to determine the optimal weights. The difference between the weighted synthetic aperture pay and wet responses is given by:

\[
\Delta S(r) = S^p(r) - S^w(r). \tag{4}
\]

The optimal weights are those that maximize the difference between the two steered synthetic aperture responses. One way to create a large difference between the responses is to use a set of weights equal to a large scalar value, which amplifies the magnitude of the response from each source in the synthetic aperture array. This type of weighting effectively increases the amount of energy radiating by the source array instead of increasing the signal from the reservoir. To ensure the energy radiated by the source array is fixed, we place the following constraint on the weights \( a_j \):

\[
\sum_j |a_j|^2 = 1, \tag{5}
\]

Now the weights are constrained in amplitude. The constrained optimization problem maximizes the difference between the pay and wet fields while constraining the total energy radiated:

\[
\max |\Delta S(r)|^2 \text{ subject to } \sum_j |a_j|^2 = 1. \tag{6}
\]

We define the optimal weights as those that create the maximum difference \( \Delta S \) in the weighted synthetic aperture pay and wet responses at receiver location \( r \). The optimization gives higher amplitude to the sources with more information about the reservoir. We select the quadratic objective function in equation 6, rather than the ratio between the pay and wet fields, because it gives a linear system of equations for the weight \( a_j \). The optimization method we outline above is similar to linear constrained optimization beamformers for synthetic aperture radar (Van Veen and Buckley, 1988). The common way to solve this type of constraint optimization problem is to use Lagrangian multipliers (Boas, 1983; Aster et al., 2005). However, because of the linearity of the objective function, we apply a different solving method. The quadratic term \( |\Delta S(r)|^2 \) in equation 6 is the equation for an ellipsoid, which describes a sphere. The optimal weights occur at the intersection of the sphere and the ellipsoid which is the principal axis of the ellipsoid. Figure 1 depicts the problem with 2D shapes. We can rewrite the inversion problem in quadratic form as

\[
\max a^\top H a^*, \tag{7}
\]

where * denotes the complex conjugate, \( a \) is the vector of optimal weights, and \( H \) is a Hermitian matrix. The components of \( H \) are \( \Delta F_j \Delta F_k^\top \), the difference between the unweighted pay and wet fields, for \( j = 1, \ldots, n \) and \( k = 1, \ldots, n \) with \( n \) equal to the number of sources in the synthetic aperture source array. The matrix is diagonalized to rotate to the principal axes of the ellipsoid by decomposing the Hermitian matrix into \( H = U \Lambda U^\top \). The eigenvector \( u_j \) corresponding to the largest eigenvalue \( \lambda_j \) is the vector of optimal weights \( a \). We meet the weighting constraint by normalizing the vector of weights. The difference between wet and pay fields is equivalent to the imprint of the reservoir on the response. The amplitude of this signal is several magnitudes larger at small source-receiver offsets than at larger source-receiver offsets. The inversion focuses on the locations with higher magnitude in the difference of response because the goal is to maximize the difference. However, there is valuable information in the signal at larger offsets. To force the inversion to value all the differences between responses evenly, the responses are weighted by the inverse of the amplitude of the wet field \( F_j^w(r) \) as shown below:

\[
W_j(r) = 1/|F_j^w(r)|. \tag{8}
\]

We apply the weighting to each response from a source in the synthetic aperture array. This type of weighting is commonly used in inversion of CSEM data to equalize the amplitudes (Weitemeyer et al., 2010). The difference with evenly valued data is given by

\[
\Delta S_j(r) = W_j(r)(S_j^p(r) - S_j^w(r)). \tag{9}
\]

Now the optimization scheme finds the optimal weights that highlight the reservoir for each individual source, even those at large offsets. The optimization method solves for data-dependent weights that create an adaptive beamformer to maximize the signal from the reservoir encoded in the electromagnetic fields. The only inputs are a component of the electric or magnetic fields.
of the sources in the synthetic aperture array. The user decides on the length and width of the source array, which allows the method to work with any survey geometry. The optimization also independently switches from steering to focusing, depending on the geometry, without additional information from the user. To show the impact of these characteristics of the optimization method and the benefits of weighted synthetic aperture, we present two examples from modeled electromagnetic fields of two shallow reservoirs in a marine setting.

4 SYNTHETIC EXAMPLES

We present examples from a synthetic model to demonstrate the benefits of an optimized, steered synthetic aperture source array. The synthetic electromagnetic fields were generated with the IBCEM3D code, modeling software for 3D electromagnetic fields (Endo and Zhandanov, 2009). We modeled a shallow water situation (water depth of 200 m) with two reservoirs that are laterally separated. The model has an anisotropic layered background with typical vertical resistivities found in shallow water locations, shown in Figure 2. The two reservoirs are both 1.5 km below the seafloor and 50 m thick with a resistivity of 50 \( \Omega \text{m} \). The two reservoirs are separated 1.5 km laterally as shown in Figure 2. The source is a 270 m horizontal electric dipole with a frequency of 0.2 Hz. The survey has five towlines spaced 1.5 km apart, each with 186 source locations. The 61 receivers are along one line, centered in the crossline direction, and spaced 500 m apart in the inline direction. A map view of the survey design is shown in Figure 3. To make our examples more realistic, we add a typical noise floor of \( 10^{-15} \text{ V/Am}^2 \) independent random noise to the simulated electromagnetic fields (Constable, 2010). A benefit of the outlined weighted synthetic aperture technique is the flexibility of the method to work for several different applications. Here we present two different applications of weighted synthetic aperture. The first example is for a situation where a higher level of detectability is required. To increase the magnitude of the recorded anomaly, we apply a 2D weighted synthetic aperture source array. The second example is for a situation where more information about the structure is needed. Resolving the two reservoirs in the model is best done with crossline steering only because the inline steering spatially averages the two anomalies. For these examples, we use only the inline component of the electric field. To view the electromagnetic fields, we use common midpoint versus offset plots, which show data points with common offsets along the same horizontal line. Displaying the response from the synthetic aperture array this way creates a pseudo-depth section (Silva Crepaldi et al., 2011). The difference is the measure of the response maximized in the optimization method; however, it is more informative to view the normalized difference, which is the difference divided by the absolute value of the background field. Figure 4 shows the normalized difference of the modeled inline electric pay and wet fields with noise added for no synthetic aperture (panel a), 2D steered synthetic aperture (panel b), and steered crossline only synthetic aperture (panel c). For the original data, the anomaly from the reservoir appears at 7 km offset and the maximum of the anomaly is 27%, which includes the signal from the reservoir but also noise. A typical criterion for detectability in CSEM surveys is a normalized difference of around 20% (Constable, 2010). With a 2D weighted synthetic aperture source array (shown in Figure 4(b)), the anomaly increases in magnitude and spatial area. However, the structural information about the two reservoirs is obscured. Applying synthetic aperture only in the crossline direction preserves the two anomalies (Figure 4(c)). The details of each example are described in the sections below.

We apply synthetic aperture to increase the anomaly from a reservoir in imaging. However, the only way to recover a model of the subsurface from CSEM data is through inversion. Figure 4(b) obscures the true structure of the model in the image, but an inversion of the weighted 2D synthetic aperture source may produce a more accurate model of the subsurface than the original data. Future work will focus on the effects of synthetic aperture to inversion of CSEM data.
The normalized difference of the inline electric pay field and the inline electric wet field with 10^{-15} V/Am^2 independent random noise added for the original data (panel a), the optimal 2D steered synthetic aperture source array (panel b), and the optimal crossline steered synthetic aperture source array (panel c).

4.1 Increasing detectability

If the goal of applying synthetic aperture is to increase the signal from the reservoir then the best method is to use a 2D source array because information from both the crossline and inline directions is included. To apply 2D synthetic aperture and find the optimal weights, we first decide on a length and width for the 2D synthetic aperture source. A larger source array creates higher detectability but the averaging in the inline direction also increases, which can obscure structure. For this example, we arbitrarily use 21 sources in the inline direction and all five sources in the crossline direction. The resulting synthetic aperture source array is 5.7 km long and 6 km wide. The source spacing in the inline direction (270 m apart) is denser than the spacing in the crossline direction (1.5 km apart). Even with this discrepancy, we achieve coherent focusing in the crossline direction. We apply the optimization scheme to find the 105 weights that maximize the difference between the pay and wet fields for one source array location. We move the synthetic aperture source array around the entire survey footprint to simulate towing the 2D synthetic aperture source array. The normalized difference of the inline electric fields for the 2D steered synthetic aperture source is shown in Figure 4(b). With the application of the optimal weighted 2D synthetic aperture source, the anomaly from the reservoir has increased in magnitude and spatial extent. The maximum normalized difference is 46% which is an increase from the 27% anomaly in the original noisy data. Additionally, the noise, shown in the large offsets in the image without synthetic aperture, Figure 4(a), does not appear in the normalized difference of the noisy inline electric fields from the steered 2D synthetic aperture source. There is still some noise in the image but the addition of multiple sources in the synthetic aperture source array increases the magnitude of the coherent signal and almost completely stacks out the random noise.

To understand the adaptive nature of the optimization scheme, it is useful to look at the optimal weights for different source array locations. The optimal weights for one source array location and receiver can be viewed in phase and amplitude plots. Figures 5(a) and 5(b) display contour plots of the amplitude and phase, respectively, of the steering coefficients for the source array centered at -8.26 km and the receiver located in the center of the survey between the two reservoirs. The optimal weights for this location steer the field toward the center by giving a higher phase shift to the source farther away in the inline direction. In the crossline direction, the weights focus toward the center with a parabolic phase shift where the outer towlines are weighted higher. The amplitude plot (Figure 5(a)) shows that the sources closer to the nearest reservoir, for this source array position, have a higher weight. The sources given lower amplitude weight contain less information about the reservoirs than those weighted higher. Figures 6(a) and 6(b) display contour plots of the amplitude and phase, respectively, for the source array centered at 3.08 km. This source array location is directly over one reservoir, and the amplitude plot shows that more emphasis is placed on the source locations with larger offsets. Less emphasis on the sources over the reservoir is congruent with the expected weighting because small source-receiver offsets are dominated by the background signal (Constable, 2010). The phase shifts (Figure 6(b)) are similar to those from the other source array location but with a larger phase shift across the source array in the inline direction. Figures 5(a) and 6(a) show there is a section of the synthetic aperture array that has amplitude weighting close to zero, which shows that the sources in that part of the source array are not contributing to the increase in the anomaly from
the reservoir. The optimization method essentially implemented a smaller synthetic aperture for these two examples, which indicates we could have chosen a smaller synthetic aperture length. However, because the method is able to recognize when to reduce the length, there is no detriment from the longer synthetic aperture.

The optimal 2D weighted synthetic aperture source array increases the magnitude and spatial area of the anomaly. However, 2D steering averages the two anomalies into one large anomaly which conceals the fact that there are two reservoirs present. The ability to discern if two reservoirs are present is difficult in CSEM data. To increase the anomaly and retain the information about the structure, we apply a different steering method.

4.2 Increasing lateral resolution

It is often difficult with CSEM field responses to determine if a reservoir is one unit or two separate resistors. If the goal is to differentiate two bodies, then crossline synthetic aperture is the best choice because the inline steering averages the two anomalies from the two receivers and the anomaly appears to be from one reservoir. We use sources from all five towlines to create a 6 km synthetic aperture in the crossline direction. We apply the optimization method for the crossline synthetic aperture source array for all source array locations and receivers. The process is the same as for the 2D source array but now we solve for five optimal weights for each source array instead of 105. We apply the optimal weights to the inline electrical component of the pay and wet fields for each source in the crossline source array.

Figure 4(c) shows the normalized difference of the crossline weighted inline electric fields. The two reservoirs are more discernible with crossline weighted synthetic aperture than in the original data (Figure 4(a)) or the normalized difference of the 2D steered inline electric fields (Figure 4(b)). The crossline only synthetic aperture increases magnitude and spatial localization of each individual reservoir and does not blur the two separate anomalies into one large anomaly. To quantify the improvement, we take the spatial average of the normalized difference from 6 km to 8 km offset and 0.5 km to 4 km CMP. The average normalized difference for the crossline weighted synthetic aperture field in this area is 21%; while the same spatial average of the original field is 17%. The noise is more visible in the crossline synthetic aperture fields than in the 2D steered fields because fewer sources are in the synthetic aperture source array, but the noise level is smaller than it is for the original data (Figure 4(a)). The optimal weights in the crossline direction create a focus by giving the sources farthest away larger phase shifts and amplitudes. Figure 7(a) shows the parabolic phase shifts for the optimal weights for the crossline source array located at -17.44 km. We did not require the optimization to create a focus, but the inversion found the best weights for the situation. We can verify if these weights are reasonable by analytically calculating the phase shifts for each of the five sources to focus the field on the reservoir. The equation for a phase shift to create a focus is given by (Fan et al., 2011):

$$\Phi(x, y, z) = k\sqrt{(x-x_f)^2 + (y-y_f)^2 + (z-z_f)^2 - D},$$

where \((x_f, y_f, z_f)\) is the location of focus, \(k\) is the wavenumber, and \(D\) is a distance used to normalize the phase shift. To use equation 10, we assume a homogeneous field and a single resistivity. We set the resistivity equal to 3 \(\Omega m\) (the resistivity of the second-to-last layer in our model) and set the depth of the focus at 1.51 km, which is the depth of the reservoirs. The focus point varies for each source array location. We choose one source array location to compare the optimal focusing with the calculated focusing. The calculated focus point
that produces a curvature matching the optimal weights is \((x_f, y_f, z_f) = (-2.21 \text{ km}, 0 \text{ km}, 1.51 \text{ km})\). The phases of the optimal weights and the calculated weights are shown in Figure 7(a), and the location of the sources and calculated focus point are shown in Figure 7(b). The focus point is only an estimate of where the optimal weights focus point is located because we assume in the calculation a homogeneous resistivity model for the calculation. We find the phase for the calculated focus point that almost identically matches the curvature of the phase of the optimal weights and the spatial location of calculated focus point is reasonable for the geometry of the survey, which demonstrates the optimal weights agree with the analytical focusing. The optimization method thus solves for the weights that correspond to the optimal focus point for each source array location without any additional inputs from the user. In this example, there are five towlines symmetric about the reservoirs which make focusing the best weighting option. The steering or focusing created by the optimal weights depends on the geometry of the survey, the information within the responses, and the size of the array.

5 CONCLUSION

Locating smaller, deeper or shallower targets with CSEM in complicated environments is becoming more challenging. We demonstrated the benefits of applying the technique of synthetic aperture which virtually increases the length and/or width of the source. Applying weights to the synthetic aperture source array allows us to steer or focus the array in the inline and crossline directions. With complex settings and more intricate survey geometry, the best type of weighting is no longer intuitive. We presented a method to optimize the weights for synthetic aperture source arrays, which acts as an adaptive beamformer by adjusting to the information about the reservoir encoded in the CSEM data. A 2D synthetic aperture source array applied to CSEM data increases the detectability of the reservoir and reduces noise but may obscure structure. We found that applying crossline weighting to noisy inline electric fields from a model with two laterally separated reservoirs preserved the structure, increased the magnitude of the anomalies from the reservoirs, and reduced the noise. However, the impact of weighted synthetic aperture on inversion results is unknown; a 2D source array may have more information about the structure when inverted. Future work will explore if applying weighted synthetic aperture before inversion increases the accuracy of the recovered model. We will also continue to work with synthetic aperture for the forward problem by testing the technique on more complicated models.

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Event locating using deconvolution: Experimental application and analysis

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ABSTRACT

Time reversal techniques are used in ocean acoustics, medical imaging, seismology, and non-destructive evaluation to backpropagate recorded signals to the source of origin. We demonstrate experimentally a technique which improves the temporal focus achieved at the source location by utilizing deconvolution. One experiment consists of propagating a signal from a transducer within a concrete block to a single receiver on the surface, and then applying time reversal or deconvolution to focus the energy back at the source location. Another two experiments are run to study the robust nature of deconvolution by investigating the effect of changing the stabilization constant used in the deconvolution and the impact multiple sources have upon deconvolutions’ focusing abilities. The results show that we are able to generate an improved temporal focus at the source transducer using deconvolution while maintaining the robust nature of time reversal. Additionally, deconvolution’s costs are negligible due to it being a preprocessing step to the recorded data. The technique can be applied for detailed investigation of the source mechanisms (e.g. cracks) but also for monitoring purposes.

Key words: Time Reversal, Deconvolution, Signal Focusing, Experimental, Robust Nature

1 INTRODUCTION

Several methods are used to evaluate acoustic signal generated by events in media such as water, rocks, metals or concrete. Most of them are summarized as acoustic emission methods (AE), which are used to localize and characterize the point of origin of the events. Sophisticated methods have been developed in seismology to localize and characterize earthquakes. Time Reversal (TR) has been a topic of much research in acoustics due to its robust nature and ability to compress the measured scattered waveforms back at the source point in both space and time Parvulescu and Clay (1965); Fink (1997); Anderson et al. (2008); Larmat et al. (2010). This has led to TR being applied in a wide variety of fields such as medicine, communication, ocean acoustics, seismology or nondestructive evaluation. However, continued work is being done to improve TR’s ability to focus energy. Some newly developed techniques use an array of input transducers, measure the wave field with an array near the desired focal spot, and then optimize the spatial and temporal focusing Tanter et al. (2000, 2001); Montaldo et al. (2004); Vignon et al. (2006); Roux and Fink (2000); Aubry et al. (2001); Jonsson et al. (2004). Other methods use an array of input transducers and optimize the temporal focusing at an output transducer Daniels and Heath (2005); Qiu et al. (2006); Blomgren et al. (2008); Zhou et al. (2006); Zhou and Qiu (2006).

In this paper, we design and execute evaluation experiments to compare conventional time reversal to an improved variant which uses deconvolution (DC). We explore the application of DC, which is a primitive though robust version of the inverse filter Tanter et al. (2000); Gallot et al. (2011), to calculate the optimal signal for backpropagation. The ultrasound experiments are performed on a concrete block which has sources embedded within. Instead of using a large array of receivers, the experiments use only a single receiver to record the scattered waveforms. TR or DC is then applied to the measured scattered waveforms to calculate the TR and DC signal. The calculated signals are then backpropagated from a transducer on the surface of the block into the medium and recorded at the original source location transducer. By this experiment, we are able to explore and compare the capabilities of TR
and DC to focus the measured waveforms at a point in both space and time. We show that DC significantly improves the temporal focus compared with TR. We also run two different experiments to study the robust nature of deconvolution by investigating the effect of changing the stabilization constant used in the deconvolution and the impact multiple sources has upon deconvolutions’ focusing abilities.

2 DECONVOLUTION THEORY

Time reversal (TR) is a process used to compress the measured scattered waveforms at a point in both space and time to ideally a Dirac delta function δ(t). It uses the recorded impulse response which can be represented by a Green function $G_{AB}$ that accounts for the wave propagation between two points $A$ and $B$. TR then simply reverses the signal in time and propagates it back from the receiver location into the same medium. By doing so, one expects the energy to focus at the source location. The TR process can be represented by the following equation,

$$\int_{-\infty}^{\infty} G_{AB}(\tau)G_{AB}(\tau - t) \, d\tau = \delta(t), \quad (1)$$

where reciprocity has been used to replace the Green’s function $G_{BA}$ with $G_{AB}$. According to equation 1, the TR process, which is equivalent to the autocorrelation of $G_{AB}(t)$, should ideally lead to equal a delta function. In practice, however, one cannot truly recreate a Dirac delta function focus due to one or more conditions, necessary to satisfy Eq. 1, not upholding. In order for it to work perfectly, one must record for infinite time, Green’s functions are assumed to have a flat, infinite bandwidth, the medium is not attenuative, and one must have full coverage of the wavefield at a surface surrounding the points $A$ & $B$. These requirements are not upheld during an experiment. This led us to explore the application of deconvolution.

We can rewrite Eq. 1 in a more generalized form (using a convolution notation, rather than the integral form) as

$$F(t) = g(t) \ast R(t) \approx \delta(t), \quad (2)$$

where $\ast$ denotes convolution, $F(t)$ is the focal signal or source reconstruction, $R(t) = G_{AB}(t) \ast S(t)$ is the recorded signal measured at the receiver location $B$ from the initial source propagation, and $g(t)$ is the signal necessary to be back propagated for focusing. We are able to go from Eq. 1 to Eq. 2 because we only investigate signals between the two points $A$ and $B$, and remove the Green function notation to indicate we do not have infinite bandwidth. Thus, we remove some of the unrealistic conditions that are required for Eq. 1 to hold. For a TR process, the signal for backpropagation is purely the time reversed recorded signal: $g(t) = R(-t)$. Our goal is to calculate the optimal signal $g(t)$ such that the focal signal $F(t)$ approximately equals a Dirac delta function $\delta(t)$. Deconvolution equates to inverse filtering by transforming to the frequency domain, thus Eq. (2) becomes

$$F(\omega) = g(\omega)G(\omega)S(\omega) \approx 1. \quad (3)$$

Equation (3) is used to solve for $g(\omega)$,

$$g(\omega) = \frac{1}{G(\omega)S(\omega)} \frac{G(\omega)^* S(\omega)^*}{|G(\omega)S(\omega)|^2}, \quad (4)$$

where $*$ denotes complex conjugation. Eq. 4 is, however, unrealistic for experimental use in the event that there is a limited bandwidth, significant background noise, or more specifically, if $R(\omega) = 0$ at any frequency. To avoid the associated singularity, we add a constant to the denominator of the last term of eqn. 4 to ensure that we never divide by 0, hence Eq. 4 becomes,

$$g(\omega) = \frac{G(\omega)^* S(\omega)^*}{|G(\omega)S(\omega)|^2 + \epsilon}, \quad (5)$$

where $\epsilon$ is a constant related to the original received signal as

$$\epsilon = \gamma \text{ mean}(|G(\omega)S(\omega)|^2). \quad (6)$$

The quantity $\gamma$, which is sometimes referred to as the waterlevel parameter Clayton and Wiggins (1976), is a constant chosen to optimally reduce the effect of noise introduced through the DC procedure. Here we use $\gamma = 0.9$ for all experiments. The value 0.9 was chosen based on optimizing the focusing energy in a process similar to that developed by Clayton et al. Clayton and Wiggins (1976). Equation 5 gives the solution for $g(\omega)$. One only has to inverse Fourier transform this result to retrieve the “optimal” DC signal in the time domain to be backpropagated such that one gets a approximate Dirac delta function focus.

3 EXPERIMENTAL SET UP

A laboratory experiment was created, and run in the Civil Engineering lab of Colorado School of Mines. A 30x30x37 cm$^3$ concrete block was cast from 72 kg Quickcrete mix (No. 1101, max aggregate grain size < 4 mm), with about 60 liters of water and 5 kg of additional gravel (5-15 mm grain size). The block contains only minimal reinforcement as shown in Figure 1(a). Three ultrasonic piezo transducers (type Acsys SO807, center frequency 60 kHz, labeled ES in Figure 1) were attached to the minimal reinforcement in order to cast them within the concrete block. These sources (transducers) are visible in Figure 1(a). These sources were oriented differently in order to generate more complex waveforms and to study the effect of varying source orientation. Source 3 was oriented perpendicular to the
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Source set up and reinforcement placed within the concrete block

Acquisition workflow

Backpropagation workflow

**Figure 1.** Diagram indicating tools and methods used during acquisition and backpropagation.

For backpropagation, the setup is reversed. The BAM US device is removed. The transmitter signal generator is replaced by the digital/analog converter integrated in the data acquisition device, sending the computed, time reversed/deconvolved waveforms to the external sensor. The embedded sensor is used as receiver, again using the preamplifier before AD conversion and recording. This reversed setup is shown in Figure 1(c). We have used a sampling frequency of 2 MHz and 20,000 samples per trace (10 ms recording time). A 4,000 sample (2 ms) pre-trigger interval was set. The reason for the longer pre-trigger interval is explained in Ulrich et al. Ulrich et al. (2013). Amplitude resolution is 16 bit. True zero time of the transmitter could be identified by electromagnetic crosstalk between transmitter and receiver cables, generating a small but easy to recognize impulse in the receiver data. The laboratory contained other noise due to multiple experiments being run simultaneously. Due to a high noise lab environment and a lack of a power amplifier for the backpropagated transmitter signal, we apply an additional 2 kHz low-pass Butterworth filter on all data.

In order to test the stability of deconvolution for different values of the regularization parameter $\gamma$, we run the exact same experiment as for a single source described above and shown in Figure 1(b). Once the signal was recorded, we applied DC multiple times with different gamma values in order to generate the different signals to be backpropagated. We then ran the same workflow as shown in Figure 1(c) for each DC signal separate, recording the focused wavefield at the source location each time.

For multiple sources, we executed the workflow described above and shown in Figure 1(b) three times (once for every source). This was necessary because we did not have the equipment capabilities in this laboratory to generate a source function at all three source locations at different onset times. We recorded these three generated wavefields separately and then superimposed them. Due to the experiment being run separately three times, each recorded signal was normalized independently. This caused our recorded signals for all three sources to vary between amplitudes of $-1$ and $1$. Thus, it destroyed the relative amplitude variation one would expect for three different sources at different locations and orientations. Once superimposed, TR or DC was applied and we carried out the same backpropagation workflow as shown in Figure 1(c). During the backpropagation, one restores the relative amplitudes in the focus achieved because of reciprocity.
6.5
6.5
10
10
9.5
8
8
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10
9.5
9
73x79
DC calculated signal. The temporal focus achieved using TR while Figure 2 recorded at the source location where Figure 2 (a) is an example at the direction of source emission. Once our wave field was recorded at the single receiver, TR or DC was applied to calculate the back propagating signals.

For a single source, deconvolution ideally achieves an improved temporal focus. This is due to there being a single term in the denominator as shown in Eq. 5, which leads to the following,

\[ g(\omega)G(\omega) \approx \frac{G(\omega)G^*(\omega)S^*(\omega)}{|G(\omega)S(\omega)|^2 + \epsilon} \approx \frac{1}{S(\omega)}, \]  (7)

where \( G(\omega) \) represents the Green’s function describing the propagation between source and receiver, \( S(\omega) \) represents the source function in the frequency domain, and \( g(\omega) \) the signal we are trying to solve for with deconvolution. Equation. 7 should approach \( 1/S(\omega) \) as \( \epsilon \) approaches 0. Therefore, the focus achieved using deconvolution approaches an optimal reconstruction of the inverse of the source function and not necessarily a Dirac delta function. When the source function is a delta function, \( S(\omega) \) is constant, this leads to a delta function at the focal point.

The DC and TR signals were then backpropagated from the transducer on the surface of the block into the same medium and recorded at the original source location transducer. Figure 2 shows the refocused waves recorded at the source location where Figure 2 (a) is the temporal focus achieved using TR while Figure 2 (b) represents the temporal focus achieved using the DC calculated signal. The temporal focus achieved using TR has significant side-lobes away from the time of focus; the temporal focus achieved using DC has suppressed most of these side-lobes and was able to produce a better focus. In order to quantify this improvement, we calculate the amount of energy in a .02 ms window around the time of focus compared to the total energy of the signal. The temporal focus achieved using TR only had 41% of the total energy within this window while DC’s temporal focus had 80% of the total energy within this window. Thus, DC is able to generate a significantly better temporal focus than TR. Our source function used wasn’t a Dirac delta function but deconvolution still improved the focus significantly as it improved the reconstruction of our source function. Once we had shown that deconvolution was able to improve the focus at a point in both space and time, we continued our experimental studies to investigate the robust nature of deconvolution.

4.2 Regularizing the deconvolution

The purpose of this experiment was to study the robust nature of DC by changing the location of the receiver and investigating the effect of regularizing the deconvolution through the parameter \( \epsilon \) used in equation 5. This experiment started the same way as our previous single source experiment. We first propagated a defined 60 kHz source function from the embedded source towards the external receiver. For this experiment, the receiver’s direction of measurement was parallel to the direction of source emission. The recorded signal was then deconvolved using various values of \( \gamma \) which was the constant scalar number used to characterize the regularization term \( \epsilon \) defined in equation 6. These calculated deconvolved signals were propagated back into the medium from the receiver location and recorded at the source transducer.

Theoretically, we would expect the focused wavefield to contain significant amount of noise at low \( \gamma \) val-
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The temporal focus, defined as the amount of energy in a 0.02 ms window around the time of focus compared to the total energy of the signal, as function of $\gamma$. High temporal focus indicates most of the energy is compressed at the time of focus.

Values. As $\gamma$ increases, the temporal focus is expected to improve because we reduce the effect of noise and force our signal to generate a better approximate Dirac delta function focus. However, if $\gamma$ becomes too large, one approaches the temporal focus achieved using TR. This can be seen as follows: For time reversal, $g(t) = R(-t)$, therefore, $g(\omega) = R^*(\omega)$. If $\gamma$ is large, $\epsilon$ becomes large in the sense that $\epsilon \gg |R(\omega)|^2$, and equation 5 reduces to

$$g(\omega) \approx \frac{1}{\epsilon} G(\omega)^* S(\omega)^* = \frac{1}{\epsilon} R^*(\omega),$$

which implies that our deconvolved signal is just a scaled version of the time reversed signal.

Figure 3 shows the normalized focused wavefield at the source location for TR and DC for different values of $\gamma$. For this experiment, we quantified the temporal focus the same way as the previous experiment with the identical window size of 0.02 ms used. The optimal DC's temporal focus was 79% (for a gamma value of 0.9) while TR had a temporal focus of 47%. We would not expect to see the exact same temporal focusing numbers as in our single source experiment because we changed the direction of displacement we record and the distance between the source and receiver.

Figure 4 highlights the effect of gamma by showing the temporal focus as a function of $\gamma$. If $\gamma$ becomes small, the temporal focus achieved decreases. However, as $\gamma$ becomes large, the temporal focus approaches TR’s temporal focus of 47%. The experiment showed that the optimal value to be $\gamma = 0.9$. However, even for different $\gamma$ values, one still achieves some form of a temporal focus as shown in Figure 3.

4.3 Multi source experiment

The purpose of this experiment was to study the effect of multiple sources when using deconvolution. We began by emitting the same 60kHz source function from different source transducers within the concrete block at different onset times. The experiment was repeated three times to record each source wavefield separately which normalized the recorded signals independently. The employed normalization caused our signals for all three sources to vary between amplitudes of $-1$ and $1$. The three recorded wavefields due to the three sources were then superimposed before TR or DC was applied. Figure 5 shows the superimposed wavefield. Note the complexity of the recorded signal, one can assume that the cross-correlation of each source wave field is negligible:

$$G_i G_j^* \approx 0 \text{ for } i \neq j$$

where $G_i$ for $i = 1, 2, 3$ is the Green’s function characterizing the source wavefield for source 1, 2, or 3. Equation
8 is crucial in explaining why deconvolution is stable for multiple sources.

We apply time reversal and deconvolution to the superimposed signal consisting of the three source wavefields to generate our TR and DC signals shown in Figure 6. Deconvolution’s signal differs from time reversal signal in its acausal nature, due to our pre-trigger time, where DC adds information past 8 ms while TR has zero amplitude after 8 ms. Additionally, the three different source wavefields are still clearly visible in the DC signal. Below, we demonstrate why deconvolution is able to focus the wavefield due to multiple source at each source location.

If there are three sources, the recorded signal in the frequency domain is given by

\[ R(\omega) = G_1 S_1 + G_2 S_2 + G_3 S_3, \]

where \( R(\omega) \) is the recorded signal in the frequency domain, and the subscripts indicate the source transducer used. The inverse signal obtained by deconvolution is given by

\[ D_t(\omega) = \frac{1}{G_1 S_1 + G_2 S_2 + G_3 S_3} \left( \frac{G_1 S_1 + G_2 S_2 + G_3 S_3}{G_1 S_1 + G_2 S_2 + G_3 S_3} \right)^* \]  

(9)

We simplify the above solution and add the regulation term \( \epsilon = \gamma \text{mean}(|R(\omega)|^2) \) to get,

\[ D_t(\omega) = \frac{G_1 S_1 + G_2 S_2 + G_3 S_3}{G_1 S_1^2 + |G_2 S_2|^2 + |G_3 S_3|^2 + \text{Crosstalk} + \epsilon} \]  

(10)

where \( \text{Crosstalk} = G_1 S_1 G_2^* S_2^* + G_1 S_1 G_3^* S_3^* + G_2 S_2 G_3^* S_3^* + G_3 S_3 G_1^* S_1^* + G_3 S_3 G_2^* S_2^* \) and \( D_t(\omega) \) represents the deconvolved signal when the recorded signal contains three source wavefields.

If we recorded each source's wavefield separate and applied deconvolution first before the superposition of the wavefields, we would get:

\[ D_s(\omega) = \frac{(G_1 S_1)^*}{|G_1 S_1|^2 + \epsilon_1} + \frac{(G_2 S_2)^*}{|G_2 S_2|^2 + \epsilon_2} + \frac{(G_3 S_3)^*}{|G_3 S_3|^2 + \epsilon_3} \]  

(11)

where \( D_s(\omega) \) represents the deconvolved signal when deconvolution is applied before superposition. One might expect that for a real scenario, where multiple sources are present, deconvolution would break down due to the influence of crosstalk. Because the recorded wavefields generated by each source are extremely complex, as shown in Figure 5, terms such as \( G_1 G_2^* \) are small (equation 8). Therefore, the influence of the crosstalk terms is minimal and we can assume it vanishes. This provides us with the following solution that relates \( D_s(\omega) \) to \( D_t(\omega) \),

\[ D_s(\omega) = \frac{(G_1 S_1)^* + (G_2 S_2)^* + (G_3 S_3)^*}{|G_1 S_1|^2 + |G_2 S_2|^2 + |G_3 S_3|^2 + \epsilon} \approx 3 \frac{(G_1 S_1 + G_2 S_2 + G_3 S_3)^*}{|G_1 S_1 + G_2 S_2 + G_3 S_3|^2 + \epsilon} \]

\[ = 3 D_t(\omega), \]  

(12)

where we assume \( |G_1 S_1| \approx |G_2 S_2| \approx |G_3 S_3| \).

Figure 7 shows that the approximation (12) holds. Figure 7 shows that after normalizing \( D_s(t) \) and \( D_t(t) \), one can note that there does not seem to be a obvious difference between \( D_s(t) \) and \( D_t(t) \). Thus, the crosstalk term may be ignored and deconvolution is stable and able to focus the wavefield due to multiple sources at each source location. We were able to do this calculation because we recorded each source wavefield separately.

In order to keep the experiment realistic, we back propagated the DC signal which was calculated after the superposition of the three recorded wavefields. Figure 6 shows the wavefields calculated using TR and DC which are propagated back into the medium from the receiver location. We then used the transducers within the concrete block as our receivers. Figure 8 shows the focused wavefields at each of the three sources for TR, shown in the top panels, versus DC, shown in the bottom panels. For sources 1 and 2, deconvolution compresses the side-lobes substantially better than TR. However, for source 3, deconvolution does not significantly improve the focus compared to time reversal.

The orientation of our sources is an important factor. Source 1 and 2 were oriented perpendicular to Source 3, and as a result the recorded waves excited by Source 3 are stronger than those excited by Source 1 and 2. However, as previously stated, we ran each source wavefield propagation separately which normalized the
Event locating using deconvolution: Experimental application and analysis

and (bottom panels) Deconvolution’s source location using (top panels) Time Reversal’s signal, Figure 8. Temporal focus measured at the three embedded source location using (top panels) Time Reversal’s signal, and (bottom panels) Deconvolution’s $D_t(t)$ signal and back propagating it from the transducer on the surface of the concrete sample.

recorded signals independently causing the amplitudes of each recorded source wavefield to vary between amplitudes of $-1$ and $1$. Thus, our superimposed signal shown in Figure 5 does not show a higher amplitude for the source 3 wavefield. However, when we back propagate our TR and DC wavefield, due to reciprocity, the source 3 wavefield focus will have a higher amplitude. This causes the crosstalk terms to be negligible for the source 3 focus because,

$$|G_3(\omega)| \gg |G_1(\omega)| \text{ and } |G_3(\omega)| \gg |G_2(\omega)|.$$  \hspace{1cm} (13)

Under these conditions, eqn. 10 reduces to,

$$D_t(\omega) \approx \frac{(G_3 S_3)^*}{(G_3 S_3)^2 + \epsilon}.$$  \hspace{1cm} (14)

which is what we had before.

Figure 8 shows that equation 13 holds because, for source 3, the focus has significantly higher relative amplitude than the crosstalk terms. For source 1 and 2, the maximum amplitude of the crosstalk is closer to the maximum amplitude of its focus.

In conclusion, for multiple sources, deconvolution is able to focus the energy at the source location at the correct time. It is arguable whether time reversal or deconvolution is better in generating a focus. However, the experiment does prove the robust nature of DC in that it will not fail under the condition of multiple sources.

5 CONCLUSION

We have introduced in an experimental study a simple though robust method for determining the optimal signal for backpropagation such that one gets an improved temporal focus at the source location. Deconvolution was shown to have an optimal regularization parameter, $\gamma$, for improved temporal focusing. If one increases $\gamma$, the temporal focus approaches that of TR; if one decreases $\gamma$, one increases the effect of noise and the temporal focus decreases dramatically. However, Figure 3 shows that one still attains a focus even for different values of $\gamma$. Additionally, deconvolution does not break down when there are multiple source wavefields being propagated. This is due to the influence of the crosstalk terms being minimal. Thus, deconvolution has a robust nature comparable to that of time reversal while having the potential to dramatically improve the focus. In conclusion, the simple and robust nature of deconvolution allows it to be implement as a preprocessing step in order to improve focusing at the source location.

6 ACKNOWLEDGMENT

We thank sponsor companies of the Consortium Project whose support made this research possible. We would like to thank BAM (specifically F. Mielentz and R. Feldmann) for providing equipment and software for the experiment, and the civil engineering lab at CSM (M. Mooney, M. Gutierrez and L. Frash) for providing us with space and resources to make the concrete specimen and to carry out our experiment.

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Reverse-Time Migration of Ultrasonic-Echo Data on a Foundation Slab

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(²) Center for Wave Phenomena, Colorado School of Mines, Golden, CO 80401, USA

ABSTRACT
The ultrasonic echo technique is a frequently used method in non destructive testing for geometry determination of concrete building elements. Important tasks are thickness measurements as well as the localization and characterization of built-in components and inhomogeneities. Currently mainly the Synthetic Aperture Focusing Technique (SAFT) is used for imaging. This algorithm is closely related to the Kirchhoff migration method and has difficulties in imaging steeply dipping interfaces and complicated structures such as steps and lower boundaries of voids. A reinforced concrete foundation slab with various reinforcement contents, different thicknesses and two pile heads was used for testing the prestack reverse-time migration (RTM) method. In a first step, the RTM was evaluated with synthetic 2D data. In the second step, ultrasonic measurement data recorded with shear wave transducers at two mutually perpendicular line profiles on the foundation slab were processed. The use of an automatic scanner simplified the measurements. Experiments such as the one detailed in this paper may be of interest to evaluate seismic migration methods on analogue models. A comparison of the RTM images with those of SAFT shows a significant improvement in the imaging of the geometry of the foundation slab. Vertical borders were reconstructed and the location and structure of the lower boundary of the foundation slab was reproduced better. Limitations still exist in imaging the pile geometry. The real data images show very noisy signals from the piles due to the reinforcement, edge effects and multiple reflections at the pile shaft.

Key words: Reverse-Time Migration, Synthetic Aperture Focusing Technique (SAFT), Non-Destructive Testing, Ultrasonic-Echo-Technique

1 INTRODUCTION
The ultrasonic echo technique is an important test method used in non destructive testing (NDT) to determine the interior of concrete building elements (Krause et al., 2008; Friese and Wiggenhauser, 2008). Important NDT tasks include thickness measurements, the localization of cracks and debondings as well as the localization and characterization of built-in components and inhomogeneities. The available Synthetic Aperture Focusing Technique algorithms (SAFT, e.g. Schickert et al. (2003)) for the reconstruction of ultrasonic echo data are closely related to Kirchhoff migration and have difficulties in imaging vertical dipping interfaces as shown in Figure 1. Reverse-time migration (RTM) has the potential to produce a more complete imaging of the features in the concrete specimens and was tested in this work at a foundation slab.
The test object was a reinforced concrete foundation (Figure 2) built at the BAM test site as part of the EU research project RuFUS (Re-use of Foundations on Urban Sites). The foundation slab is embedded in compacted sand and consists of areas of different thickness, ten different reinforcement levels, a strip foundation and two piles. At the bottom of the foundation slab is a 5 cm thick layer of lean concrete. Figure 3 shows this layer and the different levels of reinforcement before concreting.

The tasks with respect to the foundation slab were thickness determination as well as imaging the vertical step, the two pile heads and the strip foundation. In addition, the dependency of the resolution on various factors (such as 3D effects and the reinforcement) had to be investigated.

Figure 3. (a) and (b): Lean concrete layer and different levels of reinforcement before concreting (cf. (Taffe, 2008))

3 REVERSE-TIME MIGRATION

We chose the Reverse-time migration (RTM) for imaging of synthetic and experimental data. RTM was introduced by McMechan (1983) and Baysal et al. (1983). RTM uses the full wave equation. The RTM implementation in this paper uses numerical solutions of the acoustic wave equation. For the ultrasonic echo measurements at the foundation slab we used horizontally polarized shear waves which do not convert to other types of waves at interfaces. Thus using an acoustic RTM code is kinematically correct.

RTM is a wavefield-continuation method in time and is able to image waves even in areas with steep dipping reflectors and with strong velocity variations. A major disadvantage is the required extensive computing power and memory capacity.

The RTM algorithm we used is based on a two-dimensional finite difference modeling code created by using the Madagascar software package (Fomel et al., 2013). RTM consists of the following main steps:

1) The source wavefield is extrapolated forward in time assuming a known source location, source wavelet and a subsurface velocity model.

2) The receiver wavefield is modeled backward in time, from known receiver locations, the recorded data and an assumed subsurface velocity model. For the reverse modeling the migration algorithm converts the receivers into sources and plays the recorded data back into the subsurface.

3) The imaging condition used here computes the zero lag of the local cross-correlation between the two simulation results at all model grid points to find positions of existing subsurface reflectors.

4 APPLICATION TO SYNTHETIC DATA

In a first step we tested RTM with synthetic 2D data to prove the concept of RTM and to obtain information about appropriate ultrasonic measurement parameters (e.g. number and positions of sources and receivers, recording time). As an example we chose the two-dimensional section marked red in Figure 4 through the foundation slab. This two-dimensional profile contains the vertical step, a pile and the reinforcement meshes A1 and B1.

Figure 4. Two-dimensional section through the foundation slab (red) (cf. (Taffe, 2008))
The material parameters used in the velocity and density models for the simulation process are listed in Table 1. The structure of the velocity and density model is shown in Figure 5.

Table 1. Shear wave velocity $v_s$ and density values $\rho$ for the simulation process

<table>
<thead>
<tr>
<th>Material</th>
<th>$v_s$ [m/s]</th>
<th>$\rho$ [kg/m$^3$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reinforced concrete</td>
<td>2750</td>
<td>2400</td>
</tr>
<tr>
<td>Compacted sand soil</td>
<td>300</td>
<td>1800</td>
</tr>
</tbody>
</table>

Figure 5. Structure of the velocity and density model for the simulation process including a vertical step and a pile head.

To simplify the simulation and RTM of the synthetic data, both models are idealized. They do not contain any scattering aggregates inherent to concrete, since their precise distribution is not known. The 5 cm thick lean concrete layer is included in the concrete part of the model. The reinforcement was neglected.

Table 2 summarizes the simulation parameters used. The model consists of 5200 x 1350 grid points with a grid spacing of 1 mm. A ricker wavelet with a main frequency of 25 kHz was used as source signal.

Table 2. Parameters for the simulation process

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model size</td>
<td>5200 x 1350 grid points</td>
</tr>
<tr>
<td>Distance between grid points</td>
<td>0.001 m</td>
</tr>
<tr>
<td>Frequency of the Ricker wavelet</td>
<td>25 kHz</td>
</tr>
<tr>
<td>Sampling interval</td>
<td>$1 \cdot 10^{-7}$ s</td>
</tr>
</tbody>
</table>

Table 3 contains the parameters for the simulation and the RTM, which led to the most accurate image. We simulated 33 shots using 242 receiver positions (distance 0.02 m). Simulated recording time was 1.11 ms with a time step of $1 \cdot 10^{-7}$ s.

Table 3. Parameter for the simulation process and RTM ($v_{sc}$ = shear wave velocity of the concrete layer, $v_{ss}$ = shear wave velocity of the sand soil layer, $\rho_c$ = density of the concrete layer, $\rho_s$ = density of the sand soil layer)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Sources</td>
<td>33</td>
</tr>
<tr>
<td>Number of Receivers</td>
<td>242</td>
</tr>
<tr>
<td>Distance between sources</td>
<td>0.15 m</td>
</tr>
<tr>
<td>Distance between receivers</td>
<td>0.02 m</td>
</tr>
<tr>
<td>Source position no. 1</td>
<td>0.02 m</td>
</tr>
<tr>
<td>Receiver position no. 1</td>
<td>0.02 m</td>
</tr>
<tr>
<td>Recording time</td>
<td>0.00111 s</td>
</tr>
<tr>
<td>Velocity - simulation</td>
<td>$v_{sc} = 2750$ m/s, $v_{ss} = 300$ m/s</td>
</tr>
<tr>
<td>Velocity - RTM</td>
<td>$v_{sc} = 2750$ m/s, $v_{ss} = 300$ m/s</td>
</tr>
<tr>
<td>Density - simulation</td>
<td>$\rho_c = 2400$ kg/m$^3$, $\rho_s = 1800$ kg/m$^3$</td>
</tr>
<tr>
<td>Density - RTM</td>
<td>$\rho_c = 2400$ kg/m$^3$, $\rho_s = 1800$ kg/m$^3$</td>
</tr>
<tr>
<td>Thickness of concrete layer</td>
<td>1.25 m</td>
</tr>
</tbody>
</table>

Figure 6 shows the structure of the velocity and density model used for the RTM. The outer limits of the foundation slab are assumed to be known. The model doesn’t contain information on the step and the pile.

Figure 6. Velocity/density model used for RTM.

The migrated image is shown in Figure 7. The receiver positions are marked in green and the source positions are marked in red. The vertical step is reconstructed perfectly in terms of width and height. The lower boundary of the foundation slab as well as the lower part of the pile shaft and the pile base are clearly imaged. The semi-circular artifacts at the source positions (1) are caused by direct waves. The events parallel to the lower boundary of the foundation slab (2) are created by multiple reflections from the lower boundary in the synthetic data. Limitations exist in imaging the upper part of the pile shaft.

The use of the two-way wave equation produces artifacts in the reconstructed images. These events exist due to the correlation of waves which are not accounted for in the cross-correlation imaging condition (Díaz and Sava, 2012). In the left part of the migration result and in the area of the pile head stronger artifacts formed (dark shade). The reason is that in these areas the position of the lower boundary of the concrete layer in the models
for the RTM matches the real depth.

Figure 7. RTM image obtained with optimized migration parameters

5 APPLICATION TO REAL DATA

5.1 Ultrasonic Measurements

The main goal of this work was to test the RTM with ultrasonic measurement data, which were recorded at two profiles on the foundation slab (Figure 8). Profile 1 corresponds to the profile that we used for testing the RTM with synthetic data. The second profile crosses both piles and the strip foundation. Due to the page limits just data from profile 1 are presented here.

Figure 8. Two line profiles on the foundation slab (cf. (Taffe, 2008))

We used a multistatic arrangement to collect the ultrasonic-echo data. Two ultrasonic transducers were moved over the surface. Both were separated from each other and changed their positions and distances. Each transducer array (Figure 9) consists of 32 dry point piezoelectric contact transducers, which excite shear waves. The measurement frequency was 25 kHz. The shear waves were polarized in the plane perpendicular to the measuring direction.

Figure 9. Ultrasonic transducer array consisting of 32 dry point contact transducers

For the measurements an automatic scanner was used (Figure 10 and 11) with which the receiving transducer could be automatically moved from measuring point to measuring point for the respective source position. The length of the scanner is 1.2 m. On both profiles we used 32 source positions and a receiver spacing of 2 cm. Source-receiver offsets were limited to a maximum of 2.3 m due to time restrictions.

Figure 10. Scanner on the foundation slab moving the receiving transducer array (E)
Reverse-Time Migration of Ultrasonic-Echo Data

Figure 11. Scanner on the foundation slab with transmitting (S) and receiving (E) ultrasonic transducer arrays

Figure 12 shows the recorded raw data at source position no. 8 of profile 1. The direct wave (1) and the reflection of the wavefield at the lower boundary (2) of the foundation slab as well as a multiple reflection (3) at this boundary are visible. Furthermore, the reflection of the direct wave at the eastern (4) and western (6) upper edge of the foundation slab are shown. An apex of a reflection hyperbola is visible at 92 cm caused by a metal bracket inside the slab (5).

Prior to the RTM we performed the following processing steps of the data: muting crosstalk, time interpolation and bandpass filtering (cut-off frequencies: 8 kHz/100 kHz).

Figure 12. Received ultrasonic-echo data for source position no. 8 at 1.15 m

5.2 RTM Images - Profile 1

The structure of the velocity and density model which we used for the RTM images presented in this section, is shown in Figure 6. For the reinforced concrete we chose a shear wave velocity of 2740 m/s, based on a preliminary velocity analysis. The other migration parameters are summarized in Table 4. The recording time was 1.7 ms. The source and receiver positions correspond to the measurement parameters.

Table 4. Parameters for RTM for profile 1 ($v_{sc}$ = shear wave velocity of the concrete layer, $v_{ss}$ = shear wave velocity of the sand soil layer, $\rho_c$ = density of the concrete layer, $\rho_s$ = density of the sand soil layer)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Model size 5200 x 1350 grid points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance between grid points</td>
<td>0.001 m</td>
</tr>
<tr>
<td>Frequency Ricker wavelet</td>
<td>25 kHz</td>
</tr>
<tr>
<td>Sampling interval</td>
<td>1. $10^{-7}$ s</td>
</tr>
<tr>
<td>Recording time</td>
<td>0.0017 s</td>
</tr>
<tr>
<td>Number of sources</td>
<td>32</td>
</tr>
<tr>
<td>Number of receivers</td>
<td>varies</td>
</tr>
<tr>
<td>Distance between sources</td>
<td>0.15 m</td>
</tr>
<tr>
<td>Distance between receivers</td>
<td>0.02 m</td>
</tr>
<tr>
<td>Source position no. 1</td>
<td>0.02 m</td>
</tr>
<tr>
<td>Receiver position no. 1</td>
<td>0.031 m</td>
</tr>
<tr>
<td>Velocity - RTM</td>
<td>$v_{sc} = 2740 \frac{m}{s}$, $v_{ss} = 300 \frac{m}{s}$</td>
</tr>
<tr>
<td>Density - RTM</td>
<td>$\rho_c = 2400 \frac{kg}{m^3}$, $\rho_s = 1800 \frac{kg}{m^3}$</td>
</tr>
<tr>
<td>Thickness of concrete layer</td>
<td>1.25 m</td>
</tr>
</tbody>
</table>

For the image shown in Figure 13 we applied a bandpass filter process to the data (cut-off frequencies: 8 kHz/100 kHz).

Figure 13. RTM image of profile 1

The lower boundary of the foundation slab is reproduced at the correct depth, showing a low amplitude at the model boundaries. The structure of the lower boundary shows some ripples and some dip in the area of the vertical step.

The pile head (gap in the lower boundary reflection) is shown in the correct position. The position of the step is shifted by about 10 cm to the right. The pile shaft
and pile base are missing. 
At x = 2100 mm and z = 300 mm a circular reflector is reconstructed, which can be assigned to a metal bracket (Figure 14).

Figure 14. Metal bracket inside the foundation slab

Figure 15 shows the RTM image after stacking the images obtained from just shot points no. 27 to 32. The lower boundary of the slab is now imaged clearly with a higher amplitude and a small depth offset. Furthermore a phase jump can be recognized, which is possibly caused by debonding in this area. The stacking of the images of just shot points no. 7 to 15 illustrates clearly that the vertical edge of the step is reproduced (Figure 16). For this image we additionally applied an AGC and trace normalization to the data.

Figure 15. RTM image after stacking the images obtained from shot points no. 27 to 32

In a next step, we carried out a 3D/2D-correction of amplitude and phase in addition to bandpass filtering, AGC and trace normalization. The lower boundary of the slab is now clearly visible at the foot of the vertical step (Figure 17).

Figure 16. RTM image after stacking the images of the shot points no. 7 bis 15 and applying an AGC and trace normalization to the data

Figure 17. RTM image after applying a 3D/2D-correction

Using the $x^2 - t^2$-method we determined an average shear wave velocity of the reinforced concrete of 2811 m/s and an approximate depth of the lower boundary of the slab of 1213.2 mm and 726.9 mm. This led to the conclusion that most of the energy of the wavefield was reflected at the lower boundary of the reinforced concrete layer rather than at the lower boundary of the lean concrete layer.

The RTM image calculated with the average shear wave velocity of 2811 m/s shows stronger artifacts on the left side of the image, due to the use of a more accurate migration velocity (Figure 18).

Figure 18. RTM image using the calculated average shear wave velocity of 2811 m/s
5.3 SAFT Image - Profile 1

Figure 20 shows the image obtained from the 3D-SAFT reconstruction of profile 1. The ultrasonic measurements were made in July 2004 with the array A1220 (Figure 19). This array consists of 12 transmitting and 12 receiving dry point contact transducers which excite shear waves (bistatic method). The measurements were carried out automatically using a scanning system. The step width was 5 cm across the surface and the measurement frequency was 33 kHz.

For the SAFT-reconstruction we chose a migration velocity of $v = 2700$ m/s. According to the state of the art in NDT, the envelope of the receiving signals is calculated. Blue/red corresponds to high and white to low values of the envelope.

Except for the vertical faces, all reflectors are imaged in the SAFT reconstruction (Figure 20). However, the lower boundary of the slab is imaged with gaps. Compared to the RTM image (section 5.2) the lower boundary of the slab shows different x-positions in the area of the vertical edge of the step (1), probably due to the bistatic measurement setup and the large distance between the measuring points of 5 cm.

6 DISCUSSION

The comparison between RTM and SAFT imaging shows a significant improvement in imaging the geometry of the foundation slab by RTM. However, it must be noted that a quantitative comparison between the two algorithms is not possible in the current stage. For the SAFT algorithms a bistatic measurement setup and a distance between shot points of 5 cm was used. All the examples for RTM were calculated with a multi-offset geometry and a distance between receiver positions of 2 cm.

The images show a horizontal displacement of the vertical step of about 10 cm. Since the pile head was reproduced at the correct position, the displacement of the step is rather an error in construction drawings than an imaging artefact.

The imaging of the pile geometry wasn’t fully successful, since the measured data show noisy signals from the piles due to the reinforcement, edge effects, multiple reflections at the pile shaft, 3D effects, and the attenuation of the waves in concrete.

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7 CONCLUSIONS

We used synthetic and measured data sets to check the applicability of RTM to image ultrasonic echo data collected on a reinforced concrete foundation slab. The tests yielded promising results and showed that RTM is a step forward for the ultrasonic echo technique used in nondestructive testing.

The imaging of the location and structure of the lower boundary of the slab could be improved with RTM compared to conventional imaging techniques (SAFT). By using RTM vertical borders could be imaged clearly and more flaws could be found. To gain a more precise comparison between the images produced by SAFT and RTM, ultrasonic echo measurements using a distance of 2 cm between shot points will soon be performed and evaluated.

By optimizing the measurement equipment, a higher amplitude resolution should be achieved to improve the reconstruction of the pile geometry. A different scanner allowing larger aperture should be utilized for realizing a better resolution of the features inside the slab as well as reduction of artefacts and measurement uncertainties.

Furthermore, a core sample should be taken for an anal-
ysis of the depth offset and phase jump of the lower boundary of the slab at the specimen boundaries. RTM artefacts have to be analyzed and eliminated. For this task alternatives to the cross correlation imaging condition may be used. In addition, the algorithm should be expanded to three dimensions and the full elastic wave equation. Another topic to be addressed is to how to account for the size of the ultrasonic arrays. Experiments such as the one detailed in this paper may be of interest to evaluate seismic migration methods on analogue models.

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Scattering amplitude of a single fracture under uniaxial stress

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\textsuperscript{2}Now at the Institute of Geophysics, Department of Earth Sciences, ETH Zurich, 8092 Zurich, Switzerland.
\textsuperscript{3}Physical Acoustics Laboratory, Department of Physics, The University of Auckland, Private Bag 92019, Auckland, New Zealand.
\textsuperscript{4}Center for Wave Phenomena, Colorado School of Mines, Golden, CO 80401, USA.

ABSTRACT
Remotely sensing the properties of fractures has applications ranging from exploration geophysics to hazard monitoring. Newly-developed capabilities to measure the in-plane component of dense laser-based ultrasound wave fields allow us to test the applicability of a linear slip model to describe fracture properties. In particular, we estimate the size, and the normal and tangential compliance of a fracture from the measured scattering amplitudes of P- and S-waves in the laboratory. Finally, we show that the normal compliance decreases linearly with increasing uniaxial static stress in the plane of the fracture, but that our measurements of the SV scattered field do not show significant changes in the tangential compliance.

Key words: Wave scattering and diffraction – Fractures and flow – Microstructure.

1 INTRODUCTION

Fully characterizing a fracture assuming linear slip behavior involves estimating both the normal and tangential components of the compliance. In exploration geophysics, the ratio between normal and tangential compliance is used as a proxy for the presence of fluids in the fracture (Hudson et al., 1997; Liu et al., 2000; Lubbe et al., 2008).

Hydrocarbon reservoirs or aquifers are subjected to changes in the local stress as a result of production. Time-lapse monitoring of stress through changes in fracture properties can help assess reservoir conditions. Similarly, in volcanic environments the stress is related to volcanic activity, and dikes, local fractures, as well as the volcanic conduit all respond to changes in stress (Gudmundsson, 2006).

For multiple sets of parallel fractures of a small size compared to the dominant wavelength, wave propagation can be expressed in terms of effective medium theory, widely covered in existing work (Crampin, 1981; Hudson, 1981; Schoenberg & Sayers, 1995; Schoenberg & Douma, 1988; Kuchanov & Sevostianov, 2005). Conversely, the opposite case where the fracture plane is infinite leads to frequency dependent reflection and transmission coefficients (Pyrak-Nolte et al., 1990; Pyrak-Nolte & Nolte, 1992; Zhu & Snieder, 2002).

Based on the linear-slip model for a dry fracture, we derive the scattering amplitude in the frequency domain under the Born approximation for all combinations of incident and scattered wave modes, without making assumptions about the fracture size or wavelength, which can therefore be applied for a fracture of arbitrary size (Blum et al., 2011). Laser-based ultrasonic laboratory measurements of the P-wave scattered by a single fracture in clear plastic allowed us to quantify the normal compliance of a fracture, but was much less sensitive to the tangential component of compliance. With the development of a laser-based receiver that can measure the in-plane component of the wave field (Blum et al., 2010), we include scattered shear-wave modes, and show these are sensitive to the shear compliance of the fracture. After independently estimating both the normal and tangential component of the compliance, we conclude by exploring the stress dependence of these fracture properties in the laboratory. But before we introduce the experimental data, we highlight the main theoretical results of Blum et al. (2011).
2 THEORETICAL BACKGROUND

We consider a single homogeneous fracture following the linear slip model (Schoenberg, 1980), and assume that the slip discontinuity is related to the traction \( \mathbf{T} \) at the fracture by a compliance matrix \( \eta \), that can be further decomposed in normal and tangential components \( \eta_N \) and \( \eta_T \), respectively (see Blum et al. (2011) for more details). The fracture as a whole is treated as a scatterer and the slip discontinuity is related to the traction \( \mathbf{T} \) at the fracture by a compliance matrix \( \eta \) the fracture plane, and \( \alpha \) the Lamé parameters, \( A \) the area of the fracture, \( k \) the wavenumber. The angles \( \psi, \varphi, \theta \), and the unit vectors \( \hat{n} \) and \( \hat{m} \) of the direction of the incoming and outgoing waves, are defined in Figure 1.

The form factor \( F \) depends on the fracture size and shape, but in the case of a circular fracture, the form factor can be expressed as (equation (33) of Blum et al., 2011):

\[
F(k) = \frac{2}{k_{\parallel} a} J_1(k_{\parallel} a) \tag{2}
\]

where \( a \) is the radius of the fracture, \( k_{\parallel} \) the projection of the wavenumber change during scattering onto the fracture plane, and \( J_1 \) the Bessel function of order one. This derivation does not rely on assumptions about the size of the fracture with respect to the wavelength. For this work we consider the case of a fracture which size, quantified by its radius \( a \), is on the order of the elastic wavelength \( \lambda \).

Under the same conditions and assumptions, the SV to SV scattered amplitude is

\[
f_{\text{SV,SV}}(\hat{n}, \hat{p}; \hat{m}, \hat{q}) = \frac{\omega^2}{4\pi \rho \beta^2} AF(k_\beta (\hat{n} - \hat{m}))
\times \{ \mu^2 \eta_N \sin 2\psi \sin 2\theta + \mu^2 \eta_T \cos 2\psi \cos 2\theta \cos \varphi \} ,
\tag{3}
\]

where \( \beta \) is the S-wave velocity, and \( \hat{p} \) and \( \hat{q} \) the incoming and outgoing polarization unit vectors, respectively. The orientation of the vectors is shown in Figure 2.

Equation (1) shows that the P to P scattering is strongest and dependant on \( \eta_N \) only when the incoming P wave is normal to the plane of the fracture \( \psi = 0^\circ \), for an outgoing wave also normal to the fracture, either \( \theta = 0^\circ \) (forward scattering) or \( \theta = 180^\circ \). Similarly, Equation (3) shows that the SV to SV scattering is also strongest dependent on \( \eta_T \) only for the same geometry. In the following we consider the backscattering only, as the forward scattered wave however interferes with the direct wave.

3 EXPERIMENTAL SETUP

We create a single disk-shaped fracture by focusing a high power Q-switched Nd:YAG laser in a cylinder made of cast Poly(methyl methacrylate, PMMA), with a diameter of 50.8 mm and a height of 150 mm. The laser generates a short pulse \( \sim 20 \text{ ns} \) of infrared (IR) light that is absorbed by the sample material at the focal point and converted into heat. The sudden thermal expansion generates sufficient stress to form a fracture inside the plastic material (Zadler & Scales, 2008; Blum et al., 2011). Anisotropy in the elastic moduli, caused by the fabrication process, results in a fracture oriented along the cylindrical axis. The fracture studied here is approximately circular with a radius of \( \sim 3.5 \text{ mm} \) (Figure 3).
Elastic waves are excited at the surface of the sample by using the same high-power Q-switched Nd:YAG laser, operated at a much lower power, and with a beam partially focused on the surface of the cylinder. When an energy pulse from the laser hits an optically absorbing surface, part of that energy is absorbed and converted into heat. The resulting localized heating causes thermal expansion, which in turn results in elastic waves in the ultrasonic range (Scruby & Drain, 1990). Such a thermoelastic source generates waves over wide range of frequencies, depending on material properties. In this experiment, most of the elastic energy is in the 200 kHz – 5 MHz range.

We measure the elastic displacement with a laser interferometer. Our adaptive laser ultrasonic receiver is based on a doubled Nd:YAG laser, generating a Constant Wave (CW) 250 mW beam at a wavelength of 532 nm. The receiver uses two-wave mixing in a photorefractive crystal to deliver the displacement of the sample surface. This receiver measures the out-of-plane (vertical) as well as one in-plane (horizontal) component of the displacement field. It is calibrated to output the absolute displacement field in nanometers. (See Blum et al., 2010, for a complete description). The frequency response of the receiver is flat between 20 kHz and 20 MHz, and it can accurately detect displacements of the order of parts of Angstroms.

Since the cylinder is transparent for both IR and green light, we apply aluminum tape to the surface. The tape plays the role of the absorbing medium on the source side, and reflects light back for a wide range of angles to the laser receiver, allowing the measurement of both out-of-plane and in-plane components.

The cylindrical PMMA sample is mounted on a rotational stage, whereas the locations of the non-Contacting ultrasonic source and receiver are fixed in the laboratory frame of reference. The source-receiver angle \( \delta \) (defined in Figure 4) is therefore constant, and only the orientation of the fracture with respect to the frame of reference, characterized by the angle \( \psi \) between...
the normal to the fracture and direction of the incoming wave, changes. We choose to fix $\delta = 20^\circ$, which gives us the maximum backscattered amplitude, within the limitations of the experimental setup. Moreover, the source and receiver are focused on the sample in an $(x, y)$ plane normal to the cylinder axis ($z$-axis, Figure 4). While anisotropic, as mentioned above, the extruded PMMA is transversely isotropic, and its elastic properties are therefore invariant with respect to the defined angles of interest.

In order to put the sample under static stress for the second part of this study, we load it by tightening a screw pushing the top of the cylinder down. We use a bearing to accommodate the rotation of the loading screw, and insert a load gauge in between the bearing and the sample to measure the compressional stress. A picture of the laboratory setup is shown in Figure 5. We perform measurements for four different load settings; a first measurement with zero load (baseline), we next load it to a mid-load position corresponding to 5.5 Mpa, and then to full load position of 11.0 MPa, and finally a second measurement at zero load.

4 RESULTS

4.1 Unloaded sample

We first measure the scattered amplitudes for a sample under atmospheric conditions. The resulting out-of-plane and in-plane displacements are show in Figure 6. On the out-of-plane channel, the P-P scattered wave arrives around 18 $\mu$s, followed by the converted SV-P scattered event around 27 $\mu$s. The P-P wave reflected from the back of the sample arrives at 37 $\mu$s, and the converted SV-P reflection at 55 $\mu$s. The weaker unmarked events are side reflections and multiples. On the in-plane channel, an outgoing SV phase, including the P-SV scattering conversion, is visible at 27 $\mu$s, followed by the SV-
Scattering amplitude of a single fracture under uniaxial stress

Figure 7. Measured (solid lines) and fitted amplitudes (dashed lines) for the P-P and SV-SV scattered events with an unloaded sample. From the fit we get $a = 3.14 \pm 0.19$ mm, $\eta_N = 1.38 \pm 0.20 \cdot 10^{-11}$ m/Pa and $\eta_T = 2.69 \pm 0.34 \cdot 10^{-11}$ m/Pa.

Figure 8. Covariance matrix resulting from the least square inversion of the unloaded scattering data. This matrix is computed with the parameters expressed in units so that their values are between one and ten, with $a$ in mm, the compliances in $10^{-11}$ m/Pa, and $\theta_0$ in degrees.

Figure 9. Measured (solid lines) and fitted amplitudes (dashed lines) for the P-P and SV-SV scattered events with a loaded sample at 11.0 MPa. From the fit we get $a = 3.32 \pm 0.22$ mm, $\eta_N = 0.77 \pm 0.14 \cdot 10^{-11}$ m/Pa and $\eta_T = 2.14 \pm 0.29 \cdot 10^{-11}$ m/Pa.

Figure 10. Estimates of the fracture radius $a$ (in blue), and the normal and tangential compliances, $\eta_N$ (in red) and $\eta_T$ (in green), respectively, during the loading cycle. The error-bars correspond to the 95% confidence intervals from the least square fit.

SV scattered wave at 36 $\mu$s, the P-SV reflection from the back of the sample at 55 $\mu$s, and finally the SV-SV reflection from the back of the sample at 73 $\mu$s. In order to extract the scattered amplitudes, we first band-pass the data around 1 MHz. This frequency corresponds to the maximum in energy generated by the thermoelastic source, but also to wavelengths $\lambda_P = 2.6$ mm and $\lambda_S = 1.4$ mm. We are therefore in the single scattering regime described above, where the wavelength is in the order of the spatial extend of the fracture (radius $a = 3.5$ mm). We then pick the maximum amplitudes for two events of interest: the P-wave scattered from an incoming P-wave, that is detected on the out of plane channel, and the SV-wave scattered from an incoming SV-wave, that is detected on the in-plane channel. Both the scattered P and scattered SV amplitudes are normalized by the amplitude of the wave with the same mode reflected from the backwall of the sample, and corrected for geometrical spreading, effectively reducing the scattered amplitude to a fraction of the incoming amplitude.

For this geometry, the amplitude of the P-P scattered is mostly sensitive to the normal component of the compliance $\eta_N$ (see equation (1)). Conversely, amplitude of the SV-SV scattered event is mostly sensitive to the tangential component $\eta_T$, (equation (3)). We use a joint least-squares regression to obtain the parameters giving the best fit with the experimental data, as well as the corresponding confidence intervals. We invert for
the fracture radius $a$, the normal and tangential compliance, and the orientation of the fracture, given by the angle $\theta_0$ between the normal to the fracture assumed before the start of the experiment, and the normal to the fracture obtained after inversion. This last parameter does not vary significantly from one measurement to another. The measured amplitudes and corresponding fits are shown in Figure 7. The covariance matrix resulting from the inversion is shown in Figure 8.

4.2 Loading and unloading of the sample

Next, we repeat the measurements described previously, but as a function of loading, in order to investigate the change in fracture properties with increasing stress. The measured amplitudes and corresponding fits for the maximum load measurement are shown in Figure 9. A similar inversion scheme as for the unloaded sample leads to estimates of the fracture-defining parameters. Then, we repeated the process for intermediate loading values, and checked for hysteresis by repeating the no-load experiment after the loading sequence. The estimated parameters as a function of load are shown in Figure 10 and Table 1. We observe that the normal compliance $\eta_N$ decreases with increasing load. After the loading cycle, the estimated compliance is not exactly equal to the baseline value, but it is still higher than for the loaded case. The P-P and SV-SV backwall reflections stay constant in time for each loading stage, ruling out changes in the elastic properties of the homogeneous material.

5 DISCUSSION

The experimentally obtained P to P-wave scattering amplitudes are in good agreement with the theory of

Blum et al. (2011), even though the geometry of the experiments differs. On the other hand, the results presented here involve a fracture with a spatial extend on the order of the elastic wavelength, which is of importance for exploration geophysics, where the size of fractures in the reservoir are on the order of the seismic wavelength. Since the linear slip-model has been verified experimentally for both natural (Pyrak-Nolte et al., 1990,b) and simulated fractures (Hsu & Schoenberg, 1993), we expect the results shown here to also be valid for rock fractures.

The estimated components of the compliance $\eta_N$ and $\eta_T$ are in the same order of magnitude. Moreover, by recording the SV-SV scattering event we are able to estimate $\eta_T$, and observe that $\eta_N/\eta_T \sim 0.5$. Such a ratio is also noted in other studies (Worthington, 2007; Lubbe et al., 2008). The covariance matrix computed from the theoretical expressions indicates that the two components of the compliance are negatively correlated to the fracture radius, and the estimated values represent a trade-off between compliance and radius. The joint-inversion ensures that the estimate of the radius estimate is consistent for both P-P and SV-SV datasets.

As mentioned above, the reflection and scattering travel times are constant with the increase in load, and show that the mechanical properties of the sample in the plane of the measurement do not change with the load. We confirm this by also measuring the amplitude of the reflections from the back of the cylinder at zero load and maximum load. The resulting measurements in Figure 11, do not show a significant change between the two states of stress, for either the reflected P and SV waves. There is, however, an increase in PP amplitude between 0 and $10^6$, corresponding to the forward scattered wave. This is an effect of the fracture changing with stress, not the PMMA material. The higher variability of the measured reflected SS amplitude precludes us from making a similar observation for this mode.

The static uniaxial load experiments show that the estimated radius is nearly constant over the cycle of four measurements. We conclude that the effective area (area of the fracture where a discontinuity is present in the material) of the fracture does not change for such stresses, as the load is too small to modify the structure of the fracture. We observe a decrease in normal compliance $\eta_N$ as the stress increases, and the final value of the normal compliance after the loading, is slightly lower than the value before the loading. This effect could be due to a small permanent plastic change of the fracture caused by the uniaxial stress.

Most published laboratory studies of fractures involve uniaxial stresses normal to the fracture, and lead to a decrease in compliance with increasing load (Pyrak-Nolte et al., 1990), which can be explained by the fracture becoming stiffer as the stress helps “closing” it, and therefore increases the contact area of the fracture. Here, the stress axis is such that we would expect an
opening of the fracture instead, leading to an increase in compliance. For bigger openings however, the stiffness of round pores is greater than of elongated pores that more closely resemble natural fractures (Brie et al., 1985; Saleh & Castagna, 2004). By applying a load along a direction parallel to the fracture plane, the shape of the fracture changes from the ideal representation of a planar crack to a more rounded three-dimensional shape. Although this effect is small for the static load considered here, the observed change in compliance is consistent with a change in fracture shape from planar to more round-like.

Finally, it is much harder to interpret the tangential compliance estimates, compared to its normal component. The tangential values rely on the in-plane component of the wave field. This measurement is much more sensitive to the positioning of the sample with respect to the laser receiver than the out-of-plane component. As we mechanically increase the stress on the sample, small changes in position lead to a bias – in addition to the data variance – on the in-plane wave field recordings, as described in detail in Blum et al. (2010). The error bars shown in Figure 10 do not encompass the (unknown) bias. Therefore, we cannot conclude there are significant variations in the tangential component under the loads applied. Our results do show that by combining measurements of two components of the displacement field, we are able to estimate the size as well as the normal and tangential compliance of the fracture, paving the way for measurements under confining pressure (i.e., in situ subsurface conditions).

Moreover, we also look at the effects of static stress on fracture properties by applying uniaxial load parallel to the fracture plane. We monitor the fracture properties for four successive stages of stress and observe a significant decrease of the fracture normal compliance with increasing stress. We attribute this as a stiffening of the opening of a fracture. We are however unable to identify changes to the tangential compliance as a function of loading, due to limitations in the resolution of the in-plane wave field detection.

### ACKNOWLEDGMENTS

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### REFERENCES


### Table 1. Estimates of the fracture parameters with loading from the least squares fit, with 95% confidence intervals.

<table>
<thead>
<tr>
<th>load (MPa)</th>
<th>radius $a$ (mm)</th>
<th>$\eta_N$ (10^{-11} $m$/Pa)</th>
<th>$\eta_T$ (10^{-11} $m$/Pa)</th>
<th>$\theta_0$ (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00 (initial)</td>
<td>3.14 ± 0.19</td>
<td>1.38 ± 0.20</td>
<td>2.69 ± 0.34</td>
<td>1.0 ± 0.4</td>
</tr>
<tr>
<td>0.55</td>
<td>3.00 ± 0.17</td>
<td>1.04 ± 0.18</td>
<td>3.18 ± 0.36</td>
<td>1.6 ± 0.4</td>
</tr>
<tr>
<td>1.10</td>
<td>3.32 ± 0.22</td>
<td>0.77 ± 0.14</td>
<td>2.14 ± 0.29</td>
<td>1.3 ± 0.4</td>
</tr>
<tr>
<td>0.00 (final)</td>
<td>3.26 ± 0.21</td>
<td>1.14 ± 0.22</td>
<td>3.41 ± 0.44</td>
<td>1.1 ± 0.4</td>
</tr>
</tbody>
</table>


A comparative review of wavefield tomography methods

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ABSTRACT
Waveform inversion and wave-equation migration velocity analysis share one common feature: both techniques exploit information available in the same source and receiver wavefields reconstructed at all times and at all locations in space. These two wavefields are reconstructed from the same source function and observed data as solutions to the same wave-equation. Both techniques invert for the same model parameters and are formulated in the same inverse problem framework coupled with gradient calculations using the adjoint state method.

The main distinction between these techniques is in the way in which they handle the many seismic experiments representing a seismic survey. For the techniques in the waveform inversion family, an objective function is formulated by comparing wavefields for individual experiments; then, a compound objective function is formed by summing the objective functions of all the experiments. Many wavefield comparison measures are possible, including differencing and simple or oriented cross-correlations. For the techniques in the wave-equation migration velocity analysis family, an objective function is formulated after we accumulate for all experiments, e.g. through stacking, different wavefield comparison measures. Here, too, we can compare the wavefields using different measures, including simple and oriented cross-correlations. This summation over experiments produces images that simply function as proxies for the underlying seismic wavefields.

Key words: wavefield, tomography, imaging, adjoint-state method

1 INTRODUCTION
Seismic wavefield tomography methods applied to subsurface imaging are becoming increasingly popular. This trend is driven by the need to exploit every bit of information contained in the acquired data in order to constrain complex geologic structures and to extract detailed information about the geometry and physical properties of the bodies under investigation. Although much cheaper and more popular for the time being, alternative methods based on ray tracing suffer from numerical instability in complex geology and from difficulties in handling multi-pathing. Here, I restrict my attention to wavefield tomography methods and seek to analyze the various techniques in this category using a common theoretical framework.

Wavefield tomography methods have been extensively studied and can be separated into two main families:

(i) **Waveform inversion** – a family of techniques designed to characterize the subsurface by directly mapping its physical properties (e.g. the velocity). This methodology has a long history and remains an active subject of research to-date (Tarantola, 1984; Gauthier et al., 1986; Mora, 1987, 1988, 1989; Pratt and Worthington, 1990; Crase et al., 1990; Woodward, 1992; Bunks et al., 1995; Pratt, 1999; Plessix et al., 2000; Sirgue and Pratt, 2004; Operto et al., 2004; Tarantola, 2005; Plessix, 2006; Min and Shin, 2006; Shin and Min, 2006; Vigh and Starr, 2008; Shin and Ha, 2008; Plessix, 2009; Brossier et al., 2009; Virieux and Operto, 2009; van Leeuwen and Mulder, 2010; Symes, 2010; Alkhalifah and Choi, 2014).

(ii) **Wave-equation migration velocity analysis** – a family of techniques designed to alternate between identifying interfaces in the subsurface through wave-equation migration and constraining the physical properties of the medium (e.g. velocity) that allow placement of reflectors at their correct position in the subsurface. Both migration and migration velocity analysis also have a long history reported in publications covering fundamental imaging concepts (Berkhout, 1982; Chelrouth, 1985; Yilmaz, 2001; Biondi, 2006), various implementation strategies, e.g. using the acoustic wave-equation in reverse-time migration (Kosloff and Baysal, 1983; Baysal et al., 1983; McMechan, 1983), various imaging conditions (de Bruin et al., 1990; Rickett and Sava, 2002; Sava and Pomel, 2003; Biondi and Symes, 2004; Pomel, 2004; Sava and Pomel, 2005; Stolk and Symes, 2004; Biondi and Symes, 2004; Sava and Pomel, 2006; Yang and Sava, 2010; Sava and Vasconcelos, 2011; Sava and Vlad, 2011), and differ-
ent approaches for velocity analysis (Mulder and ten Kroode, 2002; Shen et al., 2003; Sava and Biondi, 2004a,b; Shen et al., 2005; Albertin et al., 2006; Maharramov and Albertin, 2007; Soubaras and Gratacos, 2007; Shen and Symes, 2008; Symes, 2009; Zhang and Wang, 2009; Yang and Sava, 2008, 2009; Biondi, 2010; Yang and Sava, 2011; Perrone et al., 2013; Biondi and Almomin, 2012, 2013; Shan and Wang, 2013; Tang and Biondi, 2013; Weibull and Arntsen, 2013; ?).

Both wavefield tomography families of method have are effective for characterizing the subsurface properties, which justifies their existing and growing popularity. However, a lot has been argued around their different implementation details, but mostly about their ability to exploit the information contained in the observed data.

My goal in this paper is to explore comparatively the wavefield tomography methods and address some of the following questions. What is really the difference between these two classes of techniques? In both cases wave propagation is governed by the same physics (i.e. the same wave-equation), the boundary conditions are the same (i.e. the same data), the model parameters are the same, etc. If we operate with the same wave-equation with the same boundary and initial conditions, i.e. we use the same wavefields, why is one class of methods considered to be “full wavefield” and the other one not so? What makes the difference, if there is one?

In the following, I proceed to describe these wavefield tomography methods from the perspective of an inverse problem. This point of view leads to a common framework for which wavefield tomography implemented in different domains become just special cases, and wavefield imaging just a proxy for analyzing seismic wavefields. This common point of view allows us to compare and contrast without ambiguity the different techniques based on their shared elements. I do not intend in this paper to provide a historic perspective on the development of the various techniques, which is why in the following I restrict references to a minimum.

The paper consists of several sections. I begin by reviewing the adjoint state method applied to the wavefields tomography problem, which is a special case of the general methodology summarized in Appendix A. Then, I discuss the wavefield tomography methods based on the separation introduced earlier:

• First, I discuss waveform inversion, i.e. a family of wavefield tomography methods that assume a-priori knowledge about locations in space where different wavefields match one another. I include in this discussion a review of wavefield comparison methods, some of which are detailed in Appendix B.

• Second, I discuss wave-equation migration velocity analysis, i.e. the family of wavefield tomography methods that do not assume knowledge about where to compare the wavefields, but extract such information from the wavefields themselves. The basic building blocks in this category are images, either conventional or extended, derived from different wavefield comparison mechanisms.

These two families of wavefield tomography methods can also be described as data-domain for the former and image-domain for the later. Although convenient, this nomenclature obscures the fact that both families of techniques derive model information from the same wavefields, and also use the same seismic wavefields to propagate information from the measurement location to all locations in space. However, this terminology highlights the fact that waveform inversion and wave-equation migration velocity analysis belong to the same general class of methods.

2 THE ADJOINT-STATE METHOD FOR SEISMIC WAVEFIELDS

The adjoint-state method summarized in Appendix A (Plessix, 2006), can be applied directly to the seismic wavefield tomography problem. In this case, we seek to find the extreme relative to model parameters of an objective function $J$ defined using wavefields obtained as solutions of a wave-equation, i.e. subject to physical constraints $F$. The wave operator $L$ could be, for example, the acoustic wave-equation

$$L = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2.$$  \hspace{1cm} (1)

The model parameters $m$ could be defined as slowness squared:

$$m(x) = \frac{1}{v^2(x)}.$$  \hspace{1cm} (2)

The objective function depends on the source and receiver wavefields, $u_s(e, x, t)$ and $u_r(x, e, t)$, obtained by solving the wave-equation 1 from a known source function and observed data, $d_s(e, x, t)$ and $d_r(e, x, t)$, respectively. The wavefields are a function of the experiment index $e$, space $x$ and time $t$. The seismic experiment could be based on a point source, a line or plane source, or other shot encodings etc.

In the wavefield tomography problem we are operating with two physical constraints, $F_s = 0$ and $F_r = 0$, accounting for the fact that both wavefields $u_s$ and $u_r$ are solutions of the wave-equation. The physical constraints are defined using the wave operator $L$ by

$$\begin{bmatrix} F_s \\ F_r \end{bmatrix} = \begin{bmatrix} L & 0 \\ 0 & L^\top \end{bmatrix} \begin{bmatrix} u_s \\ u_r \end{bmatrix} - \begin{bmatrix} d_s \\ d_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

(3)

thus capturing the idea that the variables $u_s$ and $u_r$ are wavefields obtained as solutions to a wave-equation. The wavefields $u_s$ and $u_r$ are the only ones that can be computed given the known source function $d_s$ and observed data $d_r$. In the following, I assume that the source function is known, although this may not always be the case in practice. The symbol $\top$ indicates that we are using the adjoint operator to construct the receiver wavefield, i.e. we solve the equation backward in time.

For the two mentioned physical constraints, we need the variables $a_s(e, x, t)$ and $a_r(e, x, t)$ to construct the augmented functional

$$H = J - \begin{bmatrix} F_s^\top & F_r^\top \end{bmatrix} \begin{bmatrix} a_s \\ a_r \end{bmatrix},$$

(4)
which we intend to minimize using a gradient-based optimization method. Using the definitions from Appendix A, the quantities \( u_s \) and \( u_r \) are state variables, and the quantities \( a_s \) and \( a_r \) are adjoint variables.

### 2.1 State equations

As explained in Appendix A, we find the extreme of the augmented functional \( \mathcal{H} \) by searching in the direction of its gradient relative to the model parameters \( m \) using

\[
\frac{\partial \mathcal{H}}{\partial m} = \frac{\partial \mathcal{J}}{\partial m} - \left[ \begin{array}{c} \frac{\partial F_s}{\partial m} \\ \frac{\partial F_r}{\partial m} \end{array} \right]^T \left[ \begin{array}{c} a_s \\ a_r \end{array} \right]. \tag{5}
\]

The state variables \( u_s \) and \( u_r \) are computed by solving the state equations obtained by setting the partial derivatives of the augmented functional relative to the adjoint variables \( a_s \) and \( a_r \) to zero

\[
\begin{bmatrix} \frac{\partial \mathcal{H}}{\partial a_s} \\ \frac{\partial \mathcal{H}}{\partial a_r} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{6}
\]

Using equation 4, the condition 6 is equivalent to

\[
\begin{bmatrix} \frac{\partial \mathcal{J}}{\partial a_s} \\ \frac{\partial \mathcal{J}}{\partial a_r} \end{bmatrix} = \begin{bmatrix} F_s \\ F_r \end{bmatrix}. \tag{7}
\]

Since \( \mathcal{J} \) is not a function of the adjoint variables, \( a_s \) and \( a_r \), we can write

\[
\begin{bmatrix} F_s \\ F_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{8}
\]

to obtain the state equations

\[
\begin{bmatrix} L & 0 \\ 0 & L^T \end{bmatrix} \begin{bmatrix} u_s \\ u_r \end{bmatrix} = \begin{bmatrix} d_s \\ d_r \end{bmatrix}. \tag{9}
\]

As stated earlier, the state variables \( u_s \) and \( u_r \) are wavefields obtained as solutions of the wave-equation solved forward and backward in time, respectively.

### 2.2 Adjoint equations

Likewise, the adjoint variables \( a_s \) and \( a_r \) are computed by solving the adjoint equations obtained by setting the partial derivatives of the augmented functional relative to the state variables \( u_s \) and \( u_r \) to zero;

\[
\begin{bmatrix} \frac{\partial \mathcal{H}}{\partial u_s} \\ \frac{\partial \mathcal{H}}{\partial u_r} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{10}
\]

Using equation 4, the condition 10 is equivalent to

\[
\begin{bmatrix} \frac{\partial F_s}{\partial u_s} & \frac{\partial F_r}{\partial u_s} \\ \frac{\partial F_s}{\partial u_r} & \frac{\partial F_r}{\partial u_r} \end{bmatrix} \begin{bmatrix} a_s \\ a_r \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathcal{J}}{\partial u_s} \\ \frac{\partial \mathcal{J}}{\partial u_r} \end{bmatrix}. \tag{11}
\]

Since the physical constraints \( F_s \) and \( F_r \) do not depend on the state variables \( u_s \) and \( u_r \), respectively, and the physical constraints are defined in terms of a wave-equation, we can write

\[
\begin{bmatrix} \frac{\partial F_s}{\partial u_s} & \frac{\partial F_r}{\partial u_s} \\ \frac{\partial F_s}{\partial u_r} & \frac{\partial F_r}{\partial u_r} \end{bmatrix} = \begin{bmatrix} L^T & 0 \\ 0 & L \end{bmatrix}. \tag{12}
\]

Defining adjoint sources based on the chosen objective function as

\[
\begin{bmatrix} g_s \\ g_r \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathcal{J}}{\partial u_s} \\ \frac{\partial \mathcal{J}}{\partial u_r} \end{bmatrix}, \tag{13}
\]

we can write the adjoint equations as

\[
\begin{bmatrix} L^T & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} a_s \\ a_r \end{bmatrix} = \begin{bmatrix} g_s \\ g_r \end{bmatrix}. \tag{14}
\]

The adjoint variables \( a_s \) and \( a_r \) are wavefields obtained as solutions of the wave-equation solved backward and forward in time, respectively.

### 2.3 Gradient

Finally, since we can write

\[
\begin{bmatrix} \frac{\partial F_s}{\partial m} \\ \frac{\partial F_r}{\partial m} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 u_s}{\partial \tau^2} \\ \frac{\partial^2 u_r}{\partial \tau^2} \end{bmatrix}, \tag{15}
\]

the constrained gradient of the augmented functional \( \mathcal{H} \) is

\[
\frac{\partial \mathcal{H}}{\partial m} = \frac{\partial \mathcal{J}}{\partial m} + \sum_e \left[ \left( \frac{\partial^2 u_s}{\partial \tau^2} \right) \left( \frac{\partial^2 u_r}{\partial \tau^2} \right) \right] \begin{bmatrix} a_s \\ a_r \end{bmatrix}. \tag{16}
\]

This expression implies summation over time in addition to the summation over experiments, and thus the gradient can be interpreted as the zero time-lag of the cross-correlation between state and adjoint variables. Thus, we can write the gradient as

\[
\frac{\partial \mathcal{H}}{\partial m} = \frac{\partial \mathcal{J}}{\partial m} + \sum_e \sum_{\tau} \delta(\tau) (\ddot{u}_s \ast a_s + \ddot{u}_r \ast a_r), \tag{17}
\]

where the double dot indicates second time derivative.
2.4 Model constraints

The inverse problem solution depends on our choice of objective function, $J(u_s, u_r, m)$. In the following, I consider an objective function consisting of two parts: a term defined based on the state variables $u_s$ and $u_r$, and a term defined based on the model parameters $m$:

$$J(u_s, u_r, m) = J_u(u_s, u_r) + J_m(m).$$

The term $J_m$ expresses the constraint that we seek model parameters from a multidimensional Gaussian distribution with mean $\mu_m$ and covariance $C_m$ represented by the weighting operator $W_m = C_m^{-\frac{1}{2}}$:

$$J_m = \frac{1}{2} \| r_m(x) \|^2_x,$$

where the model residual $r_m$ is

$$r_m(x) = W_m(x)(m(x) - \mu_m(x)).$$

With this notation, the model component of the constrained gradient is

$$\frac{\partial J_m}{\partial m} = W_m^T(x) r_m(x).$$

In the following, I assume that the model constraints do not change, but I consider different objective functions $J_u$ corresponding to different acquisition geometry and different measures of similarity between the source and receiver wavefields. The goal of the next sections is to evaluate the gradient of the data component of the objective function with respect to model parameters, i.e.

$$\frac{\partial J_u}{\partial m} = \sum_e \sum_{r} \delta(\tau) (\bar{u}_s * a_s + \bar{u}_r * a_r).$$

This requires that we construct appropriate adjoint source and receiver wavefields, $a_s$ and $a_r$, for various choices of the data objective function $J_u$.

2.5 Summary and discussion

Computing the gradient of an objective function $J$ for the wavefield tomography problem subject to physical constraints $F$ provided by a wave equation $\mathbf{E}$, consists in the following steps:

- Compute the state variables, i.e. the seismic wavefields $u_s$ and $u_r$, by solving the state equations, i.e. wave-equations with known source functions $d_s$ and $d_r$, equation 9.
- Compute the adjoint sources $g_s$ and/or $g_r$, which depend on the specific choice of objective function $F_u$, equation 13.
- Compute the adjoint variables, i.e. the seismic wavefields $a_s$ and $a_r$, by solving the adjoint equations which are wave-equations with source functions $g_s$ and $g_r$, equation 14.
- Compute the gradient of the objective function, constrained by the specific choice of wave-equation, equation 17.

This framework remains the same for various choices of objective functions $J_u$ and wave equations $F$. In the following, I concentrate on the cases when

(i) we define the objective function $J_u$ at a location where we know a-priori that the wavefields $u_s$ and $u_r$ should be identical with one-another, i.e. data-domain wavefield tomography, or

(ii) we define the objective function $J_u$ at locations identified based on the coherence between the wavefields as a function of experiment, e.g. similarity as a function of shot position, i.e. image-domain wavefield tomography.

I discuss these two cases separately in the following sections.

3 DATA-DOMAIN WAVEFIELD TOMOGRAPHY

Let us consider the wavefield tomography problem under the assumption that we know positions in space where the source and receiver wavefields should be identical with one-another. For example, we can do the analysis at the source or receiver coordinates, or at another point located somewhere else in the medium where this property holds. This later case could correspond, for example, to the case of imaging inside an object of known exterior shape, e.g. in non-destructive testing, but it also provides a link to the methods described in the following section.

I illustrate the methods in this section with a simple model depicted in Figure 1(a). The source is at $\{0, 0\}$ km, and the receiver is at $\{4, 0\}$ km. The state variables are the wavefields $u_s$ and $u_r$, depicted in Figures 2(a) and 2(b), respectively. Both wavefields are represented in successive frames ordered by increasing time, although the receiver wavefield $u_r$ is actually generated backward in time.

As discussed earlier, the key element for the wavefield tomography problem is the objective function which controls both the adjoint sources $g_s$ and $g_r$, in addition to the gradient itself. In all cases, I formulate the objective function using the fact that the source and receiver wavefields match each-other, if the model is accurate.

3.1 Difference

We can define an objective function using the difference between the source and receiver wavefields at the receiver position. This is equivalent to saying that we are comparing the simulated and observed data, and therefore this objective function corresponds to the conventional “full waveform inversion”.

Using the difference between the source and receiver wavefields, we can define the data residual

$$r_d(\epsilon, x, t) = W_u(\epsilon, x, t) (u_s(\epsilon, x, t) - u_r(\epsilon, x, t)),$$

and then the objective function

$$J_u = \sum_e \frac{1}{2} \| r_d(\epsilon, x, t) \|^2_{x,t},$$

where the weight $W_u$ is a combination of an operator $K_r$ which simply restricts the source and receiver wavefields to the
known receiver position, and another operator \( \mathbf{D}(t) \) weighing the data as a function of time:

\[
\mathbf{W}_u(e, x, t) = \mathbf{K}_r(e, x) \mathbf{D}(t) .
\]

Using this objective function 24, we can write the adjoint sources

\[
\begin{align*}
    g_s(e, x, t) &= \begin{bmatrix} +\mathbf{W}_u^{*}(e, x, t) \\ -\mathbf{W}_u^{*}(e, x, t) \end{bmatrix} \begin{bmatrix} r_d(e, x, t) \end{bmatrix}, \\
    g_r(e, x, t) &= \begin{bmatrix} +\mathbf{W}_u^{*}(e, x, t) \\ -\mathbf{W}_u^{*}(e, x, t) \end{bmatrix} \begin{bmatrix} r_d(e, x, t) \end{bmatrix},
\end{align*}
\]

which are just weighted versions of the difference between the source and receiver wavefields, restricted to the receiver position, Figure 1(c). As indicated earlier, the adjoint variables \((\alpha_s, \alpha_r)\), are wavefields extrapolated using the given wave-equation backward and forward in time, respectively. In this problem, only the adjoint source wavefield is relevant because it is reconstructed backward in time, thus interacting with the forward simulated source wavefield, Figure 3(a). In contrast, the adjoint receiver wavefield is reconstructed forward in time, and thus it does not interact in principle with the backward simulated receiver wavefields except at and around the receiver. Then, the gradient is computed by correlating the corresponding source and receiver state and adjoint variables, Figure 3(c).

Since the state variables are simulated with a fairly inaccurate model (15% error relative to the correct velocity), the source and receiver wavefields at the receiver are out of phase; this situation corresponds to the frequently-discussed cycle-skipping problem of full waveform inversion. The wavefield difference at the known analysis point, which coincides with the receiver position in this example, Figure 1(c), represents the superposition of pulses simulated in the true velocity, Figure 1(b), with another pulse simulated in the background (inaccurate) velocity.

All these pulses are back-propagated in the adjoint source wavefield \((\alpha_s)\), and they interfere with the forward simulated wavefield \((u_s)\) along several paths. First, wavefield coincidence occurs along the path connecting directly the source with the receiver due to the fact that one of the pulses in the back-propagated difference has actually been generated by forward propagating in the same background model. Second, wavefield coincidence also occurs along two elliptical paths corresponding to the time of the pulses seen in the observed data. The ellipse corresponding to the reflected data is the familiar migration impulse response, which indicates a straight-forward link between migration and waveform inversion.

It is apparent that at least some of the elliptical shapes do not correspond to any interface in the model, but are merely an expression of crosstalk between different waves in the extrapolated wavefields that should not correlate with one-another. For wavefield tomography, most of these elliptical shapes are artifacts that interfere with the desirable update of the background model. This can be seen, in the second example, Figure 5(a). In this case, the background model contains the reflector at \( z = 2 \) km, which generates additional pulses in the background data, and therefore in the data difference, Figure 5(c). The adjoint source wavefield carries all these pulses back into the model, Figure 7(a), and their interference with the source wavefield leads to the gradient seen in Figure 7(c). This gradient contains both the direct and reflected wave paths, but it is also contaminated with many artifacts that do not correspond to anything physical in the model.

This observation remains true if we analyze the synchronization of the source and receiver wavefields at another location where the two wavefields are expected to be similar. For example, in the setup from Figure 9(a), the data difference from Figure 9(c) enables us to generate the adjoint source and receiver wavefields depicted in Figures 11(a) and 13(a), thus leading to the gradients seen in Figures 11(c) and 13(c), respectively. The cross-talk artifacts seen here are related to the way in which we compute the data residual, i.e. as data difference. The question is whether we can compute a different data residual that is less prone to interference between unrelated wavefield paths. One possibility is to use data cross-correlation, as discussed in the following section.

### 3.2 Time correlation

Using the same source and receiver wavefields \((u_s, u_r)\), Figures 2(a) and 2(b), we can define a data residual

\[
    r_d(e, x, \tau) = \mathbf{W}_u(e, x, \tau) c(e, x, \tau)
\]

based on their cross-correlation, Figure 1(d):

\[
    c(e, x, \tau) = u_s(e, x, \tau) \ast u_r(e, x, \tau)
\]

and form the objective function

\[
    J_u = \sum_e \frac{1}{2} \| r_d(e, x, \tau) \|_{x, \tau}^2 ,
\]

where the weight \( \mathbf{W}_u \) is as a combination of an operator \( \mathbf{K}_r \) restricting the source and receiver wavefields to the known receiver position, and another operator \( \mathbf{P}(\tau) \) weighing the cross-correlation as a function of time-lag \( \tau \):

\[
    \mathbf{W}_u(e, x, \tau) = \mathbf{K}_r(e, x) \mathbf{P}(\tau) .
\]

The weight \( \mathbf{P} \) is essential in the definition of the data residual. Its role is to nullify the cross-correlation when the wavefields are identical, i.e. when the correlation function is maximum at zero time-lag.

Using the objective function 29, the adjoint sources are

\[
\begin{align*}
    g_s(e, x, t) &= \begin{bmatrix} u_r(e, x, t) \\ u_s(e, x, t) \end{bmatrix} \ast \begin{bmatrix} \mathbf{W}_u^{*}(e, x, \tau) r_d(e, x, \tau) \end{bmatrix} , \\
    g_r(e, x, t) &= \begin{bmatrix} u_r(e, x, t) \\ u_s(e, x, t) \end{bmatrix} \ast \begin{bmatrix} \mathbf{W}_u^{*}(e, x, \tau) r_d(e, x, \tau) \end{bmatrix} ,
\end{align*}
\]

where the symbol \( \ast \) indicates convolution (i.e. the adjoint of correlation) between the receiver and source wavefields, and the weighted residual. As indicated earlier, the adjoint variables \((\alpha_s, \alpha_r)\) are wavefields extrapolated using the given wave-equation backward and forward in time, respectively. As before, only the adjoint source wavefield is relevant because it is reconstructed backward in time, thus interacting with the forward simulated source wavefield, Figure 3(b). The adjoint receiver wavefield is reconstructed forward in time, and thus
it does not interact in principle with the backward simulated receiver wavefields except at or around the receiver.

Finally, the gradient is computed by correlating the corresponding source and receiver state and adjoint variables, Figure 3(d). Comparing with the gradient of the difference objective function, Figure 3(c), the correlation gradient is corrupted by fewer artifacts. The artifacts are due to the fact that the pulse contained in the data simulated in the background model correlates with individual pulses in the observed data, Figure 1(d). The adjoint source convolves the penalized correlation function with the observed data, thus leading to additional pulses in the adjoint source wavefield and consequently to artifacts resulting from cross-talk between paths that should not correlate with one-another. A better example is in Figure 7(d) where the gradient contains visible cross-talk artifacts, albeit fewer than the ones seen for the difference objective function in Figure 7(c). This observation also remains true for the case when we analyze wavefield synchronization away from either the source and receiver positions. For example, in the setup from Figure 9(a), the wavefield correlation from Figure 9(d) enables us to generate the adjoint source and receiver wavefields depicted in Figures 11(b) and 13(b), thus leading to the gradients seen in Figures 11(d) and 13(d), respectively.

We can conclude that gradient artifacts due to cross-talk between different paths present in the reconstructed wavefields are significantly reduced if we measure wavefield similarity by cross-correlation instead of differences. This reduction is due to the fewer independent branches present in the wavefields. However, when wavefields are characterized by significant multi-pathing, some non-physical artifacts still remain in the gradient even if we reduce the total number of branches to a minimum. This is because each path in the source wavefield interacts with each branch in the receiver wavefield. Further artifact reduction requires that we exploit other wavefield features, for example their directionality, as discussed in the following section.

### 3.3 Oriented correlation

Using the same source and receiver wavefields \( u_s \) and \( u_r \), Figures 2(a) and 2(b), we can define another data residual

\[
\begin{align*}
    r_d \left( e, x, \lambda, \tau \right) &= W_u \left( e, x, \lambda, \tau \right) c \left( e, x, \lambda, \tau \right) \\
    c \left( e, x, \lambda, \tau \right) &= u_s \left( e, x - \frac{\lambda}{2}t \right) * u_r \left( e, x + \frac{\lambda}{2}t \right)
\end{align*}
\]

based on their oriented cross-correlation (Appendix B)

and form an objective function

\[
\mathcal{J}_u = \sum_e \frac{1}{2} \| r_d \left( e, x, \lambda, \tau \right) \|_{\times, \lambda, \tau}^2,
\]

where the weight \( W_u \) is a combination of an operator \( K_r \) restricting the source and receiver wavefields to the known receiver position, and another operator \( P \left( \lambda, \tau \right) \) weighing the data as a function of space-lag \( \lambda \) and cross-correlation time-lag \( \tau \):

\[
W_u \left( e, x, \lambda, \tau \right) = K_r \left( e, x \right) P \left( \lambda, \tau \right).
\]

As in the preceding case, the weight \( P \) is essential in the definition of the data residual. Its role is to nullify the oriented cross-correlation when the wavefields are identical, i.e. when the correlation function is maximum at zero time-lag.

Using objective function 34, the adjoint sources are

\[
\begin{align*}
    g_s \left( e, x - \frac{\lambda}{2}t \right) &= u_r \left( e, x + \frac{\lambda}{2}t \right) \\
    g_r \left( e, x + \frac{\lambda}{2}t \right) &= u_s \left( e, x - \frac{\lambda}{2}t \right) \\
    * \left[ W_u^\top \left( e, x, \lambda, \tau \right) r_d \left( e, x, \lambda, \tau \right) \right]
\end{align*}
\]

where the symbol * indicates oriented convolution (i.e. the adjoint of oriented correlation) between the receiver and source wavefields, and the weighted residual. As indicated earlier, the adjoint variables \( (a_s, a_r) \) are wavefields extrapolated using the given wave-equation backward and forward in time, respectively. The major difference between the adjoint sources in equations 31 and 36 is that in the later case the sources are oriented according to the direction of the incoming waves, while in the former they are not. Therefore, the oriented cross-correlation enables us to beam energy in the direction of the incoming wavefield, as opposed to sending waves in all possible directions, thus further reducing the possibility of cross-talk between different wave paths present in the wavefield.

For example, the adjoint source wavefields for the example in Figure 1(a) are in Figure 3(b) for the simple correlation and in Figure 4(a) for the oriented correlation. The energy from the source obtained by the simple correlation propagates in all directions, while the energy for the source obtained by oriented correlation propagates in a specific direction, matching the direction of the incoming waves. The gradient of the oriented correlation, Figure 4(c), is correspondingly cleaner of artifacts compared with the gradient of the simple correlation, Figure 3(d).

The example in Figure 5(a) demonstrates this property even better. In this case, we are seeing waves beamed back from the adjoint source with different directions, corresponding to the different paths captured by the source wavefields, Figure 7(b) and Figure 8(a). Not surprisingly, the gradient for the oriented cross-correlation, Figure 8(c) is cleaner than the gradient of the simple cross-correlation, Figure 7(d).

These conclusions remain true if the analysis point is located away from either the source or the receiver, Figure 9(a). The oriented cross-correlation adjoint source and receiver wavefields, Figures 12(a) and 14(a), beam energy in specific directions toward the source and receivers, while the simple cross-correlation adjoint source and receiver wavefields, Figures 11(b) and 13(b), send energy in all directions.

A natural question to ask is whether we could more simply capture the directionality of the wavefield using just the zero time-lag of the correlation. This implies that we could attempt to formulate the wavefield tomography problem using the slice at \( \tau = 0 \) from the oriented cross-correlation de-
picted in Figure 1(e). This is a special case of the oriented cross-correlation
\[ c(e, \lambda) = c(e, \lambda, \tau = 0) , \]
as is also the case of the simple cross-correlation:
\[ c(e, \tau) = c(e, \lambda = 0, \tau) . \]

For the zero time-lag oriented cross-correlation, we can define the residual
\[ r_d(e, x, \lambda) = W_u(e, x, \lambda) \sum_\tau \delta(\tau) c(e, x, \lambda, \tau) , \]  \hspace{1cm} (37)
from which we can formulate an objective function using the expression
\[ J_u = \sum_e \frac{1}{2} \| r_d(e, x, \lambda) \|_{x, \lambda}^2 . \]  \hspace{1cm} (38)
The summation over \( \tau \) after multiplication with \( \delta(\tau) \) indicates that we take the zero-time lag of the cross-correlation. The weight is
\[ W_u(e, x, \lambda) = K_r(e, x) P(\lambda) , \]  \hspace{1cm} (39)
indicating that, as before, we restrict the cross-correlation at the receiver position and we also penalize it as a function of space-lag using the operator \( P \). Then, the source and receiver adjoint sources are
\[
\begin{align*}
g_s(e, x - \frac{\lambda}{2}, t) & = u_s(e, x + \frac{\lambda}{2}, t) \\
g_r(e, x + \frac{\lambda}{2}, t) & = u_s(e, x - \frac{\lambda}{2}, t) \\
[ W_u^T(e, x, \lambda, \tau) r_d(e, x, \lambda, \tau) ] & \text{.} \quad (40)
\end{align*}
\]
These expressions resemble the ones for the full oriented cross-correlation, except for the fact that the wavefields are multiplied with the residual, instead of convolved, as is the case in equation 36.

The adjoint source wavefield for the zero time-lag oriented cross-correlation is capable to capture partially the directionality of the wavefields, as seen by comparing Figure 4(b) with Figure 4(a). However, this is not always the case. The limited extent of the space-lag domain may not be sufficient to capture the correlation for events that require a longer space-lag axis. This is visible with the oriented cross-correlation from Figure 5(e). The zero time-lag captures (partially) one of the arrivals at the receiver, but not others. As a consequence, the adjoint source wavefield does not represent correctly the reflected path, Figure 8(b), thus leading to a distorted gradient, Figure 8(d). This drawback is even more striking in the example from Figure 9(a). The adjoint source and receiver wavefields, Figures 12(b) and 14(b), are essentially amplified noise and do not represent correct propagation paths, thus producing distorted gradients, Figures 12(d) and 14(d).

3.4 Summary and discussion

The derivations in this section lead to the following conclusions:
- The wavefield cross-correlation reduces the number of gradient artifacts in comparison with the wavefield difference. This is due to the reduced number of non-physical events present in the adjoint sources, and therefore in the adjoint wavefields.
- The oriented cross-correlation further reduces the number of gradient artifacts by restricting wavefield propagation in the direction from where waves are observed at the analysis point.
- Zero time-lag oriented cross-correlation can also capture wavefield directionality. However, this kind of analysis requires a larger space-lag domain, compared with the full oriented cross-correlation. In this case, we are trading time-lags (more) for space-lags (none), thus increasing the potential to interfere with events propagating in different directions.

It appears that the oriented cross-correlation provides significant advantages over the other measures of wavefield similarity considered here. This does not mean, however, that other wavefield comparison methods cannot provide other, perhaps more significant, advantages.

In all situations considered so far, I have assumed that we know where we should compare the source and receiver wavefields. Common candidates for this analysis are the source and receiver positions, as well as some interfaces with known geometry. This analysis also allows for arbitrary source functions, e.g. point sources, plane sources, randomly-encoded sources, etc.

More generally, we can ask the following question: where can we conduct a similar analysis when we do not know a-priori any point where the wavefields should match? Such analysis implies that we use additional information, as discussed next.

4 IMAGE-DOMAIN WAVEFIELD TOMOGRAPHY

The methods described in the preceding section assume that we know one or more locations where the source and receiver wavefields should match. Under this assumption, one can formulate objective functions exploiting different forms of comparison between multi-dimensional wavefields. We can construct such objective functions for each experiment separately, and then sum them all together, as is the case in Equations 24, 29, 34 and 38.

When the analysis position is unknown, we need to define the objective function by relying on the so-called semblance principle. A convenient formulation of this principle in the context of this paper is the following: the model used for wavefield reconstruction is accurate if the source and receiver wavefields match each-other for all experiments, at the same locations.

What this means in practice is that we neither define an objective function based on the wavefields themselves, nor based on the cross-correlation between the source and receiver wavefields.
wavefields of individual experiments, as is the case with all formulations described in the preceding section. Using the semblance principle, we define an objective function based on a combination of the cross-correlations between source and receiver wavefields for all experiments representing a seismic survey. In other words, we stack all wavefield cross-correlations before we define an objective function,

$$ J = F \left[ \sum_{c} c(u_{s}(e), u_{r}(e)) \right], $$

(41)

instead of stacking the objective functions defined on the cross-correlation of source and receiver wavefields of individual experiments:

$$ J = \sum_{c} F [c(u_{s}(e), u_{r}(e))]. $$

(42)

Here, $F$ is just a notation indicating that some operator acts on individual or combinations of cross-correlations $c$ of individual wavefields, $u_{s}(e)$ and $u_{r}(e)$. Stacking is, of course, just one possibility; one could use other strategies to combine various images, e.g., image warping. We can exploit this general idea in several ways, as discussed next.

### 4.1 Stack power

Starting with the correlation between the source and receiver wavefields ($u_{s}$ and $u_{r}$) at all locations in space

$$ c(e, x, \tau) = u_{s}(e, x, t) \ast u_{r}(e, x, t), $$

(43)

we can define the image as the stack over experiments of the zero time-lag cross-correlation:

$$ i(x) = \sum_{c} \sum_{x} \delta(\tau) c(e, x, \tau). $$

(44)

The multiplication by $\delta(\tau)$ followed by summation over $\tau$ indicates that we window the zero cross-correlation time-lag. Using this image obtained through a process referred-to as a conventional imaging condition, we can define a residual

$$ r_{d}(x) = W_{u}(x) i(x), $$

(45)

and then an objective function

$$ J_{u} = \frac{1}{2} \| r_{d}(x) \|_{x}^{2}. $$

(46)

maximizing the image strength, also known as the stack power.

The weight $W_{u}$ has the purpose of rebalancing the strength of the image, for example in areas of poor illumination, and is only a function of position in space:

$$ W_{u}(x) = K(x). $$

(47)

The objective function 46 is based on exactly the same wavefields as the ones exploited in the earlier sections. The fundamental difference is that the summation over experiments occurs inside the norm, as opposed to the preceding cases when the summation occurs outside the norm.

The source and receiver adjoint sources are

$$ g_{s}(e, x, t) = u_{r}(e, x, t) \left[ W_{u}^{\top}(x) r_{d}(x) \right], $$

(48)

and the adjoint source and receiver wavefields are obtained as usually by extrapolation backward and forward in time, respectively, Figures 18(a) and 20(a). Using the state and adjoint variables, we can obtain the gradient of the objective function for a specific source/receiver pair, Figures 18(c) and 20(c).

Since this objective function 46 exploits just a portion of the cross-correlation function comparing the source and receiver wavefields, we can ask whether we could not extract different or additional information by using more of the cross-correlation function?

### 4.2 Time-lag images

Using the temporal cross-correlation between the source and receiver wavefields,

$$ c(e, x, \tau) = u_{s}(e, x, t) \ast u_{r}(e, x, t), $$

(49)

we can define an alternative type of image if we preserve the entire correlation function in the output. In this case, we construct so-called time-lag images by summing this cross-correlation over experiments

$$ i(x, \tau) = \sum_{e} c(e, x, \tau), $$

(50)

defined at all positions in space. Using such time-lag images, we can construct the residual

$$ r_{d}(x, \tau) = W_{u}(x, \tau) i(x, \tau), $$

(51)

and the objective function

$$ J_{u} = \frac{1}{2} \| r_{d}(x, \tau) \|_{x, \tau}^{2}. $$

(52)

The weight

$$ W_{u}(x, \tau) = K(x) P(\tau) $$

(53)

has the purpose of restricting the analysis to some image points through the mask operator $K$, and also to apply a penalty to the correlation function through the operator $P$. We expect that there will be points in space where the cross-correlations for all experiments maximize at zero time-lag, and therefore so will their stack over experiments. Figure 16(a) shows an example of time-lag gather for the example described in Figure 15(a). Since the velocity is not correct, the maximum of the cross-correlation stack does not occur at zero time-lag or at the correct depth of the interface. We can use this information to formulate an objective function.

Using objective function 52, we can define the adjoint sources

$$ g_{s}(e, x, t) = \left[ u_{r}(e, x, t) \right] \left[ W_{u}^{\top}(x, \tau) r_{d}(x, \tau) \right], $$

(54)
which enable us to construct the adjoint source and receiver wavefields, Figures 18(b) and 20(b). The major difference with the stack power case is that in this case the source and receiver wavefields are convolved with the time-lag image stack. Figures 18(d) and 20(d) show the gradient computed by correlating the source and receiver state and adjoint variables, respectively. Not surprisingly, the gradient update originates at the depth of the strongest correlation stack, which is different from the true depth of the interface.

As in the preceding section, we can also ask the question whether we can capture directional information in the images, and then beam this energy back in the appropriate direction after the appropriate penalty has been applied. I discuss this case next.

### 4.3 Space-lag images

An approach resembling the oriented correlation discussed in the preceding section leads to so-called **space-lag images**. In this case, we exploit the oriented correlation between the source and receiver wavefields

$$c(e, x, \lambda, \tau) = u_s(e, x - \frac{\lambda}{2}, t) * u_r(e, x + \frac{\lambda}{2}, t),$$

and form an image by stacking its zero time-lag over experiments:

$$i(x, \lambda) = \sum_{e} \sum_{x} \delta(\tau) c(e, x, \lambda, \tau).$$

(56)

As suggested by the semblance principle, the zero space-lag of each individual cross-correlation must be located at the same point, if the model used for wavefield reconstruction is accurate. This is exactly the same idea used for the stack power or for the time-lag images. What is different in this case is that we construct oriented correlations capturing the propagation direction of the source and receiver wavefields, so we can use the formed image for other purposes, for example to decompose them as a function of reflection angles. Since in the ideal case the space-lag images are focused at zero space-lag, we can write the residual

$$r_d(x, \lambda) = W_u(x, \lambda) i(x, \lambda),$$

(57)

based on which we can write the objective function

$$J_u = \frac{1}{2} \|r_d(x, \lambda)\|_{x, \lambda}^2. \quad$$

(58)

As before, the weight

$$W_u(x, \lambda) = K(x) P(\lambda)$$

(59)

has the dual purpose of restricting the analysis to some image points through the mask operator K, and to penalize the image by emphasizing its components that are not concentrated at zero space-lag through the penalty operator P.

Finally, we can construct the adjoint sources

$$
\begin{align*}
g_s(e, x - \frac{\lambda}{2}, t) & = u_s(e, x + \frac{\lambda}{2}, t), \\
g_r(e, x + \frac{\lambda}{2}, t) & = u_r(e, x - \frac{\lambda}{2}, t).
\end{align*}
\]$$

(60)

which enable us to reconstruct the adjoint source and receiver wavefields, Figures 19(a) and 21(a). In this case, the wavefields are multiplied with the residual, and not convolved, as is the case in equation 54. Using the state and adjoint wavefields, we can generate the source and receiver gradients depicted in Figures 19(c) and 21(c).

The space-lag images build on the zero time-lag of the oriented cross-correlation between source and receiver wavefields. As discussed earlier, this captures the directionality of the wavefields and beam energy back from the image toward the corresponding source and receivers, but it requires that we span a fairly large space-lag domain. A more economical approach can be formulated based on complete oriented cross-correlation functions.

### 4.4 Extended images

We can consider the most general form of oriented wavefield cross-correlation

$$c(e, x, \lambda, \tau) = u_s(e, x - \frac{\lambda}{2}, t) * u_r(e, x + \frac{\lambda}{2}, t),$$

and define an extended image

$$i(x, \lambda, \tau) = \sum_{e} c(e, x, \lambda, \tau)$$

(62)

that depends both on space-lags \( \lambda \) and on time-lag \( \tau \). This type of image generalizes both the previously defined time-lag and space-lag images, Equations 50 and 56, as well as the conventional image, equation 44. Using these extended images, we can define the residual

$$\quad r_d(x, \lambda, \tau) = W_u(x, \lambda, \tau) i(x, \lambda, \tau),$$

(63)

and the objective function

$$J_u = \frac{1}{2} \|r_d(x, \lambda, \tau)\|_{x, \lambda, \tau}^2. \quad$$

(64)

where the weight represents both a mask operator \( K \), restricting the analysis to some points in the image, as well as a penalty operator \( P \), exploiting the parts of the extended image based on correlations that do not align at the same position in the subsurface:

$$W_u(x, \lambda, \tau) = K(x) P(\lambda, \tau).$$

(65)

Unlike for the special case of space-lag images, here we can exploit the extended image at specific points identified in the subsurface, as indicated by subscript \( c \) in the mask operator, and still capture the directionality of the propagating wavefields. This approach also leads to computational savings.
since we only need to compute oriented cross-correlations at specific points in the subsurface, as opposed to everywhere in space. Using the extended images, we can formulate adjoint sources

\[
\begin{align*}
g_r(e, x - \frac{\lambda}{2}, t) &= w_r(e, x + \frac{\lambda}{2}, t) \\
g_s(e, x + \frac{\lambda}{2}, t) &= w_s(e, x - \frac{\lambda}{2}, t) + [W^\top_d(x, \lambda, \tau) r_d(x, \lambda, \tau)]
\end{align*}
\]

where the symbol * indicates convolution in time. Using the source and receiver adjoint sources, we can construct the adjoint source and receiver wavefields, Figures 19(b) and 21(b), and then the source and receiver gradients, Figures 19(d) and 21(d).

4.5 Summary and discussion

This section allows us to draw the following conclusions:

- We can exploit the semblance principle to define objective functions usable for wavefield tomography. The residual underlying the objective function is defined after combining, e.g. by stacking, the cross-correlations for all the experiments in a seismic survey.
- As was the case for the analysis at known positions in the subsurface, we can compare the wavefields in many different ways, e.g. using simple or oriented time cross-correlations. Oriented correlations potentially lead to cleaner gradients, and thus to more robust inversion.
- Oriented cross-correlations can be constructed at any location in the image, but a computationally efficient process restricts such correlations to some points in space identified by the strength of the zero time-lag correlation.
- Wavefield imaging and wavefield tomography are closely related to one-another, since both use the same wavefields, and the wavefield images generated in the context of the semblance principle are just prerequisites to wavefield tomography.

Wavefield tomography formulated by assuming known positions where the source and receiver wavefields match, is often restricted to transmitted (diving) waves with analysis conducted at either the source or at the receiver positions. This often limits the depth of penetration and does not allow for model constraints deep in the subsurface. In contrast, wavefield analysis under the semblance principle enables us to probe the model at all depths, thus giving us more flexibility and better control on the model parameters deep in the model.

Key components of the analysis at arbitrary points in the subsurface are the penalty functions. These are designed to transform the images obtained after stacking over independent experiments into residuals that can be exploited in inversion. There are several possible definitions of the penalty functions:

- We can define penalties using analytic functions. These functions are designed to wipe-out ideal images, and transform everything else into a residual for optimization. Examples include the time-lag penalty \( P(\tau) = |\tau| \) and the space-lag penalty \( P(\lambda) = |\lambda| \). These common penalty functions assume idealized images (infinite frequency band, perfect illumination, etc), assumptions that are unlikely to be valid in practice.
- Alternatively, we can define penalties using the images themselves. We could simply evaluate through modeling how images constructed in a realistic frequency band behave under the influence of a velocity model and subject to partial illumination due to overburden geology or due to the acquisition geometry.

5 CONCLUSIONS

The words “wavefield tomography” describe a family of techniques that share one main feature: all use the same wavefields reconstructed using a given wave-equation from observed data. The data-domain methods (waveform inversion) and the image-domain methods (wave-equation migration velocity analysis) are thus exploiting the same wavefields, just in different ways and under different assumptions. The main difference between these methods is that in the first category we compare wavefields for each seismic experiment independently, while in the second category we compare the wavefields after summation over experiments, i.e. after we form an image. The seismic images are, therefore, just proxies for the underlying seismic wavefields and they provide a convenient mechanism for defining an objective function.

In all situations, we take advantage of similar wavefield comparison procedures, which include among others difference and (oriented) cross-correlations, although other comparison mechanisms are possible. By wavefield comparison, we formulate objective functions whose gradients reduce to the cross-correlation of state and adjoint wavefields reconstructed everywhere in the subsurface as solutions to a given wave-equation. In complex geology characterized by multipathing, one particular wavefield may contain multiple branches that should only correlate with specific branches from the matching wavefield. If this requirement is violated, the gradients are corrupted with non-physical events that can distort the model updates and hamper the optimization. Wavefield comparison by differencing is particularly prone to such artifacts; oriented cross-correlations limit artifact occurrence by directing waves in specific directions.

6 ACKNOWLEDGMENTS

Partial funding for this research is provided by KACST. The reproducible numeric examples in this paper use the Madagascar open-source software package (Fomel et al., 2013) freely available from http://www.ahay.org. This research was supported in part by the Golden Energy Computing Organization at the Colorado School of Mines using resources acquired with financial assistance from the National Science Foundation and the National Renewable Energy Laboratory.
Figure 1. Example 1: (a) Velocity model, (b) observed data, (c) data difference, (d) data cross-correlation, and (e) oriented wavefield correlation at the analysis point indicated by the box in panel (a).

Figure 2. Example 1: (a) The source wavefield $u_s$ and (b) the receiver wavefield $u_r$, ordered by time as indicated by the panel index.
Figure 3. Example 1: (a)-(b) adjoint source wavefields and (c)-(d) and gradients for objective functions defined using difference and simple time correlation, respectively

Figure 4. Example 1: (a)-(b) adjoint source wavefields and (c)-(d) and gradients for objective functions defined using oriented correlations and zero time-lag oriented correlations, respectively
Figure 5. Example 2: (a) Velocity model, (b) observed data, (c) data difference, (d) data cross-correlation, and (e) oriented wavefield correlation at the analysis point indicated by the box in panel (a).

Figure 6. Example 2: (a) The source wavefield $u_s$ and (b) the receiver wavefield $u_r$, ordered by time as indicated by the panel index.
Figure 7. Example 2: (a)-(b) adjoint source wavefields and (c)-(d) and gradients for objective functions defined using difference and simple time correlation, respectively.

Figure 8. Example 2: (a)-(b) adjoint source wavefields and (c)-(d) and gradients for objective functions defined using oriented correlations and zero time-lag oriented correlations, respectively.
Figure 9. Example 3: (a) Velocity model, (b) observed data, (c) data difference, (d) data cross-correlation, and (e) oriented wavefield correlation at the analysis point indicated by the box in panel (a).

Figure 10. Example 3: (a) The source wavefield $u_s$ and (b) the receiver wavefield $u_r$, ordered by time as indicated by the panel index.
Figure 11. Example 3: (a)-(b) adjoint source wavefields and (c)-(d) and gradients for objective functions defined using difference and simple time correlation, respectively.

Figure 12. Example 3: (a)-(b) adjoint source wavefields and (c)-(d) and gradients for objective functions defined using oriented correlations and zero time-lag oriented correlations, respectively.
Figure 13. Example 3: (a)-(b) adjoint receiver wavefields and (c)-(d) and gradients for objective functions defined using difference and simple time correlation, respectively.

Figure 14. Example 3: (a)-(b) adjoint receiver wavefields and (c)-(d) and gradients for objective functions defined using oriented correlations and zero time-lag oriented correlations, respectively.
Figure 15. Example 4: (a) Velocity model, (b) observed data, and (c) oriented wavefield correlation at the analysis point indicated by the box in panel (a).

Figure 16. Example 4: Common image gathers as a function of (a) experiment, (b) time-lag and (c) space-lag.

Figure 17. Example 4: (a) The source wavefield $u_s$ and (b) the receiver wavefield $u_r$, ordered by time as indicated by the panel index.
Figure 18. Example 4: (a)-(b) adjoint source wavefields and (c)-(d) and gradients for objective functions defined using shot-gathers and time-lag gathers, respectively

Figure 19. Example 4: (a)-(b) adjoint source wavefields and (c)-(d) and gradients for objective functions defined using zero time-lag oriented correlations and oriented correlations, respectively
Figure 20. Example 4: (a)-(b) adjoint receiver wavefields and (c)-(d) and gradients for objective functions defined using shot-gathers and time-lag gathers, respectively.

Figure 21. Example 4: (a)-(b) adjoint receiver wavefields and (c)-(d) and gradients for objective functions defined using zero time-lag oriented correlations and oriented correlations, respectively.
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7 APPENDIX A: THE ADJOINT-STATE METHOD

Consider the problem of finding the extreme of a function \( \mathcal{J}(d, m) \) under the constraints \( \mathcal{F}(d, m) = 0 \):

\[
\min \quad \mathcal{J}(d, m) \quad \text{s.t.} \quad \mathcal{F}(d, m) = 0 .
\]

(A-1)

(A-2)

The variables \( d = \{d_i\}, 1 \leq i \leq N_d \), represent the data, and the variables \( m = \{m_j\}, 1 \leq j \leq N_m \) represent the model. The variables \( N_d \) and \( N_m \) indicate the number of data and model parameters, respectively. The function \( \mathcal{F}(d, m) \) could be an objective function of an inverse problem, and the constraints \( \mathcal{F}(d, m) = \{F_i(d, m)\} \) specify the fact that the data are physical realizations of the model, for example they satisfy a wave-equation.

The constrained gradient of the selected objective function can be done efficiently using the adjoint state method (Plessix, 2006).

7.1 Augmented functional

Using the method of Lagrange multipliers, the constrained minimum of \( \mathcal{J} \) subject to the constraints given by \( \mathcal{F} = 0 \) can be found by searching in the direction of the gradient of the augmented functional \( \mathcal{H} \) defined by the expression

\[
\mathcal{H} = \mathcal{J} - \mathcal{F}^\top a .
\]

(A-3)

The parameters \( a = \{a_i\}, 1 \leq i \leq N_a \) are known as Lagrange multipliers, and the symbol \( ^\top \) indicates the adjoint operation. Since the vectors \( \mathcal{F} \) and \( a \) have equal size, the expression \( \mathcal{F}^\top a \) also indicates summation over the data parameters, i.e.

\[
\mathcal{F}^\top a = \sum_i F_i^\top a_i .
\]

(A-4)

We can represent the expression A-3 pictorially using matrices, which indicate the relative dimensions of the various variables:

\[
\begin{array}{cc}
\mathcal{H} & \mathcal{J}^\top a \\
\end{array}
\]

We can find the extreme of the augmented functional \( \mathcal{H} \) by searching in the direction of

- its gradient computed by evaluating the partial derivatives of the augmented functional relative to the model parameters \( m \) using

\[
\frac{\partial \mathcal{H}}{\partial m} = \frac{\partial \mathcal{J}}{\partial m} - \left( \frac{\partial \mathcal{F}}{\partial m} \right)^\top a ,
\]

where

- the state variables \( d \) are computed by solving the state equations obtained by setting the partial derivatives of the augmented functional \( \mathcal{H} \) relative to the Lagrange parameters \( a \) to zero

\[
\frac{\partial \mathcal{H}}{\partial a} = 0 ,
\]

and

- the adjoint variables \( a \) are computed by solving the adjoint equations obtained by setting the partial derivatives of the augmented functional \( \mathcal{H} \) relative to the state variables \( d \) to zero

\[
\frac{\partial \mathcal{H}}{\partial d} = 0 .
\]

Both \( \frac{\partial \mathcal{H}}{\partial a} = \{\frac{\partial \mathcal{H}}{\partial a_i}\} \) and \( \frac{\partial \mathcal{H}}{\partial d} = \{\frac{\partial \mathcal{H}}{\partial d_i}\} \) are vectors of size \( N_a \).

7.2 State equations

Using equation A-3, the condition A-6 is equivalent to

\[
\frac{\partial \mathcal{J}}{\partial a} - \mathcal{F} = 0 .
\]

(A-8)

Since by construction \( \mathcal{J} \) is not a function of \( a \), we have

\[
\mathcal{F} = 0 ,
\]

(A-9)

which are the original physical constraints expressing the relationships between the model and data parameters:

\[
\begin{array}{c}
\mathcal{F}^\top a = 0 \\
\end{array}
\]

which simply represent a system of \( N_d \) equations for the state variables \( d \).

7.3 Adjoint equations

Using equation A-3, the condition A-7 is equivalent to

\[
\frac{\partial \mathcal{J}}{\partial d} \mathcal{F} + \left( \frac{\partial \mathcal{F}}{\partial d} \right)^\top a = 0 ,
\]

(A-10)

therefore we can write the adjoint equations that can be used to construct the adjoint variables \( a \) given the state variables \( d \):

\[
\left( \frac{\partial \mathcal{F}}{\partial d} \right)^\top a = \frac{\partial \mathcal{J}}{\partial d} .
\]

(A-11)

The quantities \( \frac{\partial \mathcal{J}}{\partial d} \) are known as the adjoint sources since they are the forcing term for the adjoint equations used to compute the adjoint variables \( a \). Pictorially, we can represent the system A-11 as
\[ \begin{bmatrix} \frac{\partial F}{\partial d} \end{bmatrix}^\top \begin{bmatrix} a \end{bmatrix} = \begin{bmatrix} \frac{\partial J}{\partial d} \end{bmatrix} \]

thus indicating that the adjoint variables are solved by a system of similar dimensions to the one needed to solve for the state variables.

### 7.4 Gradient

Finally, we can obtain the gradient of the objective function \( J \) relative to the model \( m \) subject to the physical constraints \( F \) by

\[
\frac{\partial H}{\partial m} = \frac{\partial J}{\partial m} - \left( \frac{\partial F}{\partial m} \right)^\top a .
\]

Equation (A-12) can be represented pictorially as

\[
\begin{bmatrix} a \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial d} \end{bmatrix}^\top \begin{bmatrix} \frac{\partial J}{\partial d} \end{bmatrix} - \begin{bmatrix} \frac{\partial F}{\partial m} \end{bmatrix}^\top a .
\]

Equation (A-12) is a special case:

\[
\begin{align*}
    c(x, \lambda, \tau) &= a \left( x - \frac{\lambda}{2}, t \right) \star b \left( x + \frac{\lambda}{2}, t \right) \\
    &= \sum_t a \left( x - \frac{\lambda}{2}, t - \frac{\tau}{2} \right) b \left( x + \frac{\lambda}{2}, t + \frac{\tau}{2} \right).
\end{align*}
\]

The space-lag vector \( \lambda \) is used to reach into the wavefield away from the central point where we are interested to evaluate the similarity between the two wavefields. This symmetric definition employing the vectors \( +\lambda \) and \( -\lambda \) is just one convenient possibility.

The oriented cross-correlation between the wavefields in Figure B-1 evaluated in the neighborhood of the central point of the wavefield captures the directionality of the wavefields, as seen in Figures 2(b) and 4(b). Such oriented cross-correlation can be used to filter wavefields as a function of direction, or to capture wavefield directionality during imaging, thus enabling angle-domain imaging, for example.

### 8 APPENDIX B: ORIENTED CROSS-CORRELATION

One possible measure to evaluate the similarity between two multi-dimensional wavefields \( a(x, t) \) and \( b(x, t) \) is cross-correlation. A symmetric definition of this operation is

\[
\begin{align*}
    c(x, \tau) &= a(x, t) \star b(x, t) \\
    &= \sum_t a \left( x, t - \frac{\tau}{2} \right) b \left( x, t + \frac{\tau}{2} \right),
\end{align*}
\]

where the summation extends over the entire time domain in which the wavefields \( a \) and \( b \) are defined. The main drawback of this operation is that correlation is done point-by-point, and therefore the relation between the wavefields as a function of space is not accounted for. For example, the examples depicted in Figures B-1 and B-3 represent wavefields propagating in different directions. All synthetic wavefields are constructed such that for all 4 cases, the central trace is the same. Therefore, the central cross-correlations between the wavefields in Figure B-1 and Figures B-3 are identical, as seen in Figures 2(a) and 4(a).

Alternatively, we can formulate an oriented cross-correlation which accounts for the spatial distribution of the wavefield, for which the simple cross-correlation defined in
Wavefield tomography

Figure B-1. Example A: Synthetic plane waves propagating in different directions such that middle traces are identical.

Figure B-2. Example A: (a) Simple cross-correlation and (b) oriented cross-correlation of the wavefields depicted in Figure B-1

Figure B-3. Example B: Synthetic plane waves propagating in different directions such that middle traces are identical.

Figure B-4. Example B: (a) Simple cross-correlation and (b) oriented cross-correlation of the wavefields depicted in Figure B-3
Image-domain and data-domain waveform tomography: a case study

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ABSTRACT
Wavefield-based tomographic methods are idoneous for recovering velocity models from seismic data. The use of wavefields rather than rays is more consistent with the bandlimited nature of seismic data. Image domain methods seek to improve the focusing in extended images, thus producing better seismic images. However, image domain methods produce low resolution models due to the fact their objective functions are smooth, particularly in the vicinity of the global minimum. In contrast, data-domain methods produce high resolution models but suffer from strong non-linearity causing cycle skipping if certain conditions are not met. By combining the characteristics of each method, we can obtain models that produce better images and contain high resolution features at the same time. We demonstrate the strength of the workflow that combines both methods with an application to a marine 2D dataset with variable streamer depth.

Key words: Wavefield tomography

1 INTRODUCTION
An accurate velocity model is the main requirement for a successful imaging. In order to be consistent with the typical band-limited seismic data, one ought to use wavefield-based tomographic. The wavefield-based approach avoids shortcomings inherent in ray-based methods, such as limited model sensibility (a ray travels trough an infinitesimally narrow path inside the model) and instability around sharp boundaries in the velocity model.

Velocity analysis methods based on wavefield extrapolation are commonly referred to as Wavefield Tomography (WT) (Tarantola, 1984; Woodward, 1992; Pratt, 1999; Sava and Biondi, 2004a,b; Shen and Symes, 2008; Biondi and Symes, 2004; Symes, 2008) ; such tomographic approaches can be formulated either in the image domain, where one tries to improve image quality, or in the data domain, where one seeks consistency between modeled and observed data.

Image-domain wavefield tomography can be formulated by many means. A common approach aims to improve the flatness of angle gathers, or equivalently one can improve the focusing of space-lag gathers (Shen and Calandra, 2005). Space-lag gathers (Rickett and Sava, 2002; Sava and Fomel, 2006), also referred to as sub-surface offset gathers, measure the spatial similarity between source and receiver wavefields. Hence, during tomography, one seeks to increase the similarity of the spatial correlation for a collection of seismic experiments (Shen and Symes, 2008; Yang and Sava, 2011; Weibull and Arntsen, 2013; Yang et al., 2013; Shan and Wang, 2013). Inverse problems formulated in the image domain are generally better-posed than those formulated in the data domain. This results from the fact that the image domain objective functions are smoother than those in the data domain.

Data-domain wavefield tomography is generally formulated by improving the consistency between modeled and observed data. Originally, Tarantola (1984) introduced the data difference as a similarity estimate in the time domain. Alternatively, the problem can be solved in the frequency domain (Pratt, 1999). Contrary to the image-domain formulation, data-domain wavefield tomography is highly non-linear i.e. the objective function has many local minima. To overcome the non-linearity, a multi-scale separation approach is needed (Bunks et al., 1995). Within each scale (frequency or frequency band), the problem can be more linear if the initial model is closer to the one corresponding to the global minimum. Another loop of multi-scale can be added by introducing time damping, a method commonly referred to as Laplace-Fourier waveform inversion (Sirgue and Pratt, 2004; Shin and Ho Cha, 2009). The purpose of the time damping is to fit earlier arrivals first, and then to fit later arrivals progressively.

Both data-domain and image-domain tomographic methods share many parts of the process: both use the same extrapolation engine (the two-way wave equation), and share similarities in building the gradient of the objective func-
tion through the Adjoint State Method framework (Tarantola, 1984; Flessix, 2006; Symes, 2008). In this report, we combine image-domain and data-domain wavefield tomography approaches for optimizing the velocity model. The idea is to produce a model that improves focusing with image-domain wavefield tomography and then refine it using data-domain wavefield tomography. We apply the wavefield tomography workflow to a marine 2D dataset. The data are acquired with a variable depth streamer cable, which produces a varied notch spectrum. The increasing depth produces a better low frequency response, which can be useful in the multi-scale approach discussed earlier.

2 IMAGE DOMAIN WAVEFIELD TOMOGRAPHY

In this section, we review the image-domain wavefield tomography using space-lag gathers (Rickett and Sava, 2002; Sava and Vasconcelos, 2011). These kind of gathers highlight the spatial consistency between wavefields by exploring the focusing information in the image domain. The moveout in the gather is sensitive to velocity perturbations, and hence can be optimized. Space-lag gathers are defined as follows:

$$R(x, \lambda) = \sum_e \sum_t u_s(e, x - \lambda, t) u_r(e, x + \lambda, t),$$

where $\lambda$ is the space-lag vector, $x$ the image location, $e$ the experiment index, $u_s$ the source wavefield, and $u_r$ the receiver wavefield. The source wavefield $u_s$ is produced by forward extrapolation of the source function, whereas the receiver wavefield $u_r$ is produced by backward propagation of the data at the receiver location. In matrix notation, the process is described by

$$\begin{bmatrix} L(m, t) & 0 \\ 0 & L^T(m, t) \end{bmatrix} \begin{bmatrix} u_s \\ u_r \end{bmatrix} = \begin{bmatrix} f_s \\ f_r \end{bmatrix},$$

where $m = 1/v^2(x)$ is the medium slowness squared, $f_s$ is the source function, $f_r$ is the data at the receiver locations, $L(m)$, and $L^T(m)$ are forward and backward wave propagators, respectively. In this report we use the scalar wave equation as wave operator:

$$L(m) = m(x) \frac{\partial^2}{\partial t^2} - \nabla^2.$$  

A well-focused gather concentrates most of its energy around $\lambda = 0$. This can be used as an optimization criterion by minimizing the energy outside $\lambda = 0$. We can accomplish this by defining an objective function

$$J = \frac{1}{2} ||P(\lambda) R(x, \lambda)||^2,$$

where $P(\lambda)$ is the penalty function, which plays a vital role in the inversion. Shen and Symes (2008) propose a mix between $P(\lambda) = |\lambda|$ and $P(\lambda) = \delta(\lambda)$. The first penalty function corrects for most kinematic errors, whereas the second one improves the focusing of the image $R(x, \lambda = 0)$. Yang et al. (2013) propose a penalty operator $P(\lambda)$ that accounts for illumination, which seeks to bring the defocused data, not to $\lambda = 0$, but to a region of acceptable focusing within the limits of illumination. Depending on the choice of the the penalty operator $P(\lambda)$, equation 4 can be either minimized or maximized. In this report, we use $P(\lambda) = |\lambda|$ as penalty operator primarily for computational cost reasons. This penalty operator defines a smooth objective function and corrects for most kinematic errors in the model.

Once we have the penalized gathers (image residuals), we compute the adjoint sources (Shen and Symes, 2008; Weibull and Aartsen, 2013),

$$g_s(x, t) = \sum_{\lambda} P(\lambda)^2 R(x + \lambda, \lambda) u_r(x + 2\lambda, t)$$

for the source adjoint source, and

$$g_r(x, t) = \sum_{\lambda} P(\lambda)^2 R(x - \lambda, \lambda) u_s(x - 2\lambda, t)$$

for the receiver adjoint source.

Yang and Sava (2010) and Shan and Wang (2013) use an alternative formulation for the source side:

for all $\lambda$, do: $g_s(x - \lambda, t) + = P(\lambda)^2 R(x, \lambda) u_r(x + \lambda, t)$.

(7)

and for the receiver side:

for all $\lambda$, do: $g_r(x + \lambda, t) + = P(\lambda)^2 R(x, \lambda) u_s(x - \lambda, t)$.

(8)

It turns out that both formulations are equivalent. In the first formulation, we gather information from the vicinity of position $x$, whereas in the second one we scatter the residual in the vicinity of $x$. In order to get the equivalence between equations 5 and 7 we can simply do a change of variables $x' = x - \lambda$ in equation 7. Similarly, we can do the change of variables $x' = x + \lambda$ in equation 8 to obtain the equivalence between equations 6 and 8.

Once we have the adjoint sources, we solve

$$\begin{bmatrix} L^T(m, t) & 0 \\ 0 & L(m, t) \end{bmatrix} \begin{bmatrix} a_s \\ a_r \end{bmatrix} = \begin{bmatrix} g_s \\ g_r \end{bmatrix}.$$  

(9)

The gradient of equation 4 with respect to our model parameters is defined as follows:

$$\nabla J(x) = \sum_e \sum_t \bar{u}_s(e, x, t) a_s(e, x, t) + \bar{u}_r(e, x, t) a_r(e, x, t).$$  

(10)

3 DATA DOMAIN WAVEFIELD TOMOGRAPHY

The construction of the tomography problem in the data domain amounts to measuring the error (or residual) at the receiver locations. For data domain wavefield tomography, we normally use the data difference for the residual:

$$J = \frac{1}{2} ||u_s(x_r, \Omega) - f_r(x_r, \Omega)||^2 = \frac{1}{2} ||\Delta d||^2,$$

where $x_r$ are the receiver locations and $\Omega$ is the complex valued frequency whose purpose we will explain later. Note that
$f_r(x_r, \Omega) = u_r(x_r, \Omega)$. Since for building the residual we only need to forward propagate the source function

$$L(m, \Omega)u_s = f_s(x_s, \Omega),$$  

we have to compute one adjoint wavefield $a_s(x, \Omega)$. For data-domain wavefield tomography, computing $a_s$ involves back-propagating the data residual:

$$L^\top(m, \Omega)a_s = (\Delta d)^*.  

Here $(\Delta d)^*$ is the complex conjugate residual and $L(m, \Omega)$ is the acoustic wave equation in the frequency domain, defined as follows:

$$L(m) = -\nabla \cdot \frac{1}{\rho(x)} \nabla m(x) - \frac{m(x)}{\rho(x)} \Omega^2.  

Here $\rho(x)$ is the density of the medium. In this report, we do not invert for $\rho(x)$, instead we parametrize it as a function of the velocity following Gardner et al. (1974).

Once we obtain $u_s(x, \Omega)$ and $a_s(x, \Omega)$, we can proceed to compute the gradient:

$$\nabla J(x) = \Re \left\{ \sum_e \sum_t \Omega^2 u_s(e, x, \Omega) a^*_s(e, x, \Omega) \right\},  

here $^*$ denotes complex conjugate and $\Re \{ \}$ denotes the real part.

The data domain wavefield tomography objective function, equation 11, is highly non-linear if we operate with signals in the normal frequency band. Hence, in order to increase the chances of convergence to the global minimum, it is customary to implement the data-domain wavefield tomography in a multi-scale fashion. Bunks et al. (1995) propose to first invert lower frequencies and then move gradually to higher frequencies. The idea is that within each scale the problem looks more linear than when inverts all the bandwidth at once.

An additional outer loop in the inversion is the time damping, which leads to the so-called Laplace-Fourier domain FWI (Sirgue and Pratt, 2004; Shin and Ho Cha, 2009). The purpose of this outer loop is to first fit earlier arrivals, and then fit later arrivals. By fitting first early arrivals (shorter travel-time) we reduce the risk of large phase differences between observed and modeled data which can cause cycle-skipping. Once the travel-time differences are solved for early arrivals, we can progressively increase $\tau$. Introducing the time damping requires the following transformation: $\Omega = \omega + i/\tau$, with $\tau$ being the time damping (Kamei et al., 2013). Thus, this transformation turns the real-valued angular frequency $\omega$ into a complex-valued angular frequency $\Omega$. In order to get consistent observed data with the damped modeled data, one must also scale the observed data as $f_r(x_r, t) = d_{obs}(x_r, t) e^{-t/\tau}$ before the transformation to frequency domain.

The low frequencies of the data are sensitive to the long wavelength (smooth) components of the earth model. However, if the data do not have such frequencies, data-domain wavefield tomography is unable to update such components. In contrast, focusing in extended images is mostly sensitive to the smooth components of the model. By implementing a joint workflow using image-domain wavefield tomography for updating the smooth components of the model and later using data-domain wavefield tomography for the high resolution features of the model, we can obtain a more complete spectrum in the model. The first pass using image-domain wavefield tomography has the ability to stabilize the cycle-skipping problems in data-domain wavefield tomography.

4 APPLICATION TO A REAL 2D DATASET
In this section we apply the cascaded workflow of image-domain wavefield tomography followed by data-domain wave-
field tomography. We use image-domain wavefield tomography to correct for most kinematics errors, and then data-domain wavefield tomography to refine the model and add details. We compare 4 models: (a) the initial model, (b) the model obtained by image domain wavefield tomography, (c) the data-domain wavefield tomography obtained from (a), and (d) the data-domain wavefield tomography starting with the model obtained by image-domain wavefield tomography (b).

The dataset is a marine 2D line acquired with a variable depth cable. The towed streamer contains increasing depths as a function of offset, which enhances the frequency content of the data by producing a mixed notch response. Hence, the increased cable depths improves the low frequency content at intermediate and far offsets which can be very helpful for data-domain wavefield tomography. The cable contains offsets ranging from 0.169 to 8.256 km. Figure 1(a) shows a shot gather from $x = 14$ km and Figure 1(b) depicts the average amplitude spectrum for the same gather.
Wavefield tomography

Figure 3. (a) The data-domain wavefield tomography velocity model obtained from Figure 2(a), (b) corresponding RTM image, and (c) angle-gathers at sparse locations. The velocity ranges from 1.5km for water velocity to 4km/s in the deepest part.

We build the initial model, Figure 2(a), by performing time-domain NMO analysis followed by smoothing, RMS (stacking) conversion to interval velocity (Dix, 1955), and time to depth conversion. Figure 2(b) shows the RTM image produced by the model in Figure 2(a), and one can observe that the image is over migrated (high velocity) below 3km in depth. Figure 2(c) shows angle gathers extracted at sparse locations in the model. Note that we do not use the angle gathers for inversion; instead, we use the gathers as an independent quality control tool. The transformation from space-lag gathers $R(x, \lambda)$ to angle domain $R(x, \theta)$ follows the method of Sava and Fomel (2003). The angles vary from 0 to 45° for all the gathers shown in this report. The moveout in the gathers confirms that the velocity is too fast below 3 km. Some of the events in the gathers, however, correspond to migrated surface related multiples and their moveout is not indicative of velocity error.

For data-domain wavefield tomography, we use 7 frequency blocks with 5 frequencies each. The center frequency for each block ranges from $f = 2.6$ Hz to $f = 8.9$ Hz. For the time damping constant, we use $\tau = 1.6s$. The first step in data domain wavefield tomography involves estimating the source function $f_s(\Omega)$; later we compare the inverted source functions for each model. We use 365 shots for the inversion with a shot interval $\Delta s = 0.09375$ km. The data-domain wavefield tomography workflow is common for the two inversions.

Figure 3(a) shows the data-domain wavefield tomography model built from Figure 2(a). The data-domain wavefield tomography process slows the velocity in the shallow part of the section, close to the water bottom, introducing a sharp discontinuity in the model. In general, the velocity slows down in the right part of the model. One can see in the gathers, Figure 3(c), that in the shallow part the events get flatter with the new velocity. However, deep in the section, the model does
not correct most kinematic problems exhibited in the gathers. This area of the model corresponds to longer travel-times in the data, and these late arrivals are prone to cycle-skipping problems.

We generate the model in Figure 4(a) using the image-domain wavefield tomography approach. The idea of this tomographic step is to correct for the bulk of the kinematic errors in the model. Figure 4(a) shows that in the updated model, in general the model slows down, especially in the deep part of the section. Figure 4(b) shows the corresponding RTM image, where the focusing of the image improves significantly around $z = 3.5$ km. This observation is confirmed in Figure 4(c), where now the gathers are flatter throughout the section.

Finally, we update the image-domain wavefield tomography model with the data-domain wavefield tomography approach. Figure 5(a) depicts the updated model (compare with Figure 3(a)), which changes considerably in the interval $z = 4$ km to $z = 6$ km. Figures 5(b)-5(c) are the corresponding RTM image and angle gathers, respectively. Note that even though the velocity does not significantly vary the kinematics of the experiment, it introduces subtle structural features in the image. We can see that the structure of the line becomes flatter with the new model (see for instance the event at $z = 4$ km and $x = 18$ to 24 km).

5 DISCUSSION

In the previous section we show the imaging results from different velocity models. In this section we do a quantitative comparison between the inverted models. Figures 3(a), 4(a), and 5(a) show that the right part of the model, $x > 14$ km, does not change significantly. However, we can see in both data-domain wavefield tomography models that the shallow part of the right side of the model improves the flatness of the gathers at shallow depths (compare Figure 2(c) with Figure 3(c)).

The image-domain wavefield tomography model (Figure 4(a)) does not significantly change the kinematics of the right side of the model. We can think of two reasons for this observation: (i) the right side of the model has poorer illumination, which can be confirmed by the limited angle range in the gathers, and (ii) given that the right side of the section has a shallower water column, then we can expect several orders of surface related multiples. We can address (i) by relaxing the mute in the input shot gathers and thus improving the illumination. Another option is to use a penalty operator $P(\lambda)$ that takes into account the spatially-variable illumination of the data, as suggested by Yang et al. (2013). In relation to point (ii), the presence of multiples (surface-related or internal) violates the implicit single scattering assumption of conventional and extended images. Weibull and Arntsen (2013) suggest creating data that conform to single scattering by muting the multiples in the extended images and then demigrate the gathers. This new dataset should remove the bias of the surface related multiples in the inversion. Another option is to demultiply the data prior to inversion (e.g. SRME (Verschuur et al., 1992)). All these improvement ideas remain for ongoing and future tests.

On the left side of the section we see significant changes. Figures 6(a) to 6(d) show a detailed view of the models for $x < 14$ km. The data-domain wavefield tomography model in Figure 6(d), built from the initial model, shows some layering below the water bottom, where we can see a clear boundary in the model that is probably related to the events ranging from $z = 1.5$ km and $z = 2.5$ km. Given that the velocity is too fast, the data-domain wavefield tomography model is probably trapped in a local minimum. Hence, it cannot correct for the kinematic errors in the model. This is confirmed in the moveout of the gathers, shown in Figures 7(a)-7(b). There are not many differences to recognize from Figures 8(a)-8(b). This confirms that the data-domain wavefield tomography model did not alter the kinematics.

In contrast, when we compare previous models with the image-domain wavefield tomography model, Figure 6(c), we can appreciate a considerable correction to the velocity. Now, the slower velocity corrects for the bulk of the kinematic errors in the model. Figure 7(b) shows flat events up to $z = 4.5$ km through the detailed section. The new velocity highlights the unconformity depicted by the bright seismic event around $z = 4$ km. Also, we can see how new events get imaged between $z = 2.5$ km and $z = 4$ km in Figure 8(c). The updated model from data-domain wavefield tomography, depicted in Figure 6(d), now shows a sharp discontinuity in the velocity at $z = 3$ km. The corresponding image, Figure 8(d), shows a flatter structure after the data-domain wavefield tomography update. This is interesting because we can see how despite the added complexity in the velocity, the structure in the image is simplified.

Figures 9(a)-9(d) show the inversion of source functions for the initial, the data-domain wavefield tomography from the initial model, the image-domain wavefield tomography model, and the final data-domain wavefield tomography model, respectively. Note that the source functions inverted with the smooth model are laterally consistent. However, the consistency is improved in Figure 9(c). If we compare the source inversions from data-domain wavefield tomography models (Figures 9(b)-9(d)), we can see a higher lateral correlation, which confirms that the final data-domain wavefield tomography models better explain the kinematics of the data for direct and diving wave arrivals. Even though we invert each source individually, we use the average over source positions for the inversion. This is done because we know that in the field the air gun is shot with a constant pressure. Hence, we assume that the inconsistencies in the source functions come from the model itself.

Analyzing the focusing in space-lag gathers, or flatness of angle gatherers, is the proper quality control tool for image-domain methods. The equivalent tool for data-domain methods are the data residuals. Figures 10(a)-10(d) show the time-domain data residuals for the four models discussed in this report for a shot gather at $x = 18.75$ km. Figure 10(a) shows the data residual corresponding to Figure 2(a). One can observe large amplitude and phase residuals through the diving waves components of the data. After updating the model the residual depicted in Figure 10(b) shows that the diving waves
arrivals are better fit, specially between offsets 2.5 to 5 km. Figure 10(c) shows the residual corresponding to Figure 4(a), we can see that these residuals better explain the data than those in Figure 10(a). After updating the model, Figure 10(d) we can see how the residuals from Figure 5(a) better fit the data than any of the previous models. Now, the direct arrivals have a good match for near and intermediate offsets.

6 CONCLUSIONS
The combination between image-domain and data-domain wavefield tomography seeks to exploit the features of each method. The image-domain wavefield tomography methods are sensitive to the smooth components of the model due to the definition of the inverse problem. Once we obtain a smooth model that improves focusing in the extended images, we can proceed to further refine the model using data-domain wavefield tomography. We demonstrate the cascaded workflow using a real 2D marine dataset. Our image-domain wavefield tomography model corrects for most kinematic errors in the model, whereas the data-domain wavefield tomography model corrects early arrival phase errors in the data, and introduces discontinuities in the model directly correlated with events in the image.

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We thank sponsor companies of the Consortium Project on Seismic Inverse Methods for Complex Structures. The seismic data shown in this report is proprietary to and provided courtesy of CGG. We thank Bruce VerWest for the support with the dataset. The reproducible numeric examples in this paper use
Figure 5. (a) The data-domain wavefield tomography velocity model built from Figure 4(a), (b) corresponding RTM image, and (c) angle-gathers at sparse locations. The velocity ranges from 1.5 km/s for water velocity to 4 km/s in the deepest part.

the Madagascar open-source software package (Fomel et al., 2013) freely available from http://www.ahay.org.

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Figure 6. Detail from (a) initial velocity, (b) data-domain wavefield tomography built with initial velocity, (c) image-domain wavefield tomography velocity model, and (d) data-domain wavefield tomography model built with image-domain wavefield tomography model.
Figure 7. Detail from angle gathers from (a) initial velocity, (b) data-domain wavefield tomography built with initial velocity, (c) image-domain wavefield tomography velocity model, and (d) data-domain wavefield tomography model built with image-domain wavefield tomography model.
Figure 8. Detail from RTM images from (a) initial velocity, (b) data-domain wavefield tomography built with initial velocity, (c) image-domain wavefield tomography velocity model, and (d) data-domain wavefield tomography model built with image-domain wavefield tomography model.
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Figure 10. Data domain residuals for shot position $x = 18.75$ km using: (a) the initial velocity model, (b) the data-domain model built from the initial model, (c) the image-domain model, and (d) the data-domain model built from the image-domain model.


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Elastic full-waveform inversion of transmission data in 2D VTI media

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ABSTRACT

Full-waveform inversion (FWI) has proved effective in significantly improving the spatial resolution of seismic velocity models. However, it is implemented mostly for isotropic media and the applications to anisotropic models are typically limited to acoustic approximations. Here, we develop a foundation for elastic FWI in laterally heterogeneous VTI (transversely isotropic with a vertical symmetry axis) media. The model is parameterized in terms of the P- and S-wave vertical velocities and the P-wave normal-moveout and horizontal velocities. To derive the gradients of the objective function with respect to the VTI parameters, we employ the adjoint-state method. The iterative inversion is performed in the time domain using the steepest-descent method and a finite-difference modeling code. To test the algorithm, we introduce Gaussian anomalies in the Thomsen parameters of a homogeneous VTI medium and perform FWI of transmission data for different configurations of the source and receiver arrays. The inversion results strongly depend on the acquisition geometry and the aperture because of the parameter trade-offs. In contrast to acoustic FWI, the elastic inversion helps constrain the S-wave vertical velocity, which for our model is decoupled from the other parameters.

1 INTRODUCTION

Full-waveform inversion (FWI) is a technique for estimating subsurface properties by using entire seismic waveforms recorded at the surface or in a borehole. Depending on the problem and availability of forward-modeling algorithms, FWI can be performed in the time domain (Kolb et al., 1986; Gauthier, 1986; Mora, 1987; Bunks et al., 1995) or frequency domain (Song and Williamson, 1995; Song et al., 1995; Pratt, 1999; Pratt and Shipp, 1999). Evaluation of the gradient of the objective function is often based on the adjoint-state method, as described in Tarantola (1984a), Fichtner et al. (2006), and Liu and Tromp (2006).

Kohn et al. (2012) discuss the influence of parameterization on elastic isotropic FWI and conclude that it is preferable to describe the model in terms of the P- and S-wave velocities and density rather than the impedances. Liu and Tromp (2006) derive the gradients of the objective function with respect to the stiffness coefficients and the perform FWI for earthquake data from an elastic isotropic model. FWI has been extended to anisotropic media, but typically in the acoustic approximation (Plessix and Rynja, 2010; Gholami et al., 2011; Plessix and Cao, 2011; Shen, 2012). Such “anisotropic acoustic” algorithms, however, do not properly handle reflection amplitudes and cannot be applied to multicomponent data. Elastic FWI of synthetic multicomponent surface data (consisting of both diving waves and reflections) for VTI media is performed by Lee et al. (2010), but suboptimal parameterization in terms of the stiffness coefficients causes ambiguity in their results.

In our previous work (Kamath and Tsvankin, 2012; hereafter, referred to as Paper I), we invert multicomponent reflection data (PP- and PSV-waves) from a horizontally layered VTI model for the interval Thomsen parameters – the P- and S-wave vertical velocities ($V_{P0}$ and $V_{S0}$) and anisotropy coefficients $\epsilon$ and $\delta$. Inversion for density makes the objective function highly nonlinear, thereby causing the algorithm to get trapped in local minima. Therefore, the interval densities are fixed at the correct values. Although PP-waves alone may be sufficient to resolve $V_{P0}$, $V_{S0}$, $\epsilon$, and $\delta$, stable parameter estimation for layers at depth requires employing long-offset data (with the spreadlength-to-depth ratio reaching at least two) or the addition of PS-waves. Inversion of multicomponent data benefits from using a multiscale approach, which helps reduce the sensitivity to the choice of the initial model (Bunks et al., 1995).

Here, we introduce an expansion of elastic FWI to laterally heterogeneous VTI media. The model is parameterized in terms of $V_{P0}$, $V_{S0}$, and the P-wave normal-moveout ($V_{nma,0,\rho}$) and horizontal ($V_{hor,\rho}$) velocities. To compute the gradient of the objective function, we adapt the results of Liu and Tromp (2006) obtained with the adjoint-state method. FWI is performed in the time domain with the wavefield generated using a 2D elastic finite-difference modeling code. Then the algorithm is applied to transmission data generated for homogeneous VTI models with Gaussian anomalies in the parameters $V_{P0}$, $V_{S0}$, and $\epsilon$. 
2 METHODOLOGY

2.1 Full-waveform inversion for VTI media

FWI in the time domain is designed to minimize the following objective function:

$$ F = \frac{1}{2} \sum_{r=1}^{N} \int_{0}^{T} \| u(x_r, t) - d(x_r, t) \|^2 dt, \tag{1} $$

where $N$ is the number of receivers, $T$ is the trace length, $u(x_r, t)$ is the displacement computed for a trial model, and $d(x_r, t)$ is the recorded displacement at receiver location $x_r$. Because the relationship between the data and the model is nonlinear, the inversion is performed iteratively, with the model update at each iteration found as:

$$ \Delta m = [J^T J]^{-1} J^T \Delta d, \tag{2} $$

where $J$ is the Fréchet derivative matrix obtained by perturbing each model parameter, $J^T J$ is the approximate Hessian matrix, $T$ denotes transposition, and $\Delta d$ is the difference between the observed data and those computed for a trial model.

In Paper I, we use PP reflections or a combination of PP and PS events generated for a laterally layered VTI model to estimate the interval parameters $V_{P0}$, $V_{S0}$, $\epsilon$, and $\delta$. For the purpose of inversion, it is convenient to use quantities that are directly constrained by the data and have the same units. Hence, instead of $\epsilon$ and $\delta$ Paper I operates with the P-wave NMO velocity ($V_{omo,P} = V_{P0}\sqrt{1+2\epsilon}$) and the horizontal velocity ($V_{hor,P} = V_{P0}\sqrt{1+2\epsilon}$). If the number of layers (which is fixed during the inversion) is not large, it is possible to compute the Fréchet matrix and the approximate Hessian explicitly by perturbing each model parameter.

2.2 Inverse problem in 2D

In the case of laterally heterogeneous media, computation of the Fréchet derivatives becomes prohibitively expensive because it involves calculating as many forward models at each iteration as the number of parameters (typically defined on a grid). It is more practical to calculate the gradient $[J^T \Delta d \text{ in equation } 2]$ of the objective function with the adjoint-state method, which has been widely used in FWI (Tarantola, 1984b; Plessix, 2006; Liu and Tromp, 2006; Fichtner et al., 2006). The model update, which is a scaled version of the gradient, is then calculated using steepest-descent or conjugate-gradient algorithms. Alternatively, either the so-called BFGS (Broyden-Fletcher-Goldfarb-Shanno) method or its limited-memory equivalent, the L-BFGS method (both are quasi-Newton techniques), can be employed to scale the gradient by the inverse of an approximate Hessian matrix (Virieux and Operto, 2009).

The adjoint-state method is designed to compute the gradient of the objective function using the so-called “adjoint wavefield.” Because the wave equation is self-adjoint, it can be solved for the adjoint wavefield with the data residuals treated as sources. The residuals at each time step are injected “backward in time” (i.e., starting from the last time sample), which is commonly described as back-propagation of data residuals. For 2D multicomponent data, the vertical and horizontal displacement components of the data residuals should be injected into the medium simultaneously. The gradient is obtained by applying the imaging condition to the spatial derivatives of the forward and adjoint wavefields.

Here, we assume that the properties of the VTI medium vary in 2D and consider only in-plane polarized waves (P and SV). Hence the model is described by four stiffness coefficients (written in the Voigt notation): $C_{11}$, $C_{33}$, $C_{13}$, and $C_{55}$. However, it is certain combinations of the stiffnesses that control traveltimes and amplitudes of seismic waves (Tsvankin, 2012). In particular, description of wave propagation and inversion of seismic data can be facilitated by employing Thomsen parameters and their simple combinations (e.g., the anellipticity coefficient $\eta$). Lee et al. (2010), who parameterize the VTI model in terms of the stiffnesses, are unable to resolve the coefficient $C_{13}$, likely because of the tradeoff between $C_{13}$ and $C_{55}$ in P-wave kinematic signatures. In Paper I we could constrain the relevant Thomsen parameters ($V_{P0}$, $V_{S0}$, $\epsilon$, and $\delta$), although the algorithm operated with the vertical, NMO and horizontal velocities. Likewise, here we parameterize the model in terms of the velocities $V_{P0}$, $V_{S0}$, $V_{omo,P}$, and $V_{hor,P}$.

The gradients of the objective function (equation 1) with respect to the elements of the stiffness tensor are derived in Appendix A using the results of Liu and Tromp (2006):

$$ \frac{\partial F}{\partial c_{ijkl}} = -\int_{0}^{T} \frac{\partial u_i}{\partial x_j} \frac{\partial \psi_k}{\partial x_l} dt, \tag{3} $$

where $u$ and $\psi$ are the forward and adjoint displacement wavefields, respectively. Using the chain rule, we can find the gradient for each velocity $V_u$ ($V_{P0}$, $V_{S0}$, $V_{omo,P}$, and $V_{hor,P}$):

$$ \frac{\partial F}{\partial V_u} = \sum_{ijkl} \frac{\partial F}{\partial c_{ijkl}} \frac{\partial c_{ijkl}}{\partial V_u}. \tag{4} $$

The stiffness coefficients are expressed in terms of the velocities in equations A18–A21. Combining equations 3, 4, and A18–A21 yields the gradients with respect to velocities (equations A22–A25).

Here, FWI is implemented in the time domain, primarily because the finite-difference modeling software available to us performs time-domain computations. We employ the steepest-descent method to update the model at each iteration with equal step length for all parameters.

3 NUMERICAL TESTS FOR TRANSMISSION DATA

Next, we perform tests for simple synthetic models to verify the accuracy of the gradient computation. Because the initial stage of FWI typically involves diving waves, the data are generated for transmission experiments. The model includes Gaussian anomalies in the Thomsen parameters $V_{P0}$, $V_{S0}$, and $\epsilon$ inserted into a homogeneous VTI background between line arrays of sources and receivers (Figure 1).

In the first test, the model includes an anomaly in $\epsilon$, while
Elastic full-waveform inversion of transmission data in 2D VTI media

Figure 1. VTI model with a Gaussian anomaly (standard deviation $\sigma = 300$ m) in the anisotropy parameter $\epsilon$. The background and maximum values of $\epsilon$ are 0.1 and 0.142, respectively. The other Thomsen parameters are spatially invariant: $V_{P0} = 3000$ m/s, $V_{S0} = 1500$ m/s, and $\delta = -0.05$. The dots on the left mark the source locations and the vertical line on the right represents an array of receivers placed at each grid point (6.6 m apart).

Figure 2. (a) Vertical and (b) horizontal displacements for the model in Figure 1 generated by a shot at $z = 1.5$ km.

The other parameters are spatially invariant (Figure 1). The wavefield is generated by a point displacement source polarized in the horizontal direction with a peak frequency of 10 Hz. The vertical and horizontal displacements ("recorded data") from a shot in the center of the array are shown in Figure 2. The "modeled" data are then generated in the background medium without the anomaly, and the adjoint source is obtained as the difference between the two wavefields.

Figure 3 displays the gradients with respect to the model parameters (velocities) calculated from equations A22–A25. For the source-receiver geometry in Figure 1, waves travel relatively close to the isotropy plane, and are influenced primarily by $V_{\text{hor},P}$, which is a function of $\epsilon$. Hence, the largest gradient is that for the velocity $V_{\text{hor},P}$, which correctly identifies $\epsilon$ as the parameter that needs updating. However, the initial gradient with respect to $V_{S0}$ is also significant because $\epsilon$ influences the SV-wave amplitude.

Starting from the homogeneous background model, we perform the inversion using the steepest-descent method. We run 50 iterations or stop the inversion if the objective function flattens out sufficiently (Figure 4). The gradient for $V_{S0}$ has opposite signs in subsequent iterations, and the difference between the inverted $V_{S0}$ and the actual (background) value is negligible (Figure 5(b)). Likewise, the inverted and initial values of $V_{P0}$ and $\delta$ are close, which confirms that FWI converges toward the actual model. The updates in $V_{\text{hor},P}$, combined with negligible changes in $V_{P0}$, ensure the reconstruction of the anomaly in $\epsilon$.

The shape of the anomaly (i.e., it is stretched along the horizontal axis), however, is somewhat distorted because of the source-receiver configuration. For this acquisition geometry, spatial resolution should indeed be higher in the vertical direction than horizontally, as discussed by Wu and Toksöz (1987). Even though the objective function decreases to just 0.04% of the initial value, the maximum estimated value of $\epsilon$ is about 0.12, whereas the actual $\epsilon$ reaches 0.14. When the aperture is increased by reducing the distance between the arrays by about one-half and increasing the vertical extent of the arrays by 0.5 km (Figure 6), the shape of the anomaly is better resolved (Figure 7). In addition, because the inverted veloci-
ties are closer to the actual values, the estimated $\epsilon$ (maximum value of 0.13) is slightly more accurate than in the previous example. Because this configuration yields better inversion results, it is used in all subsequent tests.

Next, we introduce an anomaly in the P-wave vertical velocity $V_{P0}$, with the sources still polarized horizontally (Figure 8). The anomaly in $V_{P0}$ also causes perturbations in the NMO velocity $V_{nmo,P}$ and the horizontal velocity $V_{hor,P}$. As a result, the largest update (about 74% of the actual anomaly) for the given configuration is the one with respect to the horizontal velocity (Figure 9(c)), whereas the updates for $V_{P0}$ (45%) and $V_{nmo,P}$ (44%) are much smaller. An update in $V_{hor,P}$ without the corresponding change in $V_{P0}$ results in an undesired update in $\epsilon$ (Figure 9(c)) and steers the inversion away from recovering the true anomaly in $V_{P0}$ (Figure 9(a)). Because the parameter $\delta$ depends on the velocity ratio $V_{nmo,P}/V_{P0}$, and the inversion updates both the velocities proportionately, there is no significant change in $\delta$ (Figure 9(d)). These results indicate that the inversion for $V_{P0}$ fails because of a lack of propagation directions near the vertical.

In contrast, when the source-receiver configuration is rotated by 90° (Figure 10) and the wave propagation is predominantly vertical, the largest update is the one for $V_{P0}$ (Figure 11). Since the actual $\epsilon$ and $\delta$ remain unperturbed, the inversion...
Figure 5. Difference between the inverted parameters (a) $V_{P0}$, (b) $V_{S0}$, (c) $\epsilon$, and (d) $\delta$ and their initial values for the model from Figure 1. The velocities have units of km/s.

needs to update $V_{nmo,P}$ and $V_{hor,P}$ along with $V_{P0}$, which would keep the anisotropy coefficients unchanged. However, whereas the change in $V_{P0}$ is about 85% of the required update, those in $V_{nmo,P}$ and $V_{hor,P}$ are only about 12.5% and 11%, respectively. This results in relatively large incorrect updates in the parameters $\epsilon$ (-0.075) and $\delta$ (-0.07).

From these tests, it is clear that the inversion results strongly depend on how the wavefields probe the model. Because of inherent trade-offs between the parameters for some
source-receiver configurations, even elastic FWI suffers from nonuniqueness. Incorporating reflection data might provide more constraints on the VTI parameters.

The last test is performed for a Gaussian anomaly in $V_{S0}$ embedded between horizontal source and receiver arrays (Figure 12). The maximum perturbation in $V_{S0}$ with respect to the background is the same as that for $V_{P0}$ in the previous tests (Figures 8 and 10), but the percentage perturbation in $V_{S0}$ is two times higher. Hence, to avoid the problem of cycle-skipping, the peak frequency of the source is reduced to 5 Hz. The inversion results, which include an update only in $V_{S0}$ (Figure 13), indicate that there is no apparent trade-off between the model parameters. As was the case in the inversion for $\epsilon$, despite the significant decrease in the objective function (to 0.03% of the initial value), the estimated $V_{S0}$ is off by about 3%. Interestingly, the inversion for $V_{S0}$ yields similar results when the source and receiver arrays are vertical and wave propagation is predominantly horizontal. This is likely due to the fact that the SV-wave velocity in VTI media is the same in the vertical and horizontal directions.

4 CONCLUSIONS

One of the most important steps in performing full-waveform inversion is calculation of the gradient of the objective function with respect to the model parameters. Here, we employed the adjoint-state method to develop gradient computation for elastic multicomponent wavefields from 2D VTI media. Then FWI was implemented in the time domain for transmitted waves from point displacement sources.

The in-plane polarized waves (P and SV) are controlled by combinations of four stiffness coefficients: $C_{11}$, $C_{13}$, $C_{33}$, and $C_{55}$. The adjoint-state method helped derive analytic expressions for the gradients of the least-squares objective function with respect to the stiffnesses. A more convenient parameterization used here includes the P-wave vertical ($V_{P0}$), NMO ($V_{\text{nmo, } P}$), and horizontal ($V_{\text{hor, } P}$) velocities and the S-wave vertical velocity ($V_{S0}$).

Numerical tests were conducted for Gaussian anomalies in one of the Thomsen parameters added to a homogeneous VTI background. The magnitude of the update with respect to each model parameter was shown to be governed by the lo-
cation and configuration of the source and receiver arrays. An anomaly just in $\epsilon$ (with unperturbed $V_{P0}$, $V_{S0}$, and $\delta$) produces the corresponding anomaly in a single velocity estimated by FWI ($V_{\text{hor},P}$). When the source and receiver arrays are vertical, the inversion algorithm correctly updates only $V_{\text{hor},P}$, thus recovering the spatial distribution of $\epsilon$.

In another test, introducing an anomaly in $V_{P0}$ also changes the velocities $V_{\text{nmo},P}$ and $V_{\text{hor},P}$. In crosshole geometry, the inversion updates predominantly the horizontal velocity without the corresponding changes in $V_{P0}$ and $V_{\text{nmo},P}$, which results in distorted estimates of $V_{P0}$ and $\epsilon$. The ratio $V_{\text{nmo},P}/V_{P0}$, however, remains practically unchanged during the inversion, so the coefficient $\delta$ stays close to the background value (as it should be). The same anomaly in the P-wave vertical velocity, but with horizontal arrays of sources and receivers, does result in a significant update in $V_{P0}$. The velocities $V_{\text{nmo},P}$ and $V_{\text{hor},P}$, however, remain almost unchanged. Hence, although the inversion recovers most of the anomaly in $V_{P0}$, it incorrectly updates $\epsilon$ and $\delta$.

The best-constrained parameter in this series of tests proved to be the S-wave velocity $V_{S0}$, which has negligible influence on P-wave data. The inversion for an anomaly in $V_{S0}$ produces an accurate update in that parameter with small changes in $V_{P0}$, $\epsilon$, and $\delta$. In addition, estimation of $V_{S0}$ works equally well for horizontal and vertical source and receiver arrays, likely because the SV-wave velocities in VTI media coincide in the horizontal and vertical directions.

The tests confirm that increasing the aperture improves the spatial resolution and magnitude of the recovered anomalies. The observed trade-offs in the inversion results may be explained in terms of the ‘radiation pattern’ exhibited by the different model parameters, as will be discussed in a sequel paper.

5 ACKNOWLEDGMENTS

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Figure 9. Difference between the inverted parameters (a) $V_{P_0}$, (b) $V_{S_0}$, (c) $\epsilon$, and (d) $\delta$ and their initial values for the model with an anomaly in $V_{P_0}$ (Figure 8).

Figure 10. VTI model with a Gaussian anomaly in $V_{P_0}$. The model parameters are the same as in Figure 8, but the source and receiver arrays are horizontal. The sources are polarized vertically.
Elastic full-waveform inversion of transmission data in 2D VTI media

Figure 11. Difference between the inverted parameters (a) $V_{P0}$, (b) $V_{S0}$, (c) $\epsilon$, and (d) $\delta$ and their initial values for the model with an anomaly in $V_{P0}$ (Figure 10).

Figure 12. VTI model with a Gaussian anomaly in $V_{S0}$. The background and maximum values of $V_{S0}$ are 1500 m/s and 1783 m/s, respectively. The other Thomsen parameters are spatially invariant: $V_{P0} = 3000$ m/s, $\delta = -0.05$, and $\epsilon = 0.1$. The sources are polarized horizontally.


Figure 13. Difference between the inverted parameters (a) $V_{P0}$, (b) $V_{S0}$, (c) $\epsilon$, and (d) $\delta$ and their initial values for the model with an anomaly in $V_{S0}$ (Figure 12).


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APPENDIX A: GRADIENT COMPUTATION FOR VTI MEDIA USING THE ADJOINT STATE METHOD

As discussed in Plessix (2006) and Liu and Tromp (2006), the objective function in equation 1 should be minimized under the constraint that the modeled displacement \( u(x_r, t) \) satisfies the wave equation. Here, we use the elastic wave equation for heterogeneous, arbitrarily anisotropic media:

\[
\rho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j} \left( c_{ijkl} \frac{\partial u_k}{\partial x_l} \right) = f_i, \tag{A1}
\]

where \( \rho \) is the density, \( c_{ijkl} \) are the components of the stiffness tensor, and \( f \) is the body force per unit volume. All indices change from 1 to 3 and summation over repeated indices is implied. The displacement wavefield is subject to the initial conditions,

\[
u(x, 0) = 0, \quad \frac{\partial u(x, 0)}{\partial t} = 0, \tag{A2}\]

and the radiation boundary condition,

\[
u(x, t)_{x \to \infty} \to 0. \tag{A3}\]

The method of Lagrange multipliers (Strang, 1991) is used to define the Lagrangian \( \Lambda \):

\[
\Lambda = \frac{1}{2} \int_0^T \int \left[ \sum_{r=1}^N (u_i(x_r, t) - d_i(x_r, t))^2 \right] \ dt - \int_0^T \int_{\Omega} \lambda_i \left[ \rho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j} \left( c_{ijkl} \frac{\partial u_k}{\partial x_l} \right) - f_i \right] \ dV dt, \tag{A4}\]

where \( r = 1, 2 \ldots N \) denotes the receivers, \( \Omega \) is the integration domain (which includes the entire 3D space), \( \partial \Omega \) is the surface of \( \Omega \), and \( \lambda(x, t) \) is the vector Lagrange multiplier that needs to be determined. The objective is to find the stationary points of \( \Lambda \) when \( u, \lambda \), and \( c_{ijkl} \) are perturbed. After integration by parts and application of the Gauss divergence theorem, we obtain the change in the Lagrangian,

\[
\delta \Lambda = \int_0^T \int_{\Omega} \sum_{r=1}^N \left( u_i(x_r, t) - d_i(x_r, t) \right) \delta(x - x_r) \delta u_i \ dv \ dt
\]

\[
- \int_0^T \int_{\Omega} \delta c_{ijkl} \frac{\partial u_k}{\partial x_l} \frac{\partial \lambda_i}{\partial x_j} \ dV \ dt - \int_0^T \int_{\Omega} \lambda_i \left[ \rho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j} \left( c_{ijkl} \frac{\partial u_k}{\partial x_l} \right) - f_i \right] \ dV dt
\]

\[
- \int_0^T \int_{\Omega} \left[ \rho \frac{\partial^2 \lambda_i}{\partial t^2} - \frac{\partial}{\partial x_j} \left( c_{ijkl} \frac{\partial \lambda_k}{\partial x_l} \right) - f_i \right] \ dV dt
\]

\[
- \int_0^T \int_{\Omega} \left[ \rho \lambda_i \frac{\partial \delta u_i}{\partial t} - \rho \delta u_i \frac{\partial \lambda_i}{\partial t} \right] \ dv
\]

\[
+ \int_0^T \int_{\partial \Omega} \lambda_i \left[ \delta c_{ijkl} \frac{\partial u_k}{\partial x_l} + c_{ijkl} \frac{\partial (\delta u_k)}{\partial x_l} \right] n_j \ ds \ dt - \int_0^T \int_{\partial \Omega} \delta u_i c_{ijkl} \frac{\partial \lambda_k}{\partial x_l} n_j \ ds \ dt , \tag{A5}\]

where \( n \) is the vector normal to the surface \( \partial \Omega \). Perturbing \( u(x, t) \) in equations A2 and A3 yields the initial and boundary conditions for \( \delta u(x, t) \):

\[
\delta u(x, 0) = 0, \quad \frac{\partial \delta u(x, t)}{\partial t} = 0, \quad \delta u(x, t)_{x \to \infty} \to 0. \tag{A6}\]

The Lagrange multiplier \( \lambda \) is constrained by the “final” conditions (i.e., those at time \( T \)),

\[
\lambda(x, T) = 0, \quad \frac{\partial \lambda(x, T)}{\partial t} = 0, \tag{A7}\]
The wavefield \( \psi \) specific symmetries can be derived from equation A11 by substituting the appropriate stiffness tensors or matrices. This is a general result for an anisotropic medium described by the complete stiffness tensor equation, which coincides with the elastic wave equation A1. Setting the coefficient of \( c \) integrands of equation A9 go to zero. For a given model (i.e., fixed surface multiples. Instead, we impose the radiation condition to create absorbing boundaries on all sides of the model.

To simulate the Lagrange-multiplier wavefield, it is convenient to define an “adjoint wavefield” \( \psi \) (Liu and Tromp, 2006):

\[
\psi(x, t) = \lambda(x, T - t).
\] (A12)

The wavefield \( \psi \) satisfies the wave equation A10 but with the source function reversed in time:

\[
\rho \frac{\partial^2 \psi}{\partial t^2} - \frac{\partial}{\partial x_j} \left( c_{ijkl} \frac{\partial \psi}{\partial x_l} \right) = \sum_{r=1}^{N} \left[ u_i(x_r, T - t) - d_i(x_r, T - t) \right].
\] (A13)

The initial conditions for \( \psi \) (using equations A7 and A12) are as follows:

\[
\psi(x, 0) = 0, \quad \frac{\partial \psi(x, 0)}{\partial t} = 0.
\] (A14)

The wavefield \( \psi \) also satisfies the radiation boundary condition:

\[
\psi(x, t) \big|_{x \to \infty} \to 0.
\] (A15)

From equations A11 and A12, we can find the gradient of the objective function with respect to the stiffness coefficients:

\[
\frac{\partial \mathcal{F}}{\partial c_{ijkl}} = - \int_0^T \frac{\partial u_i}{\partial x_j} \frac{\partial \psi_k}{\partial x_l} dt.
\] (A16)
If, instead of $c_{ijkl}$, the model is described by parameters $m_n$, the gradient of $F$ can be found from the chain rule:

$$\frac{\partial F}{\partial m_n} = \sum_{ijkl} \frac{\partial F}{\partial c_{ijkl}} \frac{\partial c_{ijkl}}{\partial m_n}. \quad (A17)$$

Here, we parameterize the model in terms of the velocities $V_{P0}$, $V_{S0}$, $V_{nmo}$, $P$, and $V_{hor}$, $P$. The stiffness coefficients (written in the two-index notation) represent the following functions of the velocities (Tsvankin, 2012):

$$C_{11} = \rho V_{hor}^2, \quad (A18)$$

$$C_{33} = \rho V_{P0}^2, \quad (A19)$$

$$C_{13} = \rho \sqrt{(V_{P0}^2 - V_{S0}^2)(V_{nmo,P}^2 - V_{S0}^2)} - \rho V_{S0}^2, \quad (A20)$$

$$C_{55} = \rho V_{S0}^2. \quad (A21)$$

Using equations A16, A17, and A18 - A21, we obtain the following gradients of the objective function with respect to the velocities:

$$\frac{\partial F}{\partial V_{P0}} = -2V_{P0} \int_0^T \rho \left[ \frac{\partial \psi_3}{\partial x_3} \frac{\partial u_3}{\partial x_3} + \frac{1}{2} \frac{V_{nmo,P}^2 - V_{S0}^2}{V_{P0}^2 - V_{S0}^2} \left( \frac{\partial \psi_1}{\partial x_1} \frac{\partial u_3}{\partial x_3} + \frac{\partial \psi_3}{\partial x_3} \frac{\partial u_1}{\partial x_1} \right) \right] dt, \quad (A22)$$

$$\frac{\partial F}{\partial V_{S0}} = -2V_{S0} \int_0^T \rho \left\{ \frac{2V_{S0}^2 - V_{P0}^2 - V_{nmo,P}^2}{2(V_{P0} - V_{S0})} - 1 \right\} \left( \frac{\partial \psi_1}{\partial x_1} \frac{\partial u_3}{\partial x_3} + \frac{\partial \psi_3}{\partial x_3} \frac{\partial u_1}{\partial x_1} \right) dt, \quad (A23)$$

$$\frac{\partial F}{\partial V_{nmo,P}} = -2V_{nmo,P} \int_0^T \rho \frac{V_{nmo,P}^2 - V_{S0}^2}{V_{nmo,P}^2 - V_{S0}^2} \left( \frac{\partial \psi_1}{\partial x_1} \frac{\partial u_3}{\partial x_3} + \frac{\partial \psi_3}{\partial x_3} \frac{\partial u_1}{\partial x_1} \right) dt, \quad (A24)$$

$$\frac{\partial F}{\partial V_{hor,P}} = -2V_{hor,P} \int_0^T \rho \frac{\partial \psi_1}{\partial x_1} \frac{\partial u_1}{\partial x_1} dt, \quad (A25)$$
Sensitivity analysis for elastic full-waveform inversion in VTI media

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ABSTRACT

Multiparameter full-waveform inversion (FWI) is generally nonunique, and the results are strongly influenced by the geometry of the experiment and the type of recorded data. Studying the sensitivity of different subsets of data to the model parameters may help in choosing an optimal acquisition design, inversion workflow, and parameterization. Here, we derive the Fréchet kernel for FWI of multicomponent data from a 2D VTI (transversely isotropic with a vertical symmetry axis) medium. The kernel is obtained by linearizing the elastic wave equation using the Born approximation and employing the asymptotic Green’s function. The amplitude of the kernel (‘radiation pattern’) yields the angle-dependent energy scattered by a perturbation in a certain model parameter. The perturbations are described in terms of the P- and S-wave vertical velocities and the P-wave normal-moveout and horizontal velocities. The background medium is assumed to be homogeneous and isotropic, which allows us to obtain simple expressions for the radiation patterns corresponding to all four velocities. These patterns help explain the FWI results for multicomponent transmission data generated for Gaussian anomalies in the Thomsen parameters inserted into a homogeneous VTI medium.

1 INTRODUCTION

Full-waveform inversion (FWI) of seismic data is often performed under the assumption of an acoustic and isotropic earth model. In that case, the wavefield is described by a single spatially varying parameter – the P-wave velocity. Taking elasticity and/or anisotropy into account introduces additional parameter fields, and the outcome of the inversion strongly depends on the choice of model parameterization.

The sensitivity of FWI to a chosen set of parameters has been studied for acoustic VTI media. The P-waves kinematics in VTI models can be described by the vertical velocity \( V_{P0} \), the normal-moveout velocity \( V_{nmo} \) (\( V_{nmo} = V_{P0} \sqrt{\frac{1+2\epsilon}{1+2\epsilon}} \)), and the horizontal velocity \( V_{hor} \) (\( V_{hor} = V_{P0} \sqrt{\frac{1+2\epsilon}{1+2\epsilon}} \)). Employing the singular value decomposition (SVD) of the Fréchet derivative matrix (‘sensitivity matrix’), Plessix and Cao (2011) evaluate the sensitivity of the objective function to the VTI parameters. Gholami et al. (2011) carry out inversion either for a single velocity (\( V_{P0}, V_{nmo}, \) or \( V_{hor} \)) while keeping the other two at the actual values or for two velocities (\( V_{P0} \) and \( V_{hor} \)) simultaneously. They conclude that the single-parameter inversion provides a good estimate of the unknown velocity, while the multiparameter inversion suffers from trade-offs.

The Fréchet kernel of FWI relates a perturbation in the wavefield to perturbations in the model parameters. Gholami et al. (2013) use finite differences to compute the Fréchet kernel for a point diffractor embedded in a homogeneous acoustic VTI space. The amplitude of the kernel varies with the angle between the incident wave and the symmetry axis at the diffractor. This variation, called the “radiation pattern,” helps evaluate the sensitivity of the inversion to model parameters. An analytic description of the radiation patterns associated with a VTI perturbation in a homogeneous isotropic medium is developed by Alkhalifah and Plessix (2013).

In a previous paper (Kamath and Tsvankin, 2014; hereafter referred to as Paper I), we perform FWI of multicomponent transmission data computed for Gaussian anomalies in the Thomsen parameters embedded in homogeneous elastic VTI models. The gradient of the objective function is obtained with the adjoint-state method. The medium is described by the P-wave vertical velocity \( V_{P0} \), NMO velocity \( V_{nmo} \), horizontal velocity \( V_{hor} \), and the S-wave vertical velocity \( V_{S0} \). Experiments performed for horizontal and vertical arrays of sources and receivers show strong dependence of the inversion results on the acquisition geometry.

Here, we perform sensitivity analysis of FWI for 2D VTI media parameterized by the velocities \( V_{P0}, V_{S0}, V_{nmo}, \) and \( V_{hor} \). First, we derive the Fréchet kernel for elastic FWI using the Born and WKBJ approximations and an asymptotic representation of the Green’s function. Then the kernel is used to obtain the radiation patterns for the relevant stiffness coefficients. Assuming a homogeneous isotropic background makes it possible to derive simple radiation patterns for the four velocities. Finally, the radiation patterns for the P- and S-wavefields are employed to explain the results of FWI for transmission data in Paper I.
2 METHODOLOGY

We consider an elastic, homogeneous, isotropic background medium with spatially varying perturbations in the stiffness coefficients. It should be emphasized that the perturbed stiffnesses correspond to an arbitrarily anisotropic, elastic medium. Assuming the background to be isotropic and homogeneous allows us to obtain relatively simple expressions for the radiation patterns.

Following Calvet et al. (2006) and Alkhalifah and Plessix (2013), we represent the elastic wave equation in the Born approximation for a perturbation $\delta c_{ijkl}$ in the stiffness tensor:

$$\rho \frac{\partial^2 \delta u_i}{\partial t^2} - \frac{\partial}{\partial x_j} \left( c_{ijkl} \frac{\partial \delta u_k}{\partial x_l} \right) = \frac{\partial}{\partial x_j} \left( \delta c_{ijkl} \frac{\partial u_k}{\partial x_l} \right),$$

where $\delta u$ is the perturbation in the wavefield caused by the perturbation $\delta c_{ijkl}$ in the stiffness tensor. The solution of equation 1 can be expressed in terms of the Green’s functions $G_{mk}$ and $G_{ni}$ (Appendix A):

$$\delta u_i(x_r, \omega) = -\int_V \delta f_m(x_s, \omega) \delta c_{ijkl}(x') \times \frac{\partial G_{mk}(x_r, x_s, \omega)}{\partial x'_j} \frac{\partial G_{ni}(x_r, x'_s, \omega)}{\partial x'_j} dV(x'),$$

where $x_r$ and $x_s$ are the locations of the source and receiver, respectively, $f$ is the source function, and $V(x')$ is the volume that includes all scatterers. The Green’s functions are then replaced by their asymptotic representation (Vavryčuk, 2007). Taking the spatial derivative of just the exponent of $G$ (i.e., of its rapidly varying part, according to the WKBJ approximation) yields (equation A16):

$$\delta u_i(x_r, \omega) = \int_V \delta f_m(x_s, \omega) A(\omega) p_i^t \ p_j^s \ g_k^t \ g_l^s \ \delta c_{ijkl} dV(x').$$

The superscripts $s$ and $r$ denote the incident and scattered wavefields, respectively, $A(\omega)$ is a function of frequency and the backscattering velocities of the incident and scattered wavefields, and $p$ and $g$ are the unit slowness and polarization vectors, respectively. The radiation pattern is the amplitude of the kernel that varies with the incident and scattering angles. For a model parameterized in terms of the stiffnesses, the radiation pattern derived from equation 3 is (equation A18):

$$\Omega = p_i^s \ p_j^s \ g_k^t \ g_l^t.$$  

Equation 4 is valid for a perturbation $\delta c_{ijkl}$ corresponding to an elastic, arbitrarily anisotropic scatterer in 3D. In this study, however, we consider a 2D elastic VTI medium, so the indices $i$, $j$, $k$, and $l$ in equations 2 – 4 take values of 1 and 3. The chain rule can then be used to obtain the pattern $\Omega$ for a VTI medium described in terms of the velocities $V_{P0}$, $V_{S0}$, $V_{nmo}$, and $V_{hor}$.

The normalized scattering coefficients for the $P$- and SV-wavefields are given by equations A23—A26 and A27—A30, respectively. In the case of the transmitted wavefield, the incidence and scattering angles coincide. Substituting equations A31 – A38 into equations A23 – A30, we obtain the radiation patterns (normalized by $2\pi V_{P0}$) for the scattered $P$-wavefield:

$$\Omega^n_P(V_{P0}) = 2 \cos^2 \theta,$$

$$\Omega^t_P(V_{S0}) = 0,$$

$$\Omega^n_P(V_{nmo}) = \frac{1}{7} \sin^2 2\theta,$$

$$\Omega^n_P(V_{hor}) = 2 \sin^2 \theta,$$

where $\theta$ is the incidence angle. The patterns for the scattered $S$-wavefield (also normalized by $2\pi V_{P0}$) are:

$$\Omega^n_S(V_{P0}) = 0,$$

$$\Omega^n_S(V_{S0}) = -2 \frac{V_{S0}}{V_{P0}},$$

$$\Omega^n_S(V_{nmo}) = \frac{1}{7} \sin^2 2\theta,$$

$$\Omega^n_S(V_{hor}) = \frac{1}{7} \sin^2 2\theta.$$

The absolute values of radiation patterns in Figure 1 show how perturbations in the velocities $V_{P0}$, $V_{nmo}$, and $V_{hor}$ scatter $P$-wave energy for different incidence angles. In our approximation, a perturbation in $V_{S0}$ does not influence the scattered $P$-wavefield. As expected, the intensity of the wavefield scattered by a perturbation in $V_{P0}$ reaches its maximum for propagation along the symmetry axis and goes to zero in the isotropy plane (Figure 1(a)). In contrast, a perturbation in $V_{hor}$ produces the largest scattering in the isotropy plane, with a rapid decay toward the symmetry axis (Figure 1(c)). The maximum energy scattered by a perturbation in $V_{nmo}$ is four times smaller than that for $V_{P0}$ and corresponds to an angle of $45^\circ$ with the symmetry axis (Figure 1(b)).

A perturbation in the velocity $V_{S0}$ scatters the $SV$-wavefield uniformly for the entire range of incidence angles (Figure 2(a)), which is likely due to the fact that $V_{S0}$ also represents the horizontal SW-wave velocity. The radiation patterns of $V_{nmo}$ and $V_{hor}$ (Figure 2(b) and 2(c), respectively) for SW-waves are similar to that of $V_{nmo}$ for the $P$-wave (Figure 1(b)). The $SV$-wave is primarily influenced by $V_{S0}$ and the parameter $\sigma = (V_{P0}/V_{S0})^2(\epsilon - \delta)$ (Vavryčuk, 2007). Therefore, for a model described in terms of $V_{P0}$, $V_{S0}$, $V_{nmo}$, and $V_{hor}$, the $SV$-wavefield does not explicitly depend on the velocity $V_{P0}$. This explains why a perturbation in $V_{P0}$ does not scatter SW-waves (equation 9).

3 ANALYSIS OF FWI OF TRANSMISSION DATA

Next, we employ the analytic results obtained above to explain the FWI results for transmission data from Paper I. The models include Gaussian anomalies in the parameters $V_{P0}$, $V_{S0}$, and $\epsilon$ embedded in a homogeneous VTI background. Although the
patterns in Figures 1 and 2 are derived for a purely isotropic background medium, we expect them to remain qualitatively valid when the background is moderately anisotropic.

The first test is performed for an anomaly in $\epsilon$, which results in the corresponding perturbation in $V_{\text{hor}}$. A vertical array of sources generates the P- and S-wavefields, and a vertical array of receivers on the other side of the anomaly records the data. For this source-receiver geometry, the aperture is about $51^\circ$ on both sides of the isotropy plane. The amplitude of the scattered energy reduces by 50% at an angle of about $\pm 33^\circ$ from the horizontal (Figure 1(c)).

We run 50 iterations of FWI starting with the homogeneous background VTI model. As demonstrated in Paper I, there is negligible difference between the inverted and starting values of $V_{P0}$ (Figure 4(a)), $V_{S0}$ (Figure 4(b)), and $V_{\text{anmo}}$. The algorithm updates only $V_{\text{hor}}$, which helps accurately recover the anomaly in $\epsilon$ (Figure 4(c)) because $V_{P0}$ remains unchanged. The estimated parameter $\delta$, which depends on the ratio $V_{\text{anmo}}/V_{P0}$, stays equal to the background value (Figure 4(d)).

These inversion results are well explained by the radiation patterns in Figure 1. For the aperture in the test, the radiation pattern of $V_{\text{hor}}$ is decoupled from those of $V_{P0}$, $V_{S0}$, and $V_{\text{anmo}}$. Hence, FWI updates only the horizontal velocity, which results in an appropriate change in the coefficient $\epsilon$ and an accurate inverted model.

Another model from Paper I includes an anomaly in $V_{P0}$ with the same source-receiver configuration. Whereas the algorithm again updates $V_{\text{hor}}$, it cannot significantly change $V_{P0}$ and $V_{\text{anmo}}$ because the aperture is insufficient to record most energy scattered by the anomalies in those velocities. Consequently, instead of the anomaly in $V_{P0}$, FWI generates a false
anomaly in $\epsilon$. As in the previous test, the algorithm correctly keeps the coefficient $\delta$ at the background value.

When the source-receiver configuration for the same model is rotated by 90° (Figure 5), the inversion algorithm predominantly updates $V_{P0}$ (Figure 6(a)). Because the receivers do not record most of the energy scattered by the anomalies in $V_{nmo}$ and $V_{hor}$, the algorithm does not update these parameters. Hence, although the anomaly in $V_{P0}$ is recovered, the parameters $\epsilon$ and $\delta$, which depend on $V_{hor}$ and $V_{nmo}$, are distorted (Figures 6(c) and 6(d), respectively).

Whereas an anomaly in the velocity $V_{S0}$ does not influence the P-wavefield, it scatters the S-wavefield equally in all directions (Figure 2(a)). As a result, the FWI algorithm operating with SV data can recover the anomaly in $V_{S0}$ irrespective of the source-receiver configuration.

### 4 CONCLUSIONS

We presented a general approach for evaluating the sensitivity of elastic anisotropic FWI to model parameterization. A point diffractor corresponding to a perturbation in the stiffness coefficients was inserted into a homogeneous isotropic background medium. By employing the Born and WKBJ approximations along with an asymptotic representation of the Green’s functions, we obtained an analytic expression for the Fréchet kernel of FWI.

For VTI media, the amplitude of the kernel (“radiation pattern”) was expressed in terms of the P- and S-wave vertical velocities ($V_{P0}$ and $V_{S0}$), and the P-wave NMO and horizontal velocities ($V_{nmo}$ and $V_{hor}$). Assuming an isotropic homogeneous background yields simple expressions for the radiation patterns of all four velocities. This methodology can be easily
adapted to obtain radiation patterns for other model parameterizations.

The scattered P-wavefield in our approximation is insensitive to the velocity $V_{S0}$, whereas the radiation patterns of $V_{P0}$ and $V_{hor}$ are decoupled. An anomaly in $V_{P0}$ scatters most of the P-wave energy in the vicinity ($\pm 45^\circ$) of the symmetry axis, so stable estimation of $V_{P0}$ requires good wavefield sampling in that range of angles. In contrast, a perturbation in $V_{hor}$ produces the largest P-wave scattering near the isotropy plane, which is favorable for crosshole geometry.

In our parameterization, the SV-wavefield is not scattered by the velocity $V_{P0}$. A perturbation in the velocity $V_{S0}$ scatters the SV-wave energy uniformly in all directions, so elastic FWI can potentially recover $V_{S0}$ for any aperture of the experiment.

An anomaly in the velocity $V_{nmo}$ predominantly scatters both the P- and SV-wavefields near an angle of $45^\circ$, and the scattering amplitude is smaller than that of the other parameters. Hence, transmitted P- and SV-waves do not provide tight constraints on $V_{nmo}$. A better estimate of $V_{nmo}$ may be obtained from reflected waves.

Our results indicate that if the background model is known and the receivers record energy propagating in the vicinity of the symmetry axis, FWI of multicomponent transmission data should be able to resolve the parameters $V_{P0}$ and $V_{S0}$. In the case of crosshole geometry, it should be possible to estimate $V_{hor}$ and $V_{S0}$.

This study was limited to analysis of the amplitude of the Fréchet kernel. By taking the phase of the kernel into account, our approach could be extended to FWI of reflection data.

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Figure 3. VTI model with a Gaussian anomaly (standard deviation $\sigma = 300$ m) in the anisotropy parameter $\epsilon$ (after Paper I). The background and maximum values of $\epsilon$ are 0.1 and 0.142, respectively. The other Thomsen parameters are spatially invariant: $V_{P0} = 3000$ m/s, $V_{S0} = 1500$ m/s, and $\delta = -0.05$. The dots on the left mark the source locations and the vertical line on the right represents an array of receivers placed at each grid point (6.6 m apart).
Figure 4. Difference between the inverted parameters (a) $V_{P0}$, (b) $V_{S0}$, (c) $\epsilon$, and (d) $\delta$ and their initial values for the model from Figure 3 (after Paper I). The velocities have units of km/s.

Figure 5. VTI model with a Gaussian anomaly in $V_{P0}$ (after Paper I). The source and receiver arrays are horizontal.
Sensitivity analysis for elastic full-waveform inversion in VTI media

Figure 6. Difference between the inverted parameters (a) $V_{P0}$, (b) $V_{S0}$, (c) $\epsilon$, and (d) $\delta$ and their initial values for the model from Figure 5 (after Paper I).

APPENDIX A: SENSITIVITY PATTERNS FOR ELASTIC FWI IN VTI MEDIA

The radiation pattern is obtained by expressing the scattered wavefield in the Born approximation using the asymptotic Green’s function. The wave equation for heterogeneous anisotropic media has the form:

\[ \rho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j} \left( c_{ijkl} \frac{\partial u_k}{\partial x_l} \right) = f_i, \quad \text{(A1)} \]

where \( \rho \) is the density, \( c_{ijkl} \) are components of the stiffness tensor, and \( f \) is the body force per unit volume. All indices change from 1 to 3 and summation over repeated indices is implied. The displacement wavefield is subject to the initial conditions,

\[ u(x, 0) = 0, \quad \frac{\partial u(x, 0)}{\partial t} = 0, \quad \text{(A2)} \]

and the radiation boundary condition,

\[ u(x, t) \bigg|_{x \to \infty} \to 0. \quad \text{(A3)} \]

Suppose the wavefield produced by a source at \( x_s \) is scattered at \( x' \) and recorded by a receiver at \( x_r \). The background is taken to be homogeneous and isotropic, but the stiffness coefficients perturbed at \( x' \) correspond to those for the in-plane polarized waves in 2D VTI media (i.e., \( C_{11}, C_{13}, C_{33}, \) and \( C_{55} \)). A perturbation in the stiffness coefficient \( c_{ijkl} \) at the scatterer \( x' \) (Figure A1) results in a perturbation \( \delta u_i \) in the wavefield. We replace the wavefield \( u_i \) in equation A1 by \( u_i = u_i^b + \delta u_i \) and the stiffnesses \( c_{ijkl} \) by \( c_{ijkl} = c_{ijkl}^b + \delta c_{ijkl} \), where the superscript \( b \) refers to the background. Retaining only the terms linear in the perturbations in equation A1 leads to the Born approximation:

\[ \rho \frac{\partial^2 \delta u_i}{\partial t^2} - \frac{\partial}{\partial x_j} \left( c_{ijkl}^b \frac{\partial \delta u_k}{\partial x_l} \right) = \frac{\partial}{\partial x_j} \left( \delta c_{ijkl} \frac{\partial u_k^b}{\partial x_l} \right). \quad \text{(A4)} \]

Equation A1 can be solved in the frequency domain in terms of the Green’s function using the representation theorem:

\[ u_n(x_r, \omega) = \int_{V(x')} h_i(x', \omega) G_{ni}(x_r, x', \omega) \, dV(x'), \quad \text{(A5)} \]

where \( h_i(x', \omega) \) is the force density at \( x' \), \( G_{ni}(x_r, x', \omega) \) is the Green’s function for the source at \( x' \) and receiver at \( x_r \), and \( V(x') \) represents the volume that includes all sources. To solve equation A4 for the scattered field, we replace \( h_i(x', \omega) \) in equation A5.
by the right-hand side (source term) of equation A4:
\[
\delta u_n(x_r, \omega) = \int_{V(x')} \frac{\partial}{\partial x_j} \left( \delta c_{ijkl} \frac{\partial u^b_k}{\partial x_l} \right) G_{ni} \, dV(x')
\]
\[
= \int_{V(x')} \left[ \frac{\partial}{\partial x_j} \left( \delta c_{ijkl} \frac{\partial u^b_k}{\partial x_l} \right) G_{ni} \right] \, dV(x').
\] (A6)

Applying the divergence theorem to the first term of equation A6 yields:
\[
\delta u_n(x_r, \omega) = \int_{S(x')} \delta c_{ijkl} \frac{\partial u^b_k}{\partial x_l} G_{ni} \, n_j \, dS(x') - \int_{V(x')} \delta c_{ijkl} \frac{\partial u^b_k}{\partial x_l} \frac{\partial G_{ni}}{\partial x_j} \, dV(x'),
\] (A7)

where \( S(x') \) is the surface of the volume \( V(x') \), and \( n \) is the normal to \( S(x') \) pointing outward. Expanding the volume \( V(x') \) to infinity and using the radiation boundary condition (equation A3) reduces equation A7 to
\[
\delta u_n(x_r, \omega) = - \int_{V(x')} \delta c_{ijkl}(x') \frac{\partial u^b_k}{\partial x_l}(x', \omega) \frac{\partial G_{ni}(x_r, x', \omega)}{\partial x_j} \, dV(x').
\] (A8)

Here, the wavefield \( u^b \) is computed in the background medium and is generated by the force at the source location \( x_s \). Hence, \( u^b \) can be expressed in terms of the force applied at \( x_s \) and the Green’s function:
\[
u_i^b(x', \omega) = f_m(x_s, \omega) G_{km}(x'_s, x_s, \omega).
\] (A9)

Substituting equation A9 into equation A7 and using reciprocity,
\[
G_{km}(x'_s, x_s, \omega) = G_{mk}(x_s, x'_s, \omega),
\] (A10)

we find:
\[
\delta u_n(x_r, \omega) = - \int_{V(x')} f_m(x_s, \omega) \delta c_{ijkl}(x') \frac{\partial G_{mk}(x_s, x'_s, \omega)}{\partial x_l} \frac{\partial G_{ni}(x_r, x'_s, \omega)}{\partial x_j} \, dV(x'),
\] (A11)

Next, we replace the Green’s functions in equation A11 by their asymptotic representation (Vavryčuk, 2007),
\[
G_{mk} = g^s_m \tilde{g}^s_k \tilde{G}^s,
\] (A12)
\[
G_{ni} = g^r_n \tilde{g}^r_i \tilde{G}^r,
\] (A13)

where the superscripts \( s \) and \( r \) denote the source and receiver wavefields, respectively, \( g \) is the unit polarization vector, and
\[
\tilde{G}^s = \frac{1}{4\pi \rho V_{gr} R^s} \sqrt{|K|} \exp \left[ i \frac{\pi}{2} \sigma_0 + i \omega \frac{\mathbf{p}^s}{v^s} \cdot (x'^s - x') \right],
\] (A14)
\[
\tilde{G}^r = \frac{1}{4\pi \rho V_{gr} R^r} \sqrt{|K|} \exp \left[ i \frac{\pi}{2} \sigma_0 + i \omega \frac{\mathbf{p}^r}{v^r} \cdot (x'^r - x') \right].
\] (A15)

Here, \( V_{gr} \) and \( R \) are the group velocity and distance along the ray, \( K \) is the Gaussian curvature of the slowness surface, \( \sigma_0 \) is a function of \( K \) (Vavryčuk, 2007), \( \mathbf{p} \) is the unit slowness vector, and \( v \) is the phase velocity.

Under the WKBJ approximation (e.g., Aki and Richards, 2002), the spatial derivatives are evaluated only for the rapidly varying terms of the Green’s functions (i.e., the exponent in equations A14 and A15). Substituting equations A12–A15 into equation A11 yields the following expression for the perturbed wavefield:
\[
\delta u_n(x_r, \omega) = \int_{V(x')} f_m(x_s, \omega) \mathcal{A}(\omega) \rho^s_m p^s_j \tilde{g}^s_k \tilde{g}^s_i \delta c_{ijkl} \, dV(x'),
\] (A16)

where
\[
\mathcal{A}(\omega) = g^s_m g^s_i \tilde{G}^s \tilde{G}^r \omega^2 \frac{v^s}{v^s + v^r}.
\] (A17)
The radiation pattern $\Omega$ for a model parameterized in terms of the stiffnesses is obtained as the amplitude of the kernel in equation A16 (i.e., the coefficient multiplied with $\delta c_{ijkl}$):

$$\Omega = p_i^i p_j^j g_k^k g_l^l.$$  \quad (A18)

Here, we parameterize the model in terms of the velocities $V_{P0}$, $V_{S0}$, $V_{nmo}$, and $V_{hor}$. The stiffness coefficients represent the following functions of the velocities (Tsvankin, 2012):

$$C_{11} = \rho V_{hor}^2,$$  \quad (A19)

$$C_{33} = \rho V_{P0}^2,$$  \quad (A20)

$$C_{13} = \rho \sqrt{(V_{P0}^2 - V_{S0}^2)(V_{nmo}^2 - V_{S0}^2)} - \rho V_{S0}^2,$$  \quad (A21)

$$C_{55} = \rho V_{S0}^2.$$  \quad (A22)

The perturbation in the velocities can be expressed in terms of the perturbations in the stiffness coefficients using the chain rule. Since the background medium is isotropic, the background velocities satisfy $V_{nmo} = V_{hor} = V_{P0}$, and the unit polarization vectors are parallel (P-waves) or perpendicular (SV-waves) to the corresponding group velocity vector. For the P-wavefield, the 2D radiation patterns in the vertical plane are:

$$\Omega^P(V_{P0}) = 2 \rho V_{P0} \left[2(p_3^r)^2 (p_3^s)^2 + (p_3^s)^2 (p_3^r)^2 + (p_1^r)^2 (p_1^s)^2\right],$$  \quad (A23)

$$\Omega^P(V_{S0}) = 8 \rho V_{S0} \left[2p_1^s p_3^r p_3^s - (p_3^s)^2 (p_1^r)^2 - (p_1^s)^2 (p_3^r)^2\right],$$  \quad (A24)

$$\Omega^P(V_{nmo}) = 2 \rho V_{P0} (p_3^s)^2 (p_1^r)^2 + (p_3^r)^2 (p_1^s)^2,$$  \quad (A25)

$$\Omega^P(V_{hor}) = 4 \rho V_{P0} (p_3^s)^2 (p_1^r)^2.$$  \quad (A26)

The patterns for the S-wavefield have the form:

$$\Omega^S(V_{P0}) = 0,$$  \quad (A27)

$$\Omega^S(V_{S0}) = 4 \rho V_{S0} \left[(p_1^r p_3^s + p_3^s p_1^r)^2 - (p_3^s p_1^r - p_3^r p_1^s)^2\right],$$  \quad (A28)

$$\Omega^S(V_{nmo}) = -4 \rho V_{P0} p_1^s p_3^r p_3^s p_1^r,$$  \quad (A29)

$$\Omega^S(V_{hor}) = 4 \rho V_{P0} p_1^s p_3^r p_3^s p_1^r.$$  \quad (A30)

In the case of transmitted waves, because the incident and scattered angles are the same, the components of the unit slowness and polarization vectors for P-waves satisfy:

$$p_1^s = g_1^s = -\sin \theta,$$  \quad (A31)

$$p_3^s = g_3^s = \cos \theta,$$  \quad (A32)

$$p_1^r = g_1^r = \sin \theta,$$  \quad (A33)

$$p_3^r = g_3^r = -\cos \theta,$$  \quad (A34)

where $\theta$ is the phase angle with the (vertical) symmetry axis. For S-waves, the corresponding expressions are:

$$p_1^s = -g_1^s = -\sin \theta,$$  \quad (A35)

$$p_3^s = g_3^s = \cos \theta,$$  \quad (A36)

$$p_1^r = g_1^r = \sin \theta,$$  \quad (A37)

$$p_3^r = -g_3^r = -\cos \theta.$$  \quad (A38)
Gradient calculation for waveform inversion of microseismic data in VTI media*

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ABSTRACT

In microseismic data processing, source locations and origin times are usually obtained using kinematic techniques, whereas moment-tensor estimates are typically based on linear inversion of P- and S-wave amplitudes. Waveform inversion (WI) can potentially provide more accurate source parameters along with an improved velocity model by incorporating information contained in the entire trace including the coda. Here, we address one of the key issues in implementing WI for microseismic surveys — efficient calculation of the gradient of the objective function with respect to the model parameters. Application of the adjoint-state method helps obtain closed-form expressions for the gradient with respect to the source location, origin time, and moment tensor. Computation of the forward and adjoint wavefields is performed with a finite-difference algorithm that handles elastic VTI (transversely isotropic with a vertical symmetry axis) models. Numerical examples illustrate the properties of the gradient for multicomponent data recorded by a vertical receiver array placed in homogeneous and horizontally layered VTI media.

1 INTRODUCTION

Microseismicity has developed in recent years as an efficient technique for monitoring of hydraulic fractures and the surrounding reservoir volume (Maxwell, 2010; Kendall et al., 2011). Microseismic experiments typically involve recording the seismic response to hydraulic fracturing of tight reservoirs, most often shales. The location of the induced microfractures, as well as the origin time of the corresponding seismic events can be inferred from the data acquired in an observation borehole or at the surface. Usually only a single borehole is available, but it is highly beneficial to record microseismic data in several boreholes.

Accurate location of microseismic events requires knowledge of the background velocity model. This initial model is usually obtained from sonic logs and traveltimes of the direct P- and S-waves excited by perforation shots and recorded by geophones deployed in a monitor borehole. Afterwards, the model can be updated using the traveltimes or waveforms of microseismic events. Velocity analysis without adequately accounting for seismic anisotropy may lead to significant errors in event location Van Dok et al. (2011). In particular, shales are known to be transversely isotropic and may become orthorhombic or possess an even lower symmetry due to fracturing (Tsvankin and Grechka, 2011; Tsvankin, 2012).

Depending on the receiver geometry and spatial distribution of sources, microseismic data can help estimate the pertinent anisotropy parameters (Grechka and Duchkov, 2011) simultaneously with event location. For example, Grechka et al. (2011) demonstrate that anisotropic velocity models constructed from traveltimes while locating microseismic events provide more accurate source locations than those obtained without accounting for anisotropy. Also, Li et al. (2013) show that building VTI velocity models simultaneously with event location significantly reduces the traveltime residuals compared to standard location techniques that employ isotropic velocity models obtained from sonic logs and perforation shots. Grechka and Yaskevich (2013, 2014) demonstrate that for microseismic surveys with sufficient angle coverage it is possible to construct layered triclinic models and substantially improve the accuracy of event location.

Still, kinematic inversion essentially replaces a seismic trace with the δ-functions corresponding to the traveltimes of the direct arrivals, which restricts the resolution of event location according to the Rayleigh criterion (i.e., two sources are indistinguishable if the distance between them is smaller than one-half of the predominant wavelength). Considering the rapidly increasing usage of back-projection techniques and the rich information content of microseismic data, improved results can be expected from waveform inversion (WI). Indeed, WI operates with the entire trace including scattered waves, so location results are not subject to the Rayleigh criterion. Hence, for wavelengths typical in downhole microseismic surveys, one can expect substantially reduced event-location errors. Another potential benefit of WI, not explored in this paper, is an improved accuracy of the velocity model.

The source mechanism of microseismic events can reveal important information about the rupture process. Point earthquake sources are described by the second-rank symmetric...
seismic moment tensor $\mathbf{M}$. As discussed by Vavryčuk (2007), all six independent elements of $\mathbf{M}$ for microseismic events can be retrieved from the amplitudes of P-waves recorded in three boreholes or from P-, SV-, and SH-wave amplitudes measured in two boreholes. For sources and receivers located in the $[x_1, x_3]$-plane of a VTI model, the in-plane polarized waves depend on the elements $M_{11}$, $M_{13}$, and $M_{33}$, which potentially can be found by inverting P- and SV-waves recorded in a single borehole. In practice, $\mathbf{M}$ is obtained independently of other microseismic parameters by linear inversion of P- and S-wave amplitudes. Since WI incorporates waveform information, it may provide improved estimates of $\mathbf{M}$ and, at the same time, perform event location and constrain the origin time.

Waveform inversion is used to build high-resolution velocity models from seismic data using phase and, sometimes, amplitude information (Tarantola, 1984; Gauthier et al., 1986; Mora, 1987; Pratt, 1999; Virieux and Operto, 2009). Recently WI has been extended to elastic and anisotropic media (Lee et al., 2010; Kamath and Tsvankin, 2013), which makes it suitable for multicomponent reflection and microseismic data.

The objective function in WI quantifies the difference between observed and predicted data in the time or frequency domain. Efficient inversion requires application of iterative optimization schemes such as gradient-based methods, which involve calculation of the gradient of the objective function with respect to the model parameters. In principle, the gradient can be found from the Fréchet derivatives obtained by differentiating the wavefield with respect to each model parameter. However, if the number of unknowns is large, computation of the Fréchet derivatives becomes prohibitively expensive.

A computationally efficient alternative for computing the gradient without the Fréchet derivatives is the adjoint-state method (Plessix, 2006; Fichtner, 2006, 2009; Köhn, 2011). This method makes it possible to calculate the gradient using just two forward-modeling simulations: first to generate the forward wavefield (predicted data), and second to compute the adjoint wavefield.

There has been significant progress in applying adjoint methods to tomographic velocity analysis and source-parameter inversion in global seismology. Tromp et al. (2005) and Liu and Tromp (2006) employ an adjoint formulation based on the Lagrangian-multiplier method to derive the gradient for estimating the P- and S-wave velocities in isotropic media. They also analyze the sensitivity (Fréchet) kernels for 2D and 3D velocity models. Kim et al. (2011) obtain gradient expressions for the source parameters using the adjoint-state method. They also implement conjugate-gradient inversion to estimate the source location and moment tensor for two earthquakes in Southern California using a known isotropic velocity model. Morency and Mellors (2012) follow the same approach to evaluate the source parameters of a geothermal event.

Here, we discuss gradient calculation for waveform inversion of microseismic data in heterogeneous VTI media. The current algorithm is designed to estimate the source location, origin time, and moment tensor from 2D microseismic experiments.

We start by describing 2D elastic finite-difference modeling for dislocation-type sources embedded in a VTI medium. Then we discuss application of the adjoint-state method to compute the gradient for waveform inversion. Efficient gradient calculation is implemented by adapting to our problem the general expressions for the gradient obtained by Kim et al. (2011). Although the interval Thomsen parameters of layered VTI media can be estimated by WI as well, the current formulation is limited to the gradient for the source parameters. The performance of the algorithm is illustrated by synthetic tests for homogeneous and layered VTI media. The examples display multicomponent wavefields obtained by forward and adjoint finite-difference modeling and help understand the properties of the objective-function derivatives for the source location, origin time, and moment tensor.

2 GRADIENT FROM THE ADJOINT-STATE METHOD

2.1 Forward problem

The wave equation for a heterogeneous anisotropic medium can be written as:

$$\rho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j} \left( c_{ijkl} \frac{\partial u_k}{\partial x_l} \right) = f_i,$$

where $u_i(x,t)$ is the displacement field, $t$ is time, $c_{ijkl}$ is the stiffness tensor ($i, j, k, l = 1, 2, 3$), $\rho(x)$ is density, and $f_i(x,t)$ is the body force.

Dislocation-type sources are described by the seismic moment tensor,

$$\mathbf{M} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix},$$

which can be incorporated into the source term of equation 1 by using the notion of equivalent force (Aki and Richards, 1980; Dahlen and Tromp, 1998):

$$\rho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j} \left( c_{ijkl} \frac{\partial u_k}{\partial x_l} \right) = -M_{ij} \frac{\partial |\delta(x-x^*)|}{\partial x_j} S(t),$$

where $x^*$ is the source location, $S(t)$ is the source time function, and $\delta(x-x^*)$ is the spatial $\delta$-function. We use a finite-difference (FD) algorithm to obtain exact solutions of equation 3.

2.2 Modeling examples

Forward modeling for gradient calculation in VTI media is carried out with the elastic finite-difference program sfewe in MADAGASCAR. The model is described by the interval Thomsen parameters — the P- and S-wave vertical velocities $v_{p0}$ and $v_{s0}$ and the anisotropy coefficients $\varepsilon$ and $\delta$ defined as (Thomsen, 1986; Tsvankin, 2012):
Gradient calculation for waveform inversion of microseismic data in VTI media

\[ M_{11} = -\frac{\bar{u}}{2} \sin 2\theta (c_{13} - c_{11}) , \]  
\[ M_{13} = \bar{u} \cos 2\theta c_{55} , \]  
\[ M_{33} = -\frac{\bar{u}}{2} \sin 2\theta (c_{33} - c_{13}) . \]

The wavefields generated by dislocation-type sources in a homogeneous VTI medium are shown in Figure 2. The amplitude distribution and intensity of the P- and S-waves change substantially with the fault orientation. Because the magnitude of \( \sigma \) is relatively large (close to unity), the SV-wavefront exhibits triplications at oblique propagation directions (Tsvankin, 2012).

2.3 Inverse problem

As mentioned above, the P- and SV-waves recorded in the \([x_1, x_3]\)-plane depend on the components \( M_{11}, M_{13}, \) and \( M_{33} \) of the moment tensor. Here, our goal is to invert just for the three moment-tensor elements, the source coordinates \( s_1 \) and \( s_3 \), and the origin time \( t_0 \) assuming that the velocity model is known.

The observed data \( d_{\text{obs}} \) and predicted data \( d_{\text{pre}} \) are produced by two forward simulations, where \( d_{\text{pre}} \) is obtained after perturbing the source parameters used to generate \( d_{\text{obs}} \). In both cases, the elastic displacement field \( u(x, x^r, t) \) excited by a source located at \( x^s \) is recorded by \( N \) receivers located at \( x^r \) \((n = 1, 2, \ldots, N)\). The data residuals are measured by the least-squares objective function, which has to be minimized by the inversion algorithm:

\[ \mathcal{F}(m) = \frac{1}{2} \| d_{\text{pre}}(m) - d_{\text{obs}} \|^2 . \]

2.4 Application of the adjoint-state method

The objective function depends on the model parameters through the state variables, which represent the solution of the forward-modeling equations. In our case, the state variable is the elastic displacement field \( u(x, t) \) generated by a microseismic source and governed by equation 3.

Iterative optimization techniques involve calculation of the model update at each iteration. The update direction is determined by the gradient (the derivatives of the objective function with respect to the model parameters), \( \partial \mathcal{F}(m)/\partial m \). The adjoint-state method is designed to find this gradient for the entire set of model parameters in just two modeling simulations. However, because this method does not calculate the Fréchet derivatives, it is impossible to evaluate the sensitivity of the solution to perturbation errors.

The adjoint-state method involves the following main steps (Perrone and Sava, 2012):

(i) Computation of the state variable (forward wavefield).
Figure 2. Vertical displacement generated by dip-slip sources with different orientation (defined by the angle $\theta$) in a homogeneous VTI medium. The medium parameters are $\rho = 2 \text{ g/cm}^3$, $V_P = 4047 \text{ m/s}$, $V_S = 2638 \text{ m/s}$, $\varepsilon = 0.4$, and $\delta = 0$ ($\sigma = 0.94$). The moment tensor is computed from equations 9 – 11 with $\Sigma = 1 \text{ m}^3$ and (a) $\theta = 0^\circ$, (b) $\theta = 30^\circ$, (c) $\theta = 60^\circ$, and (d) $\theta = 90^\circ$.

(ii) Computation of the adjoint source.
(iii) Computation of the adjoint-state variable (adjoint wavefield).
(iv) Computation of the gradient.

In addition to equation 3, the method requires solving the adjoint wave equation:

$$\rho \frac{\partial^2 u^\dagger}{\partial t^2} - c_{ijkl} \frac{\partial^2 u^\dagger}{\partial x_j \partial x_l} = \sum_{n=1}^{N} (d_{\text{obs}} - d_{\text{pre}})(T - t) \delta(x - x^n),$$

where $u^\dagger$ is called the “adjoint wavefield.” The so-called adjoint source on the right-hand side of equation 13 is obtained by differentiating the objective function $\mathcal{F}(m)$ with respect to the forward wavefield $u$, and consists of the time-reversed data residuals. Therefore, the adjoint simulation can be carried out with the same modeling code by “injecting” the adjoint source at the receiver locations and then running the forward simulation.

The derivatives of the objective function with respect to the moment-tensor elements, source coordinates, and origin time can be found as (Kim et al., 2011):

$$\frac{\partial \mathcal{F}}{\partial M_{ij}} = \int_0^T \varepsilon_{ij}^s(x^s, t) S(T - t) \, dt,$$
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where $x$ is the trial source location, $T$ is the recording time, $\epsilon = \frac{1}{2} \left[ \nabla u^i + (\nabla u^i)^T \right]$ is defined as the adjoint strain tensor, and $M : \epsilon^i$ is the double inner product of the tensors $M$ and $\epsilon^i$. Equations 14 – 16 applied to the 2D problem yield the gradient vector $g$ for the six source parameters:

$$
g = \left\{ \frac{\partial F}{\partial M_{11}}, \frac{\partial F}{\partial M_{13}}, \frac{\partial F}{\partial M_{33}}, \frac{\partial F}{\partial x_1^s}, \frac{\partial F}{\partial x_3^s}, \frac{\partial F}{\partial t_0} \right\}. $$

It is interesting that in contrast to the gradient for velocity-related parameters (Liu and Tromp, 2006), which depends on the interaction between the forward and adjoint wavefields, equations 14 – 16 include only the adjoint wavefield. This means that there is no need to store or recalculate the forward wavefield during gradient calculation.

When the adjoint-state method is applied to velocity analysis, $u^i$ is supposed to “illuminate” the erroneous parts of the velocity model. Likewise, for the source-inversion problem, $u^i$ reveals the difference in $M$, $x^s$, and $t_0$ that produces the mismatch between the observed and predicted data.

In principle, the number of sources in the adjoint problem is unrestricted. If the forward wavefield is excited by multiple sources, the adjoint wavefield may focus at the actual as well as perturbed (trial) source locations. The areas of focusing make the main contribution to the gradient. However, for inversion purposes we only need the derivatives in equations 14 – 16 at the trial source position.

3 SYNTHETIC TESTS

Next, we present the results of synthetic tests for a homogeneous VTI medium and a stack of three constant-density VTI layers, with the wavefield modeled using the FD code mentioned above.

In the first experiment (Figure 3), the observed and initial predicted data (Figures 4 and 5) are generated for a single microseismic event recorded in a vertical “borehole” by receivers located at each grid point. Whereas the predicted field is generated by a horizontal dip-slip source ($\theta = 0^o$), the predicted field is computed with a vertical source ($\theta = 90^o$). Next, the adjoint source “injected” at the receiver locations is used to compute the adjoint wavefield (Figure 6). This wavefield focuses at the time $t = 0.44$ s near the actual source, where the model perturbation is located. The focusing time corresponds to the raypath between the source and the receivers located at the end of the array.

After applying equations 14 – 16, we obtain the following derivatives for the five source parameters: $\frac{\partial F}{\partial M_{11}} = 0.00038$, $\frac{\partial F}{\partial M_{13}} = 1.28$, $\frac{\partial F}{\partial M_{33}} = -0.0022$, $\frac{\partial F}{\partial x_1^s} = 0.004$, $\frac{\partial F}{\partial x_3^s} = 0.21$, and $\frac{\partial F}{\partial t_0} = -0.82$. The sign of the derivatives (i.e., of the components of the gradient) defines the update direction in the model space, whereas their absolute values determine the magnitude of the update of the corresponding model parameter. The units of the derivatives are not included here because we can compare only the derivatives for the same type of parameters (e.g., for the coordinates $x_1^s$ and $x_3^s$). The main contribution to the gradient comes from the focusing of the adjoint wavefield near the source location. It may look surprising that the derivatives for $x_1^s$, $x_3^s$, and $t_0$ are nonzero because the values of $x^s$ and $t_0$ for the actual and trial sources coincide. However, the derivatives for $x_1^s$ and $t_0$ depend on the components of $M$, which deviate from the actual values (equations 15 and 16). Hence, to avoid cycle-skipping problems during the joint inversion for $x^s$, $t_0$, and $M$, a good initial guess is needed for all three types of parameters.

The second test is performed for the three-layer VTI medium in Figure 7. The actual and trial models have the same source parameters as in the previous experiment. The data are generated by a source placed in the middle layer and recorded by receivers in a vertical “borehole.” The derivatives calculated using the adjoint-state method are $\frac{\partial F}{\partial M_{11}} = 0.004$, $\frac{\partial F}{\partial M_{13}} = 2.44$, $\frac{\partial F}{\partial M_{33}} = -0.0044$, $\frac{\partial F}{\partial x_1^s} = 0.82$, $\frac{\partial F}{\partial x_3^s} = -0.0071$, and $\frac{\partial F}{\partial t_0} = -0.21$. These derivatives are similar to the ones obtained for the previous test because the tensor $M$ is perturbed in the same way for both cases. The layer boundaries, however, create a number of reflected and mode-converted waves that should make waveform inversion for source parameters better constrained.

For the third experiment, we keep the actual source at the same location as in the previous tests and use the homogeneous model from Figure 3. This time, the initial trial source is moved horizontally, which generates pronounced gradient
Figure 4. Snapshots of the vertical displacement for the model in Figure 3 at $t = 0.14$ s. $\Sigma u = 1$ m$^3$. (a) The observed wavefield produced by a dip-slip source with $\theta = 0^\circ$. (b) The predicted wavefield from a source with $\theta = 90^\circ$.

Figure 5. Vertical displacement of the (a) observed and (b) predicted data for the model in Figure 3 generated with the source parameters from Figure 4.

contributions at both source locations (Figure 8). However, as mentioned above, we are interested only in the derivatives computed for the trial source location.

The obtained derivatives are $\partial F/\partial M_{11} = -0.00017$, $\partial F/\partial M_{13} = -0.25$, $\partial F/\partial M_{33} = 0.00033$, $\partial F/\partial x_s^1 = -4.28$, $\partial F/\partial x_s^3 = 0.0061$, and $\partial F/\partial t_0 = 1.51$. Although the trial source has the correct moment tensor, the derivatives for $M_{11}$, $M_{13}$, and $M_{13}$ do not vanish because the wavefield substantially changes with source location. The derivative for $t_0$ is also nonzero because moving the source creates traveltime shifts similar to those due to a change in the origin time. Still, the inversion should be able to resolve the trade-off between $x^s$ and $t_0$ because they influence the traveltimes in a different fashion.

The large value of $\partial F/\partial x_s^1$ compared to $\partial F/\partial x_s^3$ correctly indicates that the trial horizontal source coordinate $x_s^1$ should be updated more significantly than $x_s^3$. The negative sign of $\partial F/\partial x_s^1$ will lead to a smaller value of $x_s^1$ for the next iteration of inversion, so that the source will move toward its actual position ($x_s^1 = 0.3$ km.).

The last experiment demonstrates a potential application of the adjoint wavefield. The model includes two actual sources with the same tensor $\mathbf{M}$ and origin time $t_0$ that generate the wavefield $u(x,t)$ (Figure 9). However, we assume that the data $d_{\text{obs}}$ are produced by a single event and specify a single trial source. Then the adjoint wavefield $u^\dagger$ is expected to focus at two different locations, corresponding approximately to the actual and trial source position. Instead, the adjoint wavefield actually focuses at three locations (Figure 10), which helps identify the second source missing in the trial model.

In addition to finding “hidden sources,” the field $u^\dagger$ can be used to improve the initial trial source position because the adjoint wavefield focuses (for the correct velocity model) near the actual source location. This starting source position can be later refined during WI.
Gradient calculation for waveform inversion of microseismic data in VTI media

Figure 6. Snapshots of the vertical component of the adjoint wavefield for the model in Figure 3 at times (a) $t = 0.35$ s and (b) $t = 0.44$ s. The adjoint wavefield focuses at the actual and trial source locations (which are the same) on plot (b).

Figure 7. Three-layer VTI model used in the second experiment. The source-receiver geometry is the same as in Figure 3. The distance between receivers is 5 m. The parameters $\rho = 2$ kg/m$^3$, $\varepsilon = 0.4$, and $\delta = 0$ are the same in all three layers. The vertical velocities in the first layer are $V_{P0} = 4047$ m/s and $V_{S0} = 2638$ m/s; for the second layer, $V_{P0} = 4169$ m/s and $V_{S0} = 2532$ m/s; for the third layer, $V_{P0} = 4693$ m/s and $V_{S0} = 2682$ m/s.

Figure 8. Actual source (white dot), trial source (red dot) and a vertical line of receivers (spacing is 6 m) embedded in a homogeneous VTI medium. The medium parameters are the same as in Figure 3. The actual source is located at $x_1 = 0.3$ km, $x_3 = 0.75$ km and the trial source is at $x_1 = 0.32$ km, $x_3 = 0.75$ km. The moment tensor for both sources corresponds to a horizontal ($\theta = 0^\circ$) dip-slip fault with $\Sigma u = 1$ m$^3$.

4 CONCLUSIONS

Waveform inversion is a potentially powerful tool to solve simultaneously two of the most important problems in microseismic monitoring, event location and source-mechanism estimation. Here, we employed the adjoint-state method to find analytic expressions for the gradient of the WI objective function following the results of Kim et al. (2011). Then adjoint simulations were implemented to calculate the WI gradient for the source location, origin time, and moment tensor using a known VTI velocity model.

Synthetic tests were carried out for multicomponent wavefields from one and two sources recorded by a dense array of receivers in a vertical “borehole.” The first experiment was performed for a trial source with a perturbed moment tensor correctly positioned in a homogeneous VTI medium. The
adjoint wavefield focuses near the trial source location, which identifies the perturbed area. Although in this experiment the source position was correct, the derivatives for the source coordinates and origin time do not vanish because they depend on errors in the moment-tensor elements.

In the second test, a trial source with a distorted moment tensor was placed in a three-layer VTI model. The additional interfaces produce a more complicated wavefield with a number of reflected and converted waves. These additional events do not significantly influence gradient calculation but should improve the accuracy and stability of parameter estimation.

The algorithm was also tested for a source with the correct moment tensor and origin time but erroneous location in a homogeneous VTI medium. As in the previous examples, the derivatives of the objective function with respect to the correct parameters (in this case, the moment tensor and origin time) are nonzero. The dependence of the gradient on unperturbed model parameters implies that it is essential to have accurate initial guesses for all unknowns.

Finally, the wavefield was generated by two sources set off simultaneously, while the predicted wavefield was excited by a single source. Focusing of the adjoint wavefield helped us identify the approximate location of the “missing” source. In general, if the velocity model is not strongly distorted, the adjoint wavefield can provide an accurate starting trial source position for waveform inversion.

To take full advantage of the potential of WI in microseismic studies, it can be applied to anisotropic velocity model building as well. This extension will be discussed in a future publication.

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Figure 10. Snapshots of the vertical component of the adjoint wavefield for the model in Figure 9 at times (a) $t = 0.32$ s and (b) $t = 0.44$ s. The adjoint wavefield on plot (b) focuses at both actual source locations and at the trial source location.

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Waveform inversion for parameters of microseismic sources in VTI media

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ABSTRACT

Waveform inversion (WI), which has been used primarily for high-resolution velocity analysis, can also be employed to obtain the source parameters of microseismic events. Here, we implement WI to estimate the location, origin time, and seismic moment tensor of microseismic sources embedded in VTI (transversely isotropic with a vertical symmetry axis) media. The algorithm operates with multicomponent wavefields modeled using an elastic anisotropic finite-difference code. The gradient of the objective function for the three classes of parameters is calculated with the adjoint-state method. Although in the current algorithm the interval VTI parameters are assumed to be known, they can be included in WI at almost no additional cost. Synthetic tests for data recorded by vertical receiver arrays show that it is possible to tightly constrain all source parameters, if a sufficiently accurate initial model is available. In particular, the source location in layered VTI media can be estimated simultaneously with the moment tensor. The resolution of event location, however, somewhat decreases when the origin time is unknown or there is an error in one of the VTI parameters.

1 INTRODUCTION

Waveform inversion is a nonlinear optimization technique based on full-wavefield modeling, which is designed to include the entire seismic trace in building subsurface models. As first suggested by Lailly (1983) and Tarantola (1984), back-propagation in time of the data residuals followed by cross-correlation of the resulting wavefield with the forward-propagated wavefield helps iteratively produce high-resolution velocity models. An overview of the progress in WI methods can be found in Virieux and Operto (2009). Applications of WI in exploration seismology have been mostly limited to velocity tomography. Seismic waveforms, however, contain information that can be used to constrain other important quantities, such as parameters of earthquake sources.

Recently, waveform inversion has been extended to elastic and anisotropic media (Kamath and Tsvankin, 2013), which makes it appropriate for multicomponent reflection and microseismic data. Also, WI has been employed to estimate earthquake source parameters in global seismology (Tramp et al., 2005; Liu and Tromp, 2006; Kim et al., 2011) and geothermal studies (Morency and Mellors, 2012). These techniques incorporate the adjoint-state method (Lions, 1972; Plessix, 2006; Fichtner, 2006, 2009) as a practical way to calculate the gradient of the WI objective function. The main advantage of this method (Talagrand and Courtier, 1987) is that the gradient for all model parameters is computed using only two numerical-modeling simulations. Employing the approach described in Kim et al. (2011), Jarillo Michel and Tsvankin (2014; hereafter referred to as Paper I) implemented gradient calculation in WI for the source location and moment tensor of a microseismic event in a 2D VTI medium. Their results show that the adjoint wavefield can also be used to identify the presence of missed sources and, in general, improve the initial source position for WI.

Microseismic monitoring of hydraulic fractures has become an important technology in the development of unconventional shale reservoirs (Maxwell, 2010; Kendall et al., 2011). Whereas the main goal of microseismic surveys is to estimate the source locations $x_s$, it is also essential to obtain the event origin time $t_0$ and the source moment tensor $M$. Most existing techniques invert for these quantities separately; for instance, the tensor $M$ is often estimated under the assumption that $x_s$ is known. In contrast, WI has the potential of resolving all these parameters simultaneously and with high accuracy from multicomponent seismic data.

Here, we present a 2D methodology for estimating the parameters $x_s$, $t_0$, and $M$ using waveform inversion. First, calculation of the gradient of the objective function is implemented for heterogeneous VTI media using the approach developed in Paper I. Wavefields from dislocation-type microseismic sources are generated with a 2D elastic finite-difference algorithm. Then we introduce an iterative local gradient-descent algorithm for simultaneous updating of all source parameters. The inversion involves the so-called nondimensionalization approach (Kim et al., 2011) because the model parameters belong to different classes (i.e., have different units). Finally, the methodology is tested on multicomponent data generated for
four different microseismic experiments in homogeneous and horizontally layered VTI media.

2 INVERSE PROBLEM

The wave equation for a seismic source located at point \( \mathbf{x}^0 \) in heterogeneous anisotropic media can be written as (Aki and Richards, 2002):

\[
\rho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j} \left( c_{ijkl} \frac{\partial u_k}{\partial x_l} \right) = -M_{ij} \frac{\partial \delta(\mathbf{x} - \mathbf{x}^0)}{\partial x_j} S(t),
\]

where \( u_i(\mathbf{x}, t) \) is the displacement field, \( t \) is time, \( c_{ijkl} \) is the stiffness tensor \((i, j, k, l = 1, 2, 3)\), \( \rho(\mathbf{x}) \) is the density, \( M_{ij} \) is the seismic moment tensor, \( S(t) \) is the source time function, and \( \delta(\mathbf{x} - \mathbf{x}^0) \) is the spatial \( \delta \)-function; summation over repeated indices is implied. We use finite-difference (FD) algorithm \texttt{sfewo} in \texttt{Madagascar} to obtain the exact wavefield from equation 1. P- and SV-waves that propagate in the \([x_1, x_3] \)-plane depend on the components \( M_{11}, M_{13}, \) and \( M_{33} \) of the moment tensor, which can be represented as a function of the fault dip angle \( \theta \) (Figure 1; see Paper I):

\[
\begin{align*}
M_{11} &= -\frac{\Sigma \bar{a}}{2} (c_{13} - c_{11}) \sin 2\theta, \\
M_{13} &= \Sigma \bar{a} c_{55} \cos 2\theta, \\
M_{33} &= -\frac{\Sigma \bar{a}}{2} (c_{33} - c_{11}) \sin 2\theta,
\end{align*}
\]

where \( \Sigma \) is the fault area, \( \bar{a} \) is the magnitude of the slip (displacement discontinuity), and \( c_{11}, c_{13}, c_{33}, \) and \( c_{55} \) are the stiffness coefficients in the two-index Voigt notation.

Here, our goal is to invert for the source coordinates \( x_1^0, x_3^0, t_0 \), and the three relevant moment-tensor elements assuming that the velocity model is known. Hence, the vector of unknown model parameters is defined as:

\[
m = \{x_1^0, x_3^0, t_0, M_{11}, M_{13}, M_{33}\}.
\]

In synthetic tests, the observed displacement \( \mathbf{d}_{\text{obs}} \) and predicted displacement \( \mathbf{d}_{\text{pre}}(m) \) are produced by two forward simulations, with \( \mathbf{d}_{\text{pre}} \) computed after perturbing the source parameters used to generate \( \mathbf{d}_{\text{obs}} \). The elastic displacement field \( \mathbf{u}(\mathbf{x}^0, \mathbf{x}, t) \) in both simulations is excited by a single source at \( \mathbf{x}^0 \) and recorded by \( N \) receivers located at \( \mathbf{x}^n (n = 1, 2, \ldots, N) \). The data residuals are measured by the least-squares objective function \( F \), which is minimized by the inversion algorithm:

\[
F(m) = \frac{1}{2} \| \mathbf{d}_{\text{pre}}(m) - \mathbf{d}_{\text{obs}} \|^2.
\]

3 APPLICATION OF THE ADJOINT-STATE METHOD

The adjoint-state method is designed to efficiently calculate the derivatives of the objective function with respect to the model parameters, \( \frac{\partial F(m)}{\partial \mathbf{m}} \). Computation of the gradient of the objective function for our problem is discussed in Paper I. In addition to equation 1, the method requires solving the adjoint wave equation:

\[
\rho \frac{\partial^2 \mathbf{u}^\dagger}{\partial t^2} - \frac{\partial}{\partial x_j} \left( c_{ijkl} \frac{\partial \mathbf{u}^\dagger}{\partial x_l} \right) = \sum_{n=1}^{N} (\mathbf{d}_{\text{obs}} - \mathbf{d}_{\text{pre}}) (T-t) \delta(\mathbf{x} - \mathbf{x}^n),
\]

where \( \mathbf{u}^\dagger \) is called the “adjoint wavefield.” The “adjoint source” on the right-hand side of equation 7 consists of the time-reversed data residuals and is obtained by differentiating \( F(m) \) with respect to the forward wavefield \( \mathbf{u} \). The adjoint simulation to generate \( \mathbf{u}^\dagger \) can be carried out with the forward-modeling code that solves equation 1 by using the receivers as simultaneous adjoint sources.

The derivatives of the objective function with respect to the source coordinates, origin time, and moment-tensor elements can be found as (Kim et al., 2011; Paper I):

\[
\begin{align*}
g_{\mathbf{u}^\dagger} &= \frac{\partial F}{\partial \mathbf{u}^\dagger} = \int_0^T \frac{\partial \mathbf{M} : \mathbf{e}^\dagger(\mathbf{x}^n, t)}{\partial x_j} S(T-t) \, dt, \\
g_{t_0} &= \frac{\partial F}{\partial t_0} = \int_0^T \mathbf{M} : \mathbf{e}^\dagger(\mathbf{x}^n, t) \frac{\partial \delta(T-t)}{\partial t} \, dt, \\
g_{M_{ij}} &= \frac{\partial F}{\partial M_{ij}} = \int_0^T \epsilon_{ij}^\dagger(\mathbf{x}^n, t) S(T-t) \, dt,
\end{align*}
\]
where \( x^t \) is the trial source location, \( T \) is the recording time, 
\( \varepsilon^t = \frac{1}{2}[\nabla u^t + (\nabla u^t)^T] \) is the adjoint strain tensor, and \( M : \varepsilon^t \) is the double inner product of the tensors \( M \) and \( \varepsilon^t \). Equations 8 – 10 applied to the 2D problem at hand yield the gradient vector \( g \) for the six source parameters:

\[
g = \left\{ \frac{\partial f}{\partial x^t_1}, \frac{\partial f}{\partial x^t_2}, \frac{\partial f}{\partial M_{11}}, \frac{\partial f}{\partial M_{13}}, \frac{\partial f}{\partial M_{33}} \right\}.
\]

For inversion purposes, the derivatives in equations 8 – 10 are needed only at the trial source position. Note that the derivatives for \( x^t \) and \( t_0 \) (equations 8 and 9) include the double-inner product \( M : \varepsilon^t(x^t, t_0) \), which involves summation over all elements of \( M \). Due to this dependence, stable inversion for \( x^t \) and \( t_0 \) requires an accurate initial model for the moment tensor.

### 4 IMPLEMENTATION

The model parameters have different units, and local minimization of \( f(m) \) could be performed for each parameter class separately. However, here we carry out simultaneous inversion for \( x^t \), \( t_0 \), and \( M \) employing the nondimensionalization approach suggested by Kim et al. (2011), which also helps avoid the additional computational cost of multidirectional minimization. This approach eliminates the difference between the units of different parameters classes, which makes possible simultaneous parameter updating. At the first iteration, we define the following scaling coefficients \( \sigma \) for each parameter class:

\[
\sigma_{x^t} = \beta_{x^t} \frac{1}{\sqrt{g_{x^t}^2 + g_{x^t}^3}},
\]

\[
\sigma_{t_0} = \beta_{t_0} \frac{1}{|g_{t_0}|},
\]

\[
\sigma_M = \beta_M \frac{1}{\sqrt{g_{M_{11}}^2 + g_{M_{13}}^2 + g_{M_{33}}^2}},
\]

where the quantities \( \beta_{x^t} \), \( \beta_M \), and \( \beta_{t_0} \), are scaling factors that ensure that each parameter class gives a comparable contribution to the gradient. These factors may be different for each experiment and can be determined, for instance, by evaluating the change in the gradient produced by \( \beta_c \) (\( c \) indicates the parameter class) between the first and second iteration. The “nondimensionalized” model parameters are:

\[
\hat{m}_c = \frac{m_c}{\sigma_c}.
\]

The gradient becomes dimensionless after the following scaling:

\[
g_{\hat{c}} = g_c \sigma_c.
\]

As discussed in more detail below, the inverse problem

---

**Figure 2.** Dependence of the objective function \( f(m) \) on the trial source parameters (a) \( x_1^t \), (b) \( x_2^t \), and (c) \( t_0 \) for a homogeneous VTI model. The global minimum coincides with the actual parameter value. The components of the tensor \( M \) are fixed at the actual values. Here and below, we describe the medium using the Thomsen parameters — the \( P \)- and \( S \)-wave vertical velocities \( (V_{P_0} \) and \( V_{S_0} \)) and the anisotropy coefficients \( \varepsilon \) and \( \delta \) (Thomsen, 1986; Tsvankin, 2012). In this test, \( \rho = 2 \text{ g/cm}^3, V_{P_0} = 4047 \text{ m/s}, V_{S_0} = 2638 \text{ m/s, } \varepsilon = 0.4 \), and \( \delta = 0 \).
is nonlinear, and we solve it using an iterative local gradient-descent scheme. Suppose the model $\mathbf{m}^k$ is obtained after $k - 1$ iterations of the inversion algorithm. First, the forward simulation is performed to generate the predicted data $\mathbf{d}_{\text{pre}}(\mathbf{m}^k)$, which allows us to compute the objective function $F^k$. Then, we carry out the adjoint simulation to calculate the gradient $\mathbf{g}^k$ using equations 8 – 10. The next step is nondimensionalization of the model parameters (equation 15) and scaling of the gradient (equation 16). Note that the scaling coefficients are computed at the first iteration and kept constant during the inversion. Because the nondimensionalized model parameters have the same units (those of $F$) and the scaled gradient is dimensionless, the three classes of parameters can be updated simultaneously using a constant step length $\alpha$:

$$\mathbf{m}^{k+1} = \mathbf{m}^k + \alpha \mathbf{g}^k.$$  

Assuming that $\mathbf{m}^k$ is located within the basin that contains the global minimum of $F$, the step length should be sufficiently small to ensure that $\mathbf{m}^{k+1}$ stays within this basin. Using a constant step eliminates the computational cost required to perform a line search for $\alpha$. After the update, the parameters have to be “dimensionalized” again so that they can be used as inputs for the forward modeling in the next iteration:
Figure 6. Change of the moment-tensor elements (a) $M_{11}$, (b) $M_{33}$, and (c) $M_{13}$ with iterations for the model in Figure 3. The actual values are marked by dashed lines.

Figure 7. Source (white dot) and a vertical line of receivers (spacing is 6 m) embedded in a homogeneous VTI medium. The medium parameters are the same as in Figure 3. The location of the actual and trial sources is the same: $x_1 = 0.3$ km and $x_3 = 0.75$ km. For the actual source, $t_0 = 0.049$ s and $\theta = 0^\circ$, for the trial source, $t_0 = 0.042$ s and $\theta = 15^\circ$.

\[
 m_c^{k+1} = \hat{m}_c^{k+1} \sigma_c .
\]  

5 PROPERTIES OF THE INVERSE PROBLEM

Estimating the moment tensor $M$ of an earthquake from seismic amplitudes is a linear inverse problem. However, simultaneous inversion for $M$, $x_s$, and $t_0$ is nonlinear because the recorded data depend on $x_s$ and $t_0$ in a nonlinear fashion. The joint inversion for $M$, $x_s$, and $t_0$ involves complications typical for velocity estimation using WI. For example, cycle-skipping can occur if the trial model is too far from the actual one or if the step length $\alpha$ used in model updating is too large. In particular, the trial source should be within about one-half of the predominant wavelength from the actual source location.

Figure 2 shows the variation of the normalized objective function with the coordinates $x_s$ and time $t_0$. To ensure convergence to the actual values, the trial model should lie within the basin that contains the global minimum. If the inversion involves simultaneous estimation of $x_{s1}$, $x_{s3}$, and $t_0$, the basin containing the global minimum in the plots of Figure 2 becomes more narrow, which increases the risk of cycle skipping.

6 NUMERICAL RESULTS

Here, we present synthetic tests of the WI algorithm for a homogeneous VTI medium and a stack of horizontal VTI layers. In all experiments, the observed data are generated by a single microseismic event recorded by a vertical array of closely spaced receivers.
In the first experiment, we invert for the parameters \( x_1^s, x_3^s, M_{11}, M_{13}, \) and \( M_{33} \) with the origin time \( t_0 \) fixed at the actual value (Figure 3). Because the medium is homogeneous, the wavefield is composed just of the direct P- and SV-waves. The objective function becomes practically negligible after about 10 iterations (Figure 4). The moment tensor and source coordinates are estimated with high accuracy (Figures 5 and 6), with the errors in \( x_1^s \) and \( x_3^s \) on the order of centimeters. Note that the algorithm was able to resolve the moment tensor \( M \), although the data were acquired in a single vertical borehole. The pronounced variations in \( M \) during the initial iterations are due to the incorrect position of the source, which produces large changes in the amplitudes of the P- and SV-waves.

In the second test, the origin time \( t_0 \) is perturbed, while the location \( x^s \) is fixed at the correct value (Figure 7). The tensor \( M \) is perturbed in the same way as in the previous example.

Figure 8. Change of the normalized objective function \( \mathcal{F}(m) \) with iterations for the model in Figure 7.

Figure 9. Change of the origin time \( t_0 \) with iterations for the model in Figure 7.

Figure 10. Change of the moment-tensor elements (a) \( M_{11} \), (b) \( M_{33} \), and (c) \( M_{13} \) with iterations for the model in Figure 7.
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Figure 11. Actual (white dot) and trial (red dot) sources and a vertical receiver array in a five-layer VTI model. The receiver geometry is the same as in Figure 3. The parameters $\rho = 2$ kg/m$^3$, $\varepsilon = 0.4$, and $\delta = 0$ are the same in all five layers. The vertical velocities in the top layer are $V_{P0} = 4419$ m/s and $V_{S0} = 2645$ m/s; for the second layer, $V_{P0} = 4956$ m/s and $V_{S0} = 2424$ m/s; for the third layer, $V_{P0} = 4048$ m/s and $V_{S0} = 2638$ m/s; for the fourth layer, $V_{P0} = 4170$ m/s and $V_{S0} = 2320$ m/s; for the fifth layer, $V_{P0} = 4694$ m/s and $V_{S0} = 2682$ m/s. The actual source is located at $x_1 = 0.3$ km and $x_3 = 0.75$ km with $\theta = 0^\circ$. The trial source is located at $x_1 = 0.32$ km and $x_3 = 0.8$ km with $\theta = 15^\circ$. Both events occur at the same time ($t_0 = 0.035$ s).

Figure 12. Change of the normalized objective function $F(m)$ with iterations for the model in Figure 11. All source parameters are unknown.

The rate of the decrease of the objective function (Figure 8) is similar to that in the previous test. The direct P and SV arrivals recorded in a single borehole provide sufficient information to constrain the parameters $t_0$ and $M$ (Figures 9 and 10).

Next, we use the five-layer model in Figure 11 and assume that all source parameters ($x^s$, $t_0$, and $M$) are unknown.

Figure 13. Change of the source coordinates (a) $x_1^s$ and (b) $x_3^s$ with iterations for the model in Figure 11.

The multiple layers in this model produce numerous scattered waves, which help constrain the source location. Although the origin time $t_0$ for the actual and trial sources coincides, it varies with iterations as WI tries to match the observed and predicted data.

In theory, moving the source vertically shifts the apex of the P- and S-wave moveouts up or down, with the depth of the apex determining the vertical coordinate $x_3^s$. Also, variations in the horizontal distance between the source and the receiver array change the difference between the S- and P-wave traveltimes. In contrast, changing the origin time simply shifts the P- and S-wave moveouts along the time axis without moving them in depth or altering their difference. Therefore, the parameters $x^s$ and $t_0$ influence the traveltimes in a different way, which should preclude a trade-off between them.

Still, because both $x^s$ and $t_0$ are unknown, the objective function oscillates and cannot be reduced as much as in the
previous examples (Figure 12). After a number of iterations, the estimated $x^s$ and $t_0$ are close to the actual parameters. As a result, the P- and SV-traveltime shifts produced by further perturbations in the source location and origin time become too small to guide the inversion toward the actual model. Hence, the estimated horizontal source coordinate $x^s_1$ and time $t_0$ are slightly distorted (Figures 13 and 14), and the objective function for the inverted model is somewhat larger than that for the first two tests.

In the fourth experiment we use the model in Figure 11, but invert only for $M$ and $x^s$, with the origin time $t_0$ fixed at the correct value. The objective function rapidly decays with iterations (Figure 15), and the estimated source location is almost exact (Figure 16). Also, we obtain an accurate estimate of all three elements of the tensor $M$ (Figure 17).

The previous tests assumed the correct velocity model for WI. In the last example we evaluate the influence of velocity errors on the inversion results by distorting the anisotropy coefficient $\varepsilon$ for the homogeneous VTI model in Figure 3. The algorithm estimates the parameters $x^s$ and $M$, whereas the origin time $t_0$ is fixed at the actual value. In particular, an error in $\varepsilon$ changes the P-wave horizontal velocity $V_{\text{hor}}$, which should influence estimation of the horizontal source coordinate $x^s_1$. After a fast initial decrease, the objective function flattens out (Figure 18) at a larger value than that in Figure 4. Still, errors in the source coordinates are relatively small (about 5 m for $x^s_1$), and the elements $M_{11}$ and $M_{33}$ of the moment tensor are also recovered with high accuracy. However, there is a more significant error (over 20%) in the element $M_{13}$, which is most sensitive to the quality of waveform matching.
Waveform inversion for parameters of microseismic sources in VTI media

Figure 17. Change of the moment-tensor elements (a) $M_{11}$, (b) $M_{33}$, and (c) $M_{13}$ with iterations for the model in Figure 11. The origin time $t_0$ is fixed at the correct value.

Figure 18. Change of the normalized objective function $\mathcal{J}(\mathbf{m})$ with iterations for the model in Figure 3. The inversion is performed with an incorrect value of $\varepsilon$ ($\varepsilon = 0.3$ instead of the actual 0.4). The rest of the VTI parameters are unchanged.

7 CONCLUSIONS

We presented a waveform-inversion methodology for estimating the source parameters (location $\mathbf{x}^s$, origin time $t_0$, and moment tensor $\mathbf{M}$) of microseismic events from multicomponent data. The WI algorithm operates with the elastic anisotropic wave equation and is designed for dislocation-type sources embedded in 2D heterogeneous VTI media. In addition to employing the entire trace, our method simultaneously inverts for parameters that are typically obtained separately by kinematic and amplitude techniques.

The gradient of the objective function was computed with the adjoint-state method, which requires only two modeling simulations. We employed the nondimensionalization approach to handle model updating for different parameter classes. Although the VTI parameters were assumed known, they can be included in the adjoint calculation. Whereas reconstruction of the velocity model does not substantially increase the computational cost, it can create trade-offs in the inversion.

Synthetic tests were performed for data recorded by a dense vertical array of two-component receivers in homogeneous and horizontally layered VTI media. Increasing the number of layers is actually beneficial for our algorithm because multiple reflections and conversions improve the sensitivity of WI to the source parameters. If the initial model is located within the basin of convergence, WI accurately estimates the parameters $\mathbf{x}^s$, $t_0$, and $\mathbf{M}$, especially if the origin time is fixed at the correct value. Although in theory there is no trade-off between $\mathbf{x}^s$ and $t_0$, the P- and SV-wave traveltime differences responsible for resolving them are small near the global minimum. Our testing shows that simultaneous inversion for the source coordinates and origin time may lead to small distortions in $x^s_1$ and $t_0$.

To evaluate the influence of errors in the velocity model, we estimated the parameters of a source in a homogeneous
VTI medium using an inaccurate anisotropy parameter \( \varepsilon = 0.3 \) instead of the actual \( \varepsilon = 0.4 \). Predictably, a distortion in \( \varepsilon \) propagates into the horizontal source coordinate, but the error in \( x_s^1 \) is not significant; the parameters \( x_s^3 \) and \( M \) were determined with high accuracy.

To take full advantage of WI and the adjoint-state method in microseismic monitoring, they should be extended to anisotropic velocity model building. As is done in kinematic inversion, anisotropic parameter estimation can be performed simultaneously with event location. These ideas will be discussed in a future publication.

**Figure 19.** Change of the source coordinates (a) \( x_s^1 \) and (b) \( x_s^3 \) with iterations for the model in Figure 3. The inversion uses \( \varepsilon = 0.3 \) instead of the actual \( \varepsilon = 0.4 \).

**Figure 20.** Change of the moment-tensor elements (a) \( M_{11} \), (b) \( M_{33} \), and (c) \( M_{13} \) with iterations for the model in Figure 3. The inversion uses \( \varepsilon = 0.3 \) instead of the actual \( \varepsilon = 0.4 \).
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Least-squares migration in the presence of velocity errors

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Figure 1. Least-squares migration images computed for a laterally invariant velocity model by minimizing the (a) difference between predicted and observed data, and (b) difference between predicted and shifted observed data. Compared to the (c) image computed for an optimized velocity model by minimizing the difference between predicted and observed data, the image in (b) contains similar features despite the use of a much simpler velocity model during migration.

ABSTRACT
Seismic migration requires an accurate background velocity model that correctly predicts the kinematics of wave propagation in the true subsurface. Least-squares migration, which seeks the inverse rather than the adjoint of a forward modeling operator, is especially sensitive to errors in this background model, which can result in traveltime differences between predicted and observed data that lead to incoherent and defocused migration images. We propose an alternative misfit function for use in least-squares migration that measures amplitude differences between predicted and observed data, i.e., differences after correcting for nonzero traveltime shifts between predicted and observed data. We demonstrate on synthetic and field data that, when the background velocity model is incorrect, the use of this misfit function results in better focused migration images with greater amplitude fidelity. Results suggest that our method best enhances image focusing when differences between predicted and observed data can be explained by traveltime shifts.

1 INTRODUCTION
Seismic migration can be described as the adjoint of a linearized forward modeling operator applied to observed data (Claerbout, 1992). Migration produces a reflectivity image, an image of a perturbation to the background velocity model (Cohen and Bleistein, 1979), that approximates the true reflectivity insofar as the adjoint of the forward operator approximates the pseudoinverse. Typically, the adjoint is a poor approximation, and the accuracy of the computed reflectivity image can be significantly improved by using the pseudoinverse of the forward operator rather than the adjoint. The use of the pseudoinverse of the forward operator in migration is known as least-squares migration (Nemeth et al., 1999; Østmo and Plessix, 2002; Plessix and Mulder, 2004; Kühl and Sacchi, 2003; Dai, 2012).

Least-squares migration requires the inverse of the Hessian matrix (the normal operator) of second derivatives of a misfit function with respect to model parameters. The Hessian, however, is prohibitively expensive to compute and store for most practical-sized problems. Approximations of the inverse Hessian (Gray, 1997; Chavent and Plessix, 1999; Shin et al., 2001; Rickett, 2003; Guitton, 2004; Plessix and Mulder, 2004; Venciano, 2008; Symes, 2008) are more feasible, and are often used to improve the quality of final migration images or to precondition iterative least-squares migration. In this paper, we focus on iterative least-squares migration,
Figure 2. A simple example. The (a) predicted data (black) and observed data (red); (b) predicted data (black) and shifted observed data (blue); (c) normalized misfit function computed with the predicted and observed data shown in (a); and (d) normalized misfit function computed with the predicted and shifted observed data shown in (b). A local optimization method beginning at the position indicated by the white circle will converge to a local minimum in (c), but will find the global minimum, indicated by the magenta star, in (d).

which can be used in conjunction with or in place of approximations of the Hessian. An advantage of iterative migration algorithms is that they typically are straightforward to implement; a disadvantage is that they can be more computationally expensive compared to an efficient approximation of the Hessian, or compared to a single application of the adjoint operator as is done, for example, in reverse-time migration (Baysal et al., 1983; Loewenthal and Mufti, 1983; McMechan, 1983; Whitmore, 1983; Levin, 1984).

The quality and accuracy of migration images depends greatly on the accuracy of the background velocity model, and errors in this background model can lead to an incoherent, defocused image. Ideally, the background velocity model should correctly predict the traveltimes of observed data, and should be sufficiently smooth so as not to generate reflected waves. These requirements derive from the conditions under which the Born approximation is valid (Symes, 2009), and under these conditions, migration can accurately image subsurface structures. However, when these conditions are violated, migration images are degraded and become defocused and incoherent. One reason for this degradation is that migration inverts for the perturbation to the background velocity model that controls only the amplitudes of predicted data; if the background model contains errors, then the predicted data will contain errors in both traveltimes and amplitudes in the amplitude and traveltime separation.

Often, separating these types of errors, and perhaps discarding a certain type of error, can improve inversion results. For example, for full waveform inversion (Tarantola, 1984; Pratt et al., 1998), authors advocate using only phase or traveltime information (Shin and Min, 2006; Bednar et al., 2007; Choi and Alkhalifah, 2011; Kamei et al., 2011), especially to update the low-wavenumber background model that is difficult for full waveform inversion to recover from reflection seismic data (Snieder et al., 1989; Hicks and Pratt, 2001; Xu, 2012; Ma, 2012). Our task in least-squares migration is complementary to that of full waveform inversion for the background model: we seek to invert for the high-wavenumber component of the model, i.e., the perturbation to the background model. Thus, analogous to the use of phase or traveltime information to recover the low-wavenumber component of the velocity model, we propose to use amplitude information to recover the high-wavenumber component.

The utility of this amplitude and traveltime separation is easily illustrated. Consider the task of estimating a traveltime shift and an amplitude scale between two 1D signals, shown in Figure 2. In Figures 2a and 2b, the black curve represents the predicted data, the red curve represents the observed data, and the blue curve represents the observed data shifted so that its traveltime matches that of the predicted data. The conventional least-squares misfit function, i.e., the $L^2$-norm of the difference between predicted and (possibly shifted) observed data, for two model parameters (the amplitude scale and the traveltime shift) is shown in Figures 2c and 2d. Notice in Figures 2c and 2d the location of the global minimum, indicated by the magenta star. Figure 2c shows the normalized misfit function computed between predicted and observed data (Figure 2a). With
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2 METHODS

In this section, we first briefly review linearized waveform inversion and then discuss dynamic warping, the method we use to estimate traveltime shifts, before presenting our method for amplitude-only inversion.

2.1 Linearized waveform inversion

Wave propagation in the subsurface is described approximately by the constant-density acoustic wave equation,

\[ \sigma_0 \frac{\partial^2 u_0}{\partial t^2} - \Delta u_0 = f, \]

where \( u_0 \) is the wavefield, \( \sigma_0 \) is the squared background slowness, and \( f \) is the source function. Perturbing \( \sigma_0 \) by a scattering potential \( m \) and linearizing about \( m \) yields

\[ \sigma_0 \frac{\partial^2 u}{\partial t^2} - \Delta u = -m \frac{\partial^2 u_0}{\partial t^2}, \]

where \( u \) is the scattered or perturbation wavefield. Often \( m \) is referred to as the reflectivity model or simply the reflectivity.

Let \( u_s \) denote the discretized solution of equation 2 for a source function at position \( s \). The wavefield \( u_s \) is linear in the reflectivity \( m \):

\[ u_s = L_s m, \]

where \( L_s \) is a linear prediction operator describing the evolution of the scattered wavefield in equation 2. The predicted data \( p_{s,r} \) are a subset of the wavefield \( u_s \):

\[ p_{s,r} = S_r u_s, \]

where \( S_r \) is a sampling operator that extracts the wavefield at receiver position \( r \).

To solve equation 4 for the reflectivity model \( m \), we minimize, in a least-squares sense, the difference between predicted data \( p_{s,r} \) and observed data \( d_{s,r} \):

\[ \min_m J(m) = \sum_{s,r} E_{s,r}(u_s(m)), \]

where

\[ E_{s,r}(u_s) = \frac{1}{2} \| S_r u_s - d_{s,r} \|^2. \]

To minimize equation 5, we can pursue the negative of the gradient direction

\[ \frac{\partial J}{\partial m} = \sum_{s,r} L^*_s \left( \frac{\partial E_{s,r}}{\partial u_s} \right), \]

where

\[ \frac{\partial E_{s,r}}{\partial u_s} = S^*_r (S_r u_s - d_{s,r}) \]

is the data residual. The adjoint of the prediction operator \( L_s \) is a migration operator (Claerbout, 1992), and so we obtain the well-known result (Lailly, 1983; Taran-tola, 1984) that the gradient of the least-squares misfit function can be computed by a migration of the residuals.

2.2 Dynamic warping

Before we can consider an amplitude misfit function, we require a method for estimating time-varying traveltime shifts between predicted and observed data. For this purpose, we use dynamic warping (Hale, 2013). Compared to more conventional methods for estimating traveltime shifts based on windowed crosscorrelations, dynamic warping is more accurate, especially when traveltime shifts vary rapidly as a function of time (Hale, 2013).

Dynamic time warping (Sakoe and Chiba, 1978) is a method for computing integer time shifts \( \tau = (\tau_1, \tau_2, \cdots, \tau_n) \) between two sequences \( p = (p_1, p_2, \cdots, p_n) \) and \( d = (d_1, d_2, \cdots, d_n) \) such that

\[ A = \frac{1}{2} \sum_i (p_i - d_{i+\tau_i})^2 \]

is minimized.
is minimized with respect to $\tau$ subject to the constraint
$$|\tau_i - \tau_{i-1}| \leq 1/c,$$  \hspace{1cm} (10)
where $c$ is a positive integer. An attractive feature of dynamic time warping is that the algorithm is guaranteed to find the traveltime shifts $\tau$ that minimize equation 9 subject to constraint 10, and these shifts are such that $\partial A_s / \partial \tau = 0$ when the constraint is inactive.

Although we could use dynamic time warping to independently estimate traveltime shifts between all pairs of predicted and observed traces, in practice we find that using dynamic image warping (Hale, 2013) to estimate traveltime shifts between predicted and observed common shot gathers yields more accurate shifts, especially when predicted and observed data are not simply shifted versions of each other (as is often the case even with synthetic data, and certainly always is the case with recorded field data). Dynamic image warping (Hale, 2013) approximately solves the extension to higher dimensions of the constrained optimization problem specified by equations 9 and 10, and in doing so, imposes constraints both in time (equation 10) as well as in distance or offset on the estimated traveltime shifts.

### 2.3 Inversion of amplitude errors

To formulate an inversion of amplitude errors, we modify the observed data to include a time-shift operator:

$$b_{s,r} = T_{s,r} d_{s,r},$$ \hspace{1cm} (11)

where $T_{s,r}$ is a linear operator, e.g., a sinc interpolation operator, that shifts the observed data $d_{s,r}$ by the traveltime shifts $\tau_{s,r}$ estimated using dynamic warping. Note that $T_{s,r}$ depends implicitly on the model $m$, because the traveltime shifts $\tau_{s,r}$ are computed using the predicted data $p_{s,r}$, which depend on the model.

The shifted observed data $b_{s,r}$ can be viewed as a secondary dataset obtained by processing the observed data. Processing of the observed data prior to migration is standard practice, even for conventional migration. The purpose of this processing is essentially to remove from the observed data any components that are due to an inconsistent model of wave propagation in the true subsurface. For example, in order to use acoustic forward modeling to migrate elastic data, one would need to remove shear waves from observed data prior to migration. Just as an acoustic wave equation cannot explain shear waves in observed data, the linearized wave equation (equation 2) with an incorrect background model cannot explain the traveltimes of observed data. Migration using an incorrect background model is equivalent to migration using forward modeling that is inconsistent with the observed data, and so to properly migrate these data, we must first remove those components that cannot be explained by our forward modeling. Those components are the traveltimes.

Thus, we seek to minimize the difference between predicted data $p_{s,r}$ and time-shifted observed data $b_{s,r}$:

$$\min_m J_A(m) = \sum_{s,r} A_{s,r}(u_s(m), \tau_{s,r}(m)),$$ \hspace{1cm} (12)

where

$$A_{s,r} = \frac{1}{2} |S_s u_s - T_{s,r} d_{s,r}|^2.$$

Note that if the estimated traveltime shifts $\tau_{s,r}$ are accurate, then equation 12 measures only amplitude errors between predicted and observed data. If the traveltime shifts are zero, then equation 12 reduces to equation 5.

To minimize the misfit function in equation 12, we require its gradient with respect to model parameters:

$$\partial J_A / \partial m = \sum_{s,r} L_s^T \left( \frac{\partial A_{s,r}}{\partial u_s} \right),$$ \hspace{1cm} (14)

where

$$\frac{\partial A_{s,r}}{\partial u_s} = S_s^T (S_s u_s - T_{s,r} d_{s,r}).$$  \hspace{1cm} (15)

Although $A_{s,r}$ depends on the estimated traveltime shifts $\tau_{s,r}$, we need not consider this dependence when computing the residual in equation 15 because dynamic warping minimizes equation 13 (or equation 9) subject to constraint 10, so that $\partial A_{s,r} / \partial \tau_{s,r}$ is mostly zero.

We refer to equation 15 as the amplitude residual and equation 12 as the amplitude misfit function, as they measure only amplitude errors between predicted and observed data.

### 3 RESULTS

We compare conventional least-squares migration (LSM) with the proposed method of least-squares migration of amplitude errors (LSMA) on a 2D synthetic dataset, and on a 2D field dataset.

For least-squares migration, the data are linear in the reflectivity, and thus LSM images can be computed by minimizing equation 5 with (linear) conjugate gradient iterations. To compute LSMA images by solving equation 12, however, is a nonlinear problem because the reflectivity $m$ depends on the traveltime shifts $\tau_{s,r}$, but the traveltime shifts also depend on the reflectivity. We can compute LSMA images either by minimizing equation 12 using a gradient-based descent method (e.g., steepest descent or nonlinear conjugate gradient), or alternatively, by first solving equation 12 with fixed traveltime shifts $\tau_{s,r}$, then recomputing the traveltime shifts and solving equation 12 with the new shifts, repeating until convergence.

Note that when the traveltime shifts $\tau_{s,r}$ are zero, equation 12 is equivalent to equation 5. This is the case for the first nonlinear iteration or the first solution of equation 12 with fixed $\tau_{s,r}$, in which the reflectivity is zero and hence the traveltime shifts are zero. After the first nonlinear iteration or the first solution of equa-
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Figure 3. The (a) true slowness model, (b) true background slowness model, (c) true reflectivity computed as the difference between the true slowness squared and the true background slowness squared, and (d) LSM image.

Figure 4. The (a,b) difference between the true background slowness shown in Figure 3b and the background slowness used for migration; (c) LSM image computed for the background slowness with error shown in (a); (d) LSM image computed for the background slowness with error shown in (b); (e) LSMA image computed for the background slowness with error shown in (a); and (f) LSMA image computed for the background slowness with error shown in (b).
portion 12, we obtain a nonzero reflectivity image from which to predict data and to estimate possibly nonzero traveltime shifts.

3.1 Synthetic data example

The background slowness used for modeling and migration is shown in Figure 3b, and is computed by smoothing a modified Marmousi model (Lailly and Versteeg, 1990) shown in Figure 3a along both the depth and distance axes using a two-sided exponential filter with width 100 m. The true reflectivity shown in Figure 3c is then computed as the difference between the true slowness (Figure 3a) squared and the true background slowness (Figure 3b) squared. Using the true background slowness and true reflectivity, we simulate observed data by solving equations 1 and 2 for a Ricker source function with peak frequency 10 Hz. To facilitate comparison of LSM and LSMA, all migration images for these synthetic data are computed using 20 nonlinear conjugate gradient iterations. Hence, as the cost of dynamic warping is small compared to the cost of modeling and migration, the LSM and LSMA images computed for these synthetic data come at comparable costs.

The first example shown in Figure 3 demonstrates conventional LSM using the true background slowness for migration. The reflectivity image shown in Figure 3d is obtained after 20 nonlinear conjugate gradient iterations (Nocedal and Wright, 2000) of LSM using the true background slowness with 153 shots and 767 receivers evenly spaced along the surface. As expected, this computed reflectivity matches well the true reflectivity shown in Figure 3c because the background slowness model used for migration was exactly the true background slowness. In practice, we expect the background slowness model used to migrate the data to differ from the true background slowness model.

Figure 4 illustrates the effects of erroneous background slowness models on the reflectivity images obtained using LSM and LSMA. Figures 4a and 4b show the differences between the true background slowness model (Figure 3b) and the background slowness models that we use for migration. The slowness error shown in Figure 4a was computed by smoothing a random slowness model, while the error shown in Figure 4b resulted from scaling the true background slowness by 95%.

Figures 4c and 4d show the reflectivity images computed using 20 iterations of LSM with the erroneous background slowness models with errors shown in Figures 4a and 4b, respectively. Compared to the reflectivity image (Figure 3d) computed using the true background slowness, the image in Figure 4c is degraded, and
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Figure 7. Normalized (a) data and amplitude misfit and (b) model misfit for LSM and LSMA. Here, true background refers to the true background slowness shown in Figure 3b, while incorrect background refers to the background slowness with error shown in Figure 4a.

Figure 8. Normalized (a) data and amplitude misfit and (b) model misfit for LSM and LSMA. Here, true background refers to the true background slowness shown in Figure 3b, while incorrect background refers to the background slowness with error shown in Figure 4b.

shows uneven illumination and defocused reflectors, especially at greater depths where traveltime errors resulting from the erroneous background slowness are more severe. This degradation is also seen in the reflectivity image shown in Figure 4d. The quality of this image is worse than the image shown in Figure 4c because the slowness errors (Figure 4b) all have the same sign, and so traveltime errors in the predicted data accumulate more quickly than traveltime errors for data predicted with the slowness model with error shown in Figure 4a.

Figures 4e and 4f show the reflectivity images computed with 20 iterations of LSMA. Compared to the conventional LSM images (Figures 4c and 4d), the LSMA images show improved illumination of deeper portions of the model, and better focused and more continuous reflectors throughout. For example, Figures 5a, 5b, and 5c show zoomed views of the areas enclosed by yellow boxes in Figures 4c, 4e, and 3c, respectively. Compared to the LSM image shown in Figure 5a, reflectors in the LSMA image shown in Figure 5b are more focused and better match the true reflectivity shown in Figure 5c. Similarly, zoomed views shown in Figures 5a, 5b, and 5c of the areas enclosed by green boxes in Figures 4d, 4f, and 3c, respectively, demonstrate that even for a large and biased slowness error (Figure 4b), minimizing the amplitude misfit function yields an interpretable reflectivity image with features that match those apparent in the true reflectivity.

Note, however, that the positions of features in LSMA images (Figures 4e and 4f) are shifted compared to their positions in the true reflectivity (Figure 3c). For example, compare the position of the reflector located at distance 3.5 km and depth 1 km in Figure 5b, or the reflector located at distance 4.5 km and depth 1 km in Figure 6b, to their positions in the true reflectivity. This mispositioning is expected, however, since LSMA images are computed using erroneous background slowness models.
The presence of remaining traveltime shifts between predicted and observed data, as well as spatial shifts between image features in computed LSMA images and those in the true reflectivity, is confirmed by the misfit functions shown in Figures 7 and 8. Figures 7a and 8a show normalized data and amplitude misfit functions, while Figures 7b and 8b show normalized model misfit functions (the $L^2$-norm of the difference between the computed reflectivity and the true reflectivity) for LSM and LSMA images computed using either the true background slowness model shown in Figure 3b or the erroneous background slowness model with error shown in Figure 4a or Figure 4b. In Figures 7a and 8a, note that the data misfit is not used in LSMA, but more importantly, notice that the data misfit increases in iteration 7 in Figure 7a and in iteration 2 in Figure 8a. This indicates that the better-focused LSMA images shown in Figures 4e and 4f cannot be obtained with conventional LSM, which minimizes the data misfit.

The model misfits shown in Figures 7b and 8b indicate that, for the erroneous background slowness models shown in Figures 4a and 4b, the LSM images (Figures 4c and 4d) more closely match the true reflectivity (Figure 3c) than do the LSMA images (Figures 4e and 4f). Indeed, a zero-reflectivity image is closer to the true reflectivity than the LSMA image shown in Figure 4f. However, the large model misfits for LSMA images simply reflect the fact that features in these images are shifted relative to the corresponding features in the true reflectivity. Although image features in LSMA images are shifted, it is clear that the amplitudes (but not the positions) of these features better match those of the true reflectivity.

3.2 Field data example

Next we test our method for amplitude-only migration on a subset of a field dataset provided by Eni E&P. The entire 2D dataset contains 3661 shots with a shot spacing of 12.5 m, and was recorded using a streamer with 99 receivers with a receiver spacing of 12.5 m and maximum offset of 1.225 km. The subset of the data that we migrate consists of 431 shots with shot spacing of 25 m. The data have been regularized, and multiples have been attenuated. We estimate a zero-phase wavelet from the amplitude spectrum computed from a subset of the recorded data (Claerbout, 1992), and we apply a bandpass filter to both the estimated wavelet and the recorded data to remove frequency content below 10 Hz and above 40 Hz prior to migration.

We compare LSM and LSMA for two slowness models. The first slowness model, shown in Figure 9a, is laterally invariant (except near the sea floor), while the second, shown in Figure 9b, is an optimized slowness model that was provided with the recorded data. The LSM and LSMA images computed for the laterally invariant slowness model (Figure 9a) are shown in Figures 10a and 10b, respectively. Comparing these images, we observe that reflectors in the LSMA image are more continuous and better focused than corresponding reflectors in the LSM image. Moreover, image features in the LSMA image (Figure 10b) are similar to features seen in the LSM image (Figure 10c) computed for the optimized slowness model (Figure 9b), despite the use of a much simpler slowness model for LSMA. Differences between the migration images shown in Figures 10a and 10b are most apparent in the areas enclosed by yellow boxes, in which the slowness differences (Figure 9c) between the models used for migration are relatively large. Zoomed views of the areas enclosed by yellow boxes in Figures 10a, 10b, and 10c are shown in Figures 1a, 1b, and 1c, respectively. Elsewhere, where slowness errors are smaller, differences between the migration images are less significant, as one would expect.

It is worth noting that, for this example, it was necessary to use 3D dynamic warping in LSMA. For 3D warping, rather than independently warp predicted to observed shot gathers as was done for the synthetic examples shown in Figure 4, we instead warped simultaneously all predicted shot gathers to all observed shot gathers, at each iteration of LSMA. A 3D warping enables us to constrain changes in estimated travelt ime shifts with shot location, which results in more accurate shifts. For synthetic tests in which the same forward modeling code is used to simulate both predicted and observed data, this additional constraint is per-
Figure 10. The (a) LSM image and (b) LSMA image computed for the laterally invariant slowness model shown in Figure 9a, and the (c) LSM image computed for the optimized slowness model shown in Figure 9b.
Figure 11. For the shot located at distance 1.85 km, the (a) observed data, (b) predicted data computed using the laterally invariant slowness model shown in Figure 9a and the LSMA image shown in Figure 10b, and (c) traveltime shifts between (a) and (b).

haps unnecessary. For field data, however, an additional constraint on the traveltime shifts can significantly improve the accuracy of estimated shifts, especially in cases where the data quality is low.

Because we compute LSMA images by minimizing the difference between predicted and shifted observed data (equation 12), the predicted data in general will not have the same traveltimes as the original observed data. An example of these traveltime differences for data corresponding to the shot located at distance 1.85 km is shown in Figure 11. Figure 11a shows the observed data, Figure 11b shows the predicted data computed using the laterally invariant slowness model (Figure 9a) and the LSMA image (Figure 10b), and Figure 11c shows the traveltime shifts between the data shown in Figures 11a and 11b. The maximum frequency content of the data is 40 Hz, which corresponds to a period of 25 ms. Thus we observe from Figure 11c that the remaining traveltime shifts between predicted and observed data exceed one half period. This confirms that LSMA yields an image that explains the dynamics, but not the kinematics, of the observed data.

4 CONCLUSION

We have presented a method for least-squares migration that minimizes an amplitude misfit function defined with differences between predicted data and shifted observed data, with traveltime shifts between predicted and observed data estimated using dynamic warping. The use of this amplitude misfit function results in a more coherent and better focused migration image when the background slowness model used for migration contains errors. These LSMA images contain image features with amplitudes that match those of the true reflectivity, but with positions that are shifted relative to the positions of corresponding features in the true reflectivity. LSMA images thus are better suited for interpretation of geologic structures, but in order to correctly position interpreted structures, we would need to first correctly position LSMA image features. One way to correct for the mispositioning of image features is to first align features with measurements of subsurface properties obtained from well logs, and then interpolate alignment shifts between well-log locations to generate shifts for an entire image.

The improvement in LSMA images compared to conventional LSM images depends on the nature of the background slowness error, and also on the acquisition geometry. A comparison between the images shown in Figure 4e and 4f suggests that LSMA provides a greater improvement in image quality and reflector focusing for small, systematic errors in background slowness (e.g., Figure 4b), perhaps because in such situations, traveltime shifts can explain well the differences between predicted and observed data. When the background slowness error is more complex or is too large, predicted and observed data might be inconsistent, i.e., events in one dataset do not have corresponding events in the other, making it difficult to estimate accurate shifts; or, predicted and observed data might differ by significant horizontal spatial shifts in addition to vertical traveltime shifts, in which case estimating only traveltime shifts for use in LSMA might be inadequate. While dynamic warping can also be used to estimate horizontal shifts, only vertical traveltime shifts were used in the examples shown above.

The improvement in LSMA images also depends on acquisition geometry. Traveltime differences between observed and predicted data in LSM arise from errors in the background slowness model used for migration, but more specifically, they arise from inconsistencies between different images of the same subsurface geologic structures, e.g., images computed for inconsistencies between individual shots. Thus, we expect LSMA to provide greater improvement over conventional LSM when the recorded data provide redundant information about subsurface geologic structures. Conversely, we expect LSMA and
LSM images to be more similar when data provide independent information, for example, when shots are sparsely located or shot spacing is large, or when the maximum source-receiver offset or the offset-to-depth ratio is small.

Although LSMA images can provide an improved estimate of the amplitudes of the true reflectivity, ultimately we seek a complete model of the subsurface, which includes not only an accurate reflectivity model but also an accurate background slowness model. The proposed method could potentially be extended and used to aid an inversion for the background slowness. A simple approach might be to hold the reflectivity model constant following LSMA, and then invert the remaining traveltime shifts between predicted and observed data in order to update the background slowness.

5 ACKNOWLEDGEMENT

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Imaging condition for elastic RTM

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ABSTRACT
Polarity changes in converted-wave images constructed by elastic reverse-time migration cause destructive interference after stacking over the experiments of a seismic survey. The polarity reversal is due to the fact that S-mode polarization changes as a function of direction, and therefore the imaged reflectivity reverses sign at normal incidence. We derive a simple imaging condition for converted waves imaging designed to correct the image polarity and reveal the conversion strength from one wave mode to another. Our imaging condition exploits pure P- and S-modes obtained by Helmholtz decomposition. Instead of correlating components of the vector S-mode with the P-mode, we exploit the entire S wavefield at once to produce a unique image. We generate PS and SP images using geometrical relationships between the propagation directions for the P and S wavefields, the reflector orientation and the S-mode polarization direction. Compared to alternative methods for correcting PS and SP images polarity reversal, our imaging condition is simple and robust and does not add significantly to the cost of reverse-time migration. Several numerical examples demonstrate the effectiveness of our new imaging condition in simple and complex models.

Key words: elastic wavefields, reverse-time migration, imaging condition

1 INTRODUCTION
Seismic acquisition advances, as well as ongoing improvements in computational capability, have made imaging using multi-component elastic waves, or elastic migration, increasingly feasible. Elastic migration with multi-component seismic data can provide additional subsurface structural information compared to conventional acoustic migration using single-component data. Multi-component seismic data can be used, for example, to estimate fracture distributions as well as elastic and solid-related properties.

In complex geologic environments, it is desirable to use wavefield-based imaging method, e.g. reverse-time migration (RTM). Conventional reverse-time migration consists of two steps: wavefield extrapolation followed by application of an imaging condition (Claerbout, 1971). Wavefield extrapolation requires constructing source and receiver wavefields using an estimated wavelet and the recorded data, respectively. In elastic media, source and receiver wavefields are constructed using elastic (vector) wave equations.

Following wavefield extrapolation, an imaging condition is applied by combining the source and receiver wavefields to obtain images of subsurface structures. Multi-component wavefields allow for a variety of imaging conditions (Yan and Sava, 2008; Denli and Huang, 2008; Artman et al., 2009; Wu et al., 2010). A simple imaging condition for multi-component wavefields is the crosscorrelation of components of the displacement vectors in source and receiver wavefields (Yan and Sava, 2008). In 3D, this results in 9 images for different combinations of source and receiver displacement vector components. One limitation of this method is that P- and S-modes are mixed in the extrapolated wavefields; therefore, cross-talk between P- and S-modes creates artifacts that make interpretation difficult. Another imaging condition for multi-component wavefields first requires the decomposition of wavefields into different wave modes, for example, P- and S-modes. For isotropic elastic wavefields far from the source, P- and S-modes correspond to the compressional and transversal components of the wavefield, respectively (Aki and Richards, 2002). Similar to the imaging condition using displacement vector components, this imaging condition also provides multiple images by cross-correlating different wave modes present in the source and receiver wavefields (Yoon et al., 2004; Yan and Sava, 2008; Denli and Huang, 2008; Yan and Xie, 2012). However, in this case, images correspond to reflectivity for different combinations of incident and reflected P- and S-modes, e.g. PP, PS, SP and SS reflectivity, and therefore are more useful in geological interpretation.

One difficulty when imaging multi-component wavefields is that the PS and SP images change sign at certain incident angles. For example, in isotropic media, polarity reversal occurs at normal incidence (Balch and Erdemir, 1994). This sign change can lead to destructive interference multiple experiments in a seismic survey are stacked for a final image.
A simple way to correct the polarity reversal in PS and SP images assumes that the polarity change occurs at zero offset and can be corrected based on the acquisition geometry. However, this assumption fails if the reflectors are not horizontal (Du et al., 2012b). An alternative method to correct for the polarity change requires that we compute the incident angles at each image point and then reverse the polarity based on this estimated angle. One possibility is to use ray theory to simulate the incident wavefield direction, and apply this direction to the reflected wave extrapolated from the surface to every imaging point (Balch and Erdemir, 1994). This method is limited by the ray approximation and may become impractical when applied to elastic RTM in media characterized by complex multipathing. Another method to correct for polarity changes is to image in the angle domain, and then reverse the image polarity as a function of angle (Yan and Sava, 2008; Rosales et al., 2008; Yan and Xie, 2012). Constructing angle-domain common image gathers (ADCIGs) is accurate and robust, but can also be expensive. Finally, another possibility for polarity correction is to reverse the polarity in source and receiver wavefields based on the sign of the reflection coefficient, which is computed from the directions of the incident and reflected waves (Sun et al., 2006; Du et al., 2012a). These directions typically are computed using Poynting vectors, which may be inaccurate in complicated models characterized by multipathing (Dickens and Winbow, 2011; Patrikeeva and Sava, 2013).

In this paper, we propose an alternative 3D imaging condition for elastic reverse-time migration. Our new imaging condition exploits geometric relationships between incident and reflected wave directions, the reflector orientation and wavefield polarization directions. Using our new imaging condition, we are able to obtain PS and SP images without polarity reversal. The method is simple and robust and operates on separated wave modes, for example, by the method discussed by Yan and Sava (2008). Our method is also cheap to apply since it does not require complex operations, like angle-decomposition or directional decomposition. We begin by discussing the theory underlying our method and then illustrate it using several simple and complex synthetic examples.

2 THEORY

We propose a new imaging condition meant to compensate automatically for the polarity reversal characterizing conventional images. Our method builds on existing techniques which operate by first decomposing elastic wavefields in pure P- and S-modes. However, in contrasts with more conventional methods, our imaging condition does not simply correlate the P- and S-modes. However, in contrasts with more conventional methods, our imaging condition does not simply correlate the P- and S-modes. We extend and generalize this condition for elastic reverse-time migration. Our new imaging condition exploits geometric relationships between incident and reflected waves (Sun et al., 2006; Du et al., 2012a). These directions typically are computed using Poynting vectors, which may be inaccurate in complicated models characterized by multipathing (Dickens and Winbow, 2011; Patrikeeva and Sava, 2013). These directions are computed using Poynting vectors, which may be inaccurate in complicated models characterized by multipathing (Dickens and Winbow, 2011; Patrikeeva and Sava, 2013).

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Reconstructed source and receiver elastic wavefields can be separated into P- and S-modes prior to imaging (Dellinger and Etgen, 1990; Yan and Sava, 2008). In isotropic media, this separation can be performed using Helmholtz decomposition (Aki and Richards, 2002), which describes the compressional component $P$ and transverse component $S$ of the wavefield using the divergence and curl of the displacement vector field $u$:

$$P(x, t) = \nabla \cdot u(x, t) ,$$

$$S(x, t) = \nabla \times u(x, t) .$$

PS and SP images can then be obtained by cross-correlating the $P$ wavefield with each component of the $S$ wavefield (Yan and Sava, 2008). Images produced in this way have three independent components at every location in space, i.e., PS and SP images are vector images. A side motivation for our method is the fact that it is difficult to find the physical meaning of the various images constructed using this procedure. We seek to avoid vector PS and SP images by combining all components of the S wavefield with the P wavefield into an image representing the energy conversion strength from one wave-mode to another at an interface.

To formulate the imaging condition, we define the following quantities, shown in Figure 1:

- Vectors $D_P$ and $D_S$ indicating the propagation directions of the P- and S-modes, respectively;
- Vector $n$ which is the local normal to the interface represented by plane $I$;
- Vector $S$ which indicates the polarization of the S-mode.

According to Snell’s law, vectors $D_P$, $D_S$ and $n$ belong to the reflection plane $R$. The polarization vector $S$ is orthogonal to the vector $D_S$. In the following, we assume that we know the normal vector $n$, e.g., from a prior image obtained by imaging pure PP reflections.

The polarization vector $S$ represents waves reflected from incident P or S waves; therefore, the receiver wavefield polarization vector $S$ is not orthogonal, in general, to the plane $R$, and can be decomposed into vectors $S_\perp$ and $S_\parallel$ such that

$$S = S_\perp + S_\parallel .$$

Vectors $S_\perp$ and $S_\parallel$ are orthogonal and parallel to plane $R$, re-
respectively. The S waves reflected from incident pure P waves are confined to the vector field $S$, so in the analysis of PS reflections, we can simply assume that the S-mode is orthogonal to the reflection plane $R$. A similar discussion is applicable to SP reflections. As indicated earlier, the signs of PS and SP reflection coefficients change across normal incidence in every plane, which causes the polarity of the reflected waves to change at normal incidence. Since normal incidence depends on the acquisition geometry, as well as the geologic structure, this polarity reversal can occur at different positions in space, thus making it difficult to stack images for an entire multicomponent survey.

In order to address this challenge, we propose the following imaging condition for PS and SP images:

$$I^{PS}(x) = \sum_{e} \sum_{t} (-\nabla P(x, t) \times \mathbf{n}(x, t)) \cdot S(x, t), \tag{4}$$

$$I^{SP}(x) = \sum_{e} \sum_{t} ((\nabla \times S(x, t)) \cdot \mathbf{n}(x)) P(x, t). \tag{5}$$

Here, $P(x, t)$ and $S(x, t)$ represent the scalar P- and vector S-modes after Helmholtz decomposition, respectively, and $I^{PS}(x)$ and $I^{SP}(x)$ are the PS and SP images, respectively.

The physical interpretation of our new imaging condition, equations 4 and 5, is as follows:

- **PS imaging** (equation 4): The vector $\nabla P$ characterizes the propagation direction of the P-mode and can be calculated directly on the separated P wavefield. The cross-product with the normal vector $\mathbf{n}$ constructs a vector orthogonal to the reflection plane $R$; as indicated earlier, this direction is parallel with the polarization vector of the S-mode, $S$. Therefore, the dot product of $\nabla P \times \mathbf{n}$ and $S$ is just a projection of the vector $S$ wavefield on a direction which depends on the incidence direction of the P wavefield. For a P-mode incident in opposite direction, the vector $\nabla P \times \mathbf{n}$ is reverses direction, thus compensating for the opposite polarization of the S-mode. Consequently, the PS imaging condition has the same sign regardless of incidence direction, and therefore PS images can be stacked without canceling each-other at various positions in space.

- **SP imaging** (equation 5): The vector $S$ characterizes the polarization direction of the S-mode, and for the situation considered here it is orthogonal to the reflection plane $R$. Therefore, vector $\nabla \times S$ is contained in the reflection plane. The scalar product with the normal vector $\mathbf{n}$ produces a scalar field characterizing the magnitude of the S-mode, but signed according to its relation with respect to the normal $\mathbf{n}$. This scalar quantity can be simply correlated with the scalar reflected P wavefield, thus leading to a scalar image without sign change as a function of the incidence direction. Therefore, SP images produced in this fashion can also be stacked without canceling each other at various positions in space.

In 2D, the imaging condition simplifies. The $S$ has only one non-zero component, $S_y$, since the vector $S$ is orthogonal to the local reflection plane $R$. Therefore, the PS and SP imaging conditions are:

$$I^{PS} = - \sum_{e} \sum_{t} \left( \frac{\partial P}{\partial x} n_x - \frac{\partial P}{\partial z} n_z \right) S_y, \tag{6}$$

$$I^{SP} = \sum_{e} \sum_{t} \left( \frac{\partial S_y}{\partial x} n_x - \frac{\partial S_y}{\partial z} n_z \right) P, \tag{7}$$

where $I = I(x, z), P = P(x, z, t), S_y = S_y(x, z, t), n_x = n_x(x, z)$ and $n_z = n_z(x, z)$. For a horizontal reflector, i.e. $n = (0, 0, 1)$, we can write:

$$I^{PS} = - \sum_{e} \sum_{t} \frac{\partial P}{\partial x} S_y, \tag{8}$$

$$I^{SP} = \sum_{e} \sum_{t} \frac{\partial S_y}{\partial x} P, \tag{9}$$

which simply indicates that in the imaging condition we correlate the P or S wavefields with the $x$ derivative of the S or P wavefields, respectively. That is, of course, just a special case of the more general relation in equations 4 and 5.

We illustrate our new imaging condition with the synthetic model shown in Figure 3(a), which contains one horizontal reflector embedded in constant velocity. Figures 3(b) and 3(c) show the source P-mode and receiver S-mode for the single source indicated in Figure 3(a). As expected, the S-mode changes polarity as a function of the propagation direction. Figure 4(a) is the PS image obtained using the conventional imaging condition, i.e. the cross correlation of the source P wavefield (Figure 3(b)) with the receiver S wavefield (Figure 3(c)), and inherits the polarity change from the S-mode. In contrast, our new imaging condition leads to the image in Figure 4(b) without polarity reversal. This correction allows us to stack multiple elastic images constructed for different seismic experiments.
3 EXAMPLES

We also illustrate our method using two synthetic models.

The first model, Figure 5(a), consists of semi-parallel gently dipping layers. We use 40 sources evenly distributed along the surface, and 500 receivers located at the surface of the model. The source function is represented by a Ricker wavelet with peak frequency of 35 Hz. Figures 5(b) and 5(c) are snapshots of the source P and receiver S wavefields, respectively, and Figures 5(d) and 5(e) are snapshots of the source S and receiver P wavefields, respectively. In both cases, we observe polarity changes in S wavefield.

Using the conventional imaging condition (i.e. cross-correlation of the source and receiver wavefields), we obtain the PS and SP image shown in Figures 6(c) and 9(c). The reflectors on the left side of the model, are not well imaged, due to the fact that the polarity of individual images changes, thus causing destructive interference during summation over shots. The PS and SP common image gathers at $x = 1.5$ km, shown in Figures 7(a) and 8(a), show this polarity reversal causing image destruction.

In contrast, Figures 6(d) and 9(d) show PS and SP images using our new imaging condition. In this case, the interfaces are more continuous compared to the image constructed by the simple cross-correlation imaging condition. Moreover, the PS and SP common image gathers at $x = 1.5$ km, shown in Figures 7(b) and 8(b), confirm that there is no polarity change as a function of shot position.

The second example, Figure 10(a), is a modified Marmousi model. The Marmousi model was created in 1988 by...
the Institut Français du Pétrole (IFP). It contains several major faults and semi-parallel dipping layers. We use 60 explosive sources evenly distributed along the surface, and 576 multi-component receivers located at \( z = 0.2 \text{ km} \). The source function is represented by a Ricker wavelet with peak frequency of 35 Hz.

The source S wavefield is weak, due to the minor energy conversion at the top of the model, therefore the SP image is also weak. So in the following, we show only the PS image. Using the conventional and our new imaging conditions, we obtain the PS images shown in Figure 12(c) and 12(d), respectively. The conventional PS image is much weaker than the corresponding image obtained with our new imaging condition. This is due to the fact that the polarity changes as a function of source position occur at different locations in subsurface, and stacking leads to destructive interference between images obtained for different shots. In contrast, our new imaging condition corrects for this effect and leads to images better representing the reflection strength as a function of position in space.

4 CONCLUSIONS

We derive a new 3D imaging condition for PS and SP images constructed by elastic reverse-time migration. As for more conventional methods, P- and S-modes are obtained using Helmholtz decomposition. However, our imaging condition does not correlate various components of the S wavefield with the P wavefields; instead, our method uses geometrical relationships between the wavefields, their propagation direction, the reflector orientation and polarization directions to construct a single image characterizing the PS or SP reflectivity. Our method is simple and robust and leads to accurate images without the need to decompose wavefields into directional components, or to construct costlier images in the angle domain.

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Figure 6. PS images obtained using (a), (c) the conventional imaging condition and (b), (d) our new imaging condition. Panels (a) and (b) depict single-shot images, and panels (c) and (d) depict images obtained after stacking over shot.
Figure 7. PS common image gather at $x = 1.5$ km obtained from (a) the conventional imaging condition and (b) our new imaging condition.

Figure 8. SP common image gather at $x = 1.5$ km obtained from (a) the conventional imaging condition and (b) our new imaging condition.

Figure 9. SP images obtained using (a), (c) the conventional imaging condition and (b), (d) our new imaging condition. Panels (a) and (b) depict single-shot images, and panels (c) and (d) depict images obtained after stacking over shot.
Figure 10. (a) Marmousi model; (b) source P wavefield and (c) receiver S wavefield for a single shot.

Figure 11. PS common image gather at $x = 2$ km obtained from (a) the conventional imaging condition and (b) our new imaging condition.

Figure 12. PS single-shot image using (a) the conventional imaging condition and (b) our new imaging condition. PS stack images using (c) the conventional imaging condition and (d) our new imaging condition.
Elastic imaging with OBS receiver-side multiples

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ABSTRACT
Receiver-side water-column multiples acquired with ocean-bottom seismic sensors can be used for elastic imaging of the subsurface, which can provide additional information relative to more conventional acoustic imaging. These multiples can be separated from primaries contained in ocean-bottom seismic data using techniques such as up/down decomposition. In elastic imaging, the down-going wavefield consists of only P-waves because multiples propagate through the water-column. Receiver-side water-column multiples can be imaged by backpropagation from virtual receivers located at symmetric positions relative to the ocean surface, thereby increasing the imaging aperture. We perform imaging in the angle domain, which enables us to reverse the reflection polarity change at normal incidence. In addition to physical reflections, a strong solid/liquid interface at the ocean bottom generates unwanted mode conversions which are present in the images in the form of cross-talk between physical and non-physical events. However, this cross-talk can be easily identified in shot-domain or image-domain common-image gathers, and can therefore be removed after imaging. In the end, our procedure leads to a collection of images accounting for different combinations of incident and reflected P and S waves. These images exploit surface-related multiples, and are therefore the expression of different subsurface illumination relative to the analogous reflections using primary reflections only. Such elastic images can be used to infer accurate lithological information about the subsurface using angle-dependent reflectivity for different combinations of incident and reflected waves.

Key words: elastic wavefields, reverse-time migration, OBS

1 INTRODUCTION
Seismic data acquisition using ocean-bottom seismic (OBS) sensors is a rapidly developing technology that can address significant challenges in marine acquisition. For example, OBS acquisition can provide wide-azimuth recording geometries (Dash et al., 2009). Typical OBS systems consist of a hydrophone and a geophone that record pressure and three components of particle velocity, respectively. In contrast to more conventional acquisition geometries, OBS acquisition typically consists of a sparse set of receivers on the ocean bottom and a relatively dense network of sources at the ocean surface. The OBS sensors are sparsely distributed on the ocean bottom due to their high cost and also to the difficulty of their deployment. However, the small number of OBS sensors is compensated for by a dense network of sources at the ocean surface. This setup enables acquisition of wide-azimuth data, which is invaluable in imaging complex geologic structures (Grion et al., 2007; Dash et al., 2009; Berg et al., 2010; Wong et al., 2011). Furthermore, four-component data acquisition facilitates separation of acquired data into up-going and down-going waves, as well as into P- and S-modes (Schalkwijk, 2001). This separation enables imaging of the subsurface using elastic waves, which can potentially provide access to lithological information.

Although usually strong, receiver-side first-order water-column multiples are often considered noise and removed from the acquired data, after which imaging is performed using primaries separated at the ocean bottom. However, multiples contain additional information beyond primaries and provide increased illumination of the subsurface (Ronen et al., 2005; Dash et al., 2009; Tu et al., 2011; O’Brien et al., 2013). Imaging using multiples can be carried out by assuming that receivers are located on a virtual horizon that mirrors the ocean bottom across the ocean surface (Ronen et al., 2005; Dash et al., 2009; Wong et al., 2011). These multiples provide a wider illumination of the subsurface, especially in deep water.

OBS imaging with receiver-side water-column multiples is usually performed under the acoustic assumption. In this paper, we develop a method for elastic imaging with OBS data in order to generate images of the subsurface using both P-
and S-modes separated from receiver-side water-column multiples. For this purpose, we first make use of the fact that the P-modes propagating into the water column convert to S-modes when crossing the ocean bottom. Then, we perform elastic reverse-time migration (RTM) using an imaging condition based on wave-mode separation with Helmholtz decomposition (Dellinger and Etgen, 1990; Yan and Sava, 2008) to produce images corresponding to all possible combinations of P- and S-modes extracted from the source and receiver wavefields.

Our methodology relies on mode conversions at the ocean bottom in order to use converted waves for imaging. We reconstruct the receiver wavefield using the down-going waves observed in the recorded data. In the reconstructed receiver wavefield, some P-to-S conversions at the ocean bottom correspond to physical events, i.e., the ray paths of the conversions can be tracked in the reconstructed source wavefield. However, other P-to-S conversions are not physical, but are merely artifacts of the elastic wavefield extrapolation used in migration. These non-physical conversions generate artifacts in the image that interfere with the true geologic reflectors in the subsurface, and, therefore, look similar to the artifacts due to multiples or crosstalk between separate physical experiments. However, the spatial positions of these artifacts are not consistent in images computed for different shot positions, and thus they can be identified in common-image gathers (CIGs) and can be differentiated from physical reflection events by their nonzero moveout.

The novelty of our paper is that we combine related methods to solve the problems that occur in elastic migration with OBS water-column multiples in order to improve the quality of the image. In the following sections, we describe the sequence of steps necessary for elastic migration using water-column multiples, and then we illustrate our methodology with a modified Marmousi model.

2 THEORY
Our procedure for elastic imaging using water-column multiples consists of several steps, as outlined here. First, we decompose the recorded data into up- and down-going waves in order to isolate the water-column multiples. We then apply elastic imaging and compute PS and SP angle gathersto reverse the reflectivity polarity change across images obtained for different experiments. Finally, we remove artifacts caused by wave-mode conversions at the ocean bottom and the sparse receiver geometry. In the following, we detail our specific implementation.

2.1 Up/down separation
An OBS seismometer contains a hydrophone that records the scalar pressure field as well as a geophone that records three components of particle velocity (Berg et al., 2010). The pressure represents the spatial derivatives of the particle displacement, while the velocity represents the time derivatives of the displacement. With this information, four-component OBS data provides the ability to decompose the wavefield into primaries and multiples at the receiver locations. Following wavefield decomposition, we can use the separated multiples and primaries for migration. In our paper, we focus only on migration with multiples.

Wavefield decomposition methods can be classified into two general categories: (1) methods based on theoretical analysis of wave equations at a liquid-solid boundary, and (2) methods that predict multiples from primary reflections. Techniques in the first category usually make certain assumptions in order to obtain a computationally efficient expression for the multiple and primary separation (Barr and Sanders, 1989; Amundsen and Reitan, 1995; Osen and Amundsen, 2001; Schalkwijk, 2001). For example, Barr and Sanders (1989) assume that the reflection and transmission coefficients are known, and Amundsen and Reitan (1995) use the plane-wave assumption. The second category of methods can predict both source- and receiver-side water-column multiples in OBS data (Xia et al., 2006; Ma et al., 2010; Jin and Wang, 2012) under certain assumptions. For example, one could assume that the amplitude of the direct arrivals is preserved following pre-processing procedures (Ma et al., 2010) or that the wavefield is densely recorded or interpolated in both the receiver and shot grids (Jin and Wang, 2012).

Our method belongs in the first category and uses PZ summation. The term PZ summation refers to the summation and subtraction of the P (pressure) and Z (the vertical velocity) data, after appropriate weighting, to compute up- and down-going waves, respectively. Assuming that the ocean bottom is horizontal, receiver-side water-layer multiples are down-going waves propagating toward the ocean bottom in the water layer, while primaries propagate upward toward the ocean bottom. Also, down-going waves in pressure data have opposite polarity compared to down-going waves in vertical velocity data. These different characteristics of up-going (U) and down-going (D) waves allow for their separation using these expressions (Grion et al., 2007):

\[
U = \frac{1}{2}P + \alpha V_z, \tag{1}
\]
\[
D = \frac{1}{2}P - \alpha V_z, \tag{2}
\]

where \( P \) and \( V_z \) represent the vertical component of the pressure and particle velocity vector, respectively, and \( \alpha \) is a scale factor that depends on certain assumptions, e.g., that the OB reflectivity is known (Schalkwijk, 2001; Xia et al., 2006) or that the amplitudes of hydrophone and geophone traces are not affected by the data processing procedures prior to wavefield decomposition (Hoffe et al., 1999).

We demonstrate up/down separation, in addition to other procedures discussed below, using the synthetic model shown in Figures 1(a)-(c), which contains one horizontal reflector in the subsurface below the ocean bottom. Our acquisition geometry is designed to be similar to that of a typical OBS survey, and consists of 11 receivers sparsely located on the ocean bottom at \( z = 0.8 \) km and 91 sources evenly distributed along the ocean surface. Figure 2(a) shows a shot gather contain-
Elastic imaging with OBS receiver-side multiples

2.2 Mirror imaging

Figure 1. The 2D test model; (a) P-wave velocity (b) S-wave velocity and (c) density profiles in the 2D synthetic model.

Figure 2. (a) A shot gather containing up- and down-going waves; (b) The decomposed down-going waves.

ing direct waves, primary reflections, internal multiples, and water-column multiples. After up/down separation, we obtain only the down-going waves (Figure 2(b)), which represent the receiver-side water-column multiples.

Note that both up- and down-going modes may potentially contain other multiples (higher-order surface-related multiples or internal multiples). However, we assume that these multiples are lower in amplitude and thus do not significantly contribute to the migration image. This is, of course, a limitation of our method. If this assumption is violated in practice, then the migration images will be contaminated by additional cross-talk noise, similar to the case discussed later in this paper.

2.3 Imaging condition

Reconstructed source and receiver elastic wavefields can be separated into P- and S-modes prior to imaging (Dellinger and Etgen, 1990; Yan and Sava, 2008). In isotropic media, this separation can be achieved using the Helmholtz decomposition theorem (Aki and Richards, 2002), which describes the P- and S-modes in terms of the displacement vector $u(x, t)$:

$$ P = \nabla \cdot u, $$
$$ S = \nabla \times u. $$

In isotropic media, the S-mode consists of two degenerate waves which are indistinguishable from one-another. In the case of 2D wave propagation, the S-mode has only one nonzero component, and thus can be treated as a scalar. We can then use the imaging condition formulated by Yan and Sava (2008) to combine different wave-modes from source and receiver wavefields to obtain independent images for different combinations of incident and reflected wave modes:

$$ R^{ij}(x, t) = \sum_i W_s^i(x, t) W_r^j(x, t). $$

Here, indices $i$ and $j$ indicate the wave-mode used in imaging.

Figures 5(a)-6(b) show PP, SS, PS, and SP images computed from all the shots using mirror imaging, i.e., with receiver multiples back propagated from virtual receivers mirrored across the ocean surface. For all images in Figures 5(a)-6(b), we observe the reflector at its true depth of $z = 1.2$ km, but we also observe cross-talk artifacts due to interference between non-physical P- and S-modes. However, separating the true reflection events from cross-talk artifacts is not triv-
Figure 3. Different illumination patterns in the subsurface for primary (dashed lines) and water layer multiples (solid lines).

(a) Figure 4. Schematic representation of the receiver-side first-order surface-related multiples. The solid lines represent P-waves and the dashed lines represent S-waves. From (a) to (d), the panels correspond to PP, PS, SP and SS reflections.

Figure 5. The (a) PP and (b) SS migrated images of the OBS receiver-side multiples. The vertical line on the migrated image indicates the position of the shot-domain CIGs.

Figure 6. (a) PS and (b) SP migrated images of the OBS receiver-side multiples. The vertical line on the migrated image indicates the position of the angle-domain CIGs.

2.4 Polarity reversal

In isotropic media, the signs of PS and SP reflection coefficients change across normal incidence. Additional sign changes can occur at large angles of incidence, but in isotropic media, energy reflected at large incidence angles is generally as the artifacts can be stronger in amplitude than the true events. For example, in Figure 5(b), the true reflector (at depth \( z = 1.2 \text{ km} \)) is difficult to distinguish from the much stronger artifacts. Also, notice in the PS and SP images shown in Figure 6(a) and 6(b), respectively, that the reflector is less coherent near the center of the model. This lack of coherence is due to the change in sign of PS and SP reflection coefficients across normal incidence. By stacking over all shots, some of the migrated images have opposite polarity and interfere destructively. In the next sections, we discuss how to deal with polarity change, and attenuate the cross-talk artifacts.
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weaker, so the effects of these additional sign changes are negligible. The normal incidence change can be observed in angle-domain CIGs. For the synthetic model shown in Figures 1(a)-(c), we obtain the PS and SP images shown in Figures 6(a) and 6(b), respectively, by applying the imaging condition in equation 5. Polarity reversal can be performed using ray theory (Balch and Erdemir, 1994), wavenumber-domain separation (Du et al., 2011), or angle-domain imaging conditions (Yan and Sava, 2008; Du et al., 2011; Yan and Xie, 2012). In this paper, we follow Yan and Sava (2008) and reverse the polarity change for PS and SP images in the angle-domain. To obtain angle-domain CIGs, we first compute space-lag extended images (Rickett and Sava, 2002; Sava and Vasconcelos, 2009):

\[
R_i^s(x, \lambda) = \sum_t W_s^r(x, \lambda, t) W_i^s(x + \lambda, t). \tag{6}
\]

Here, \(x = \{x, y, z\}\) represents the space coordinates and \(\lambda = \{\lambda_x, \lambda_y, \lambda_z\}\) represents the space-lag vector. Then, we map the extended images to the angle domain \(R_i^\theta(x, \theta, \phi)\) using conventional methods (Sava and Fomel, 2003; Sava and Vlad, 2011). As discussed by Sava and Vlad (2011), it is not necessary to compute all space lags in order to obtain the opening angle \(\theta\) and azimuth angle \(\phi\). We can compute just two of the three space lags and then reconstruct the third lag with the given reflector normal.

The right panels of Figures 6(a) and 6(b) show angle gathers for the synthetic model (Figures 1(a)-(c)) at \(x = 3.0\) km after we reverse the polarity changes for PS and SP reflections, respectively. After applying the polarity reversal at each image point, we obtain the PS and SP images (Figures 6(a) and 6(b)) by stacking over angle gathers.

However, even after polarity reversal, the angle gathers (right panels of Figures 6(a) and 6(b)) still contain artifacts due to the sparse distribution of receivers and the nonphysical wave conversions. These artifacts might stack out during imaging, but we choose to eliminate them so that we can, for example, better use the gathers for velocity analysis. The following section details how we remove the sampling artifacts.

2.5 Artifact attenuation

Elastic reverse-time migration requires numerical reconstruction of source and receiver wavefields. For the source wavefield, we simulate an elastic wavefield using a pressure source located at a known position near the surface. For the receiverside wavefield, we simulate an elastic wavefield by using the recorded data, separated into primaries and multiples, as sources in the water column. Injected P-modes not only propagate through the ocean bottom as P-modes, but also convert into S-modes as they cross the liquid-solid interface. Some conversions correspond to physical S-modes that originate from reflectors in the subsurface, but other non-physical conversions lead to fake modes that do not represent wave propagation. Such fake modes produce artifacts in the migrated images. For example, in Figure 5(b), notice the artifacts that are apparent above the true reflector location (at depth \(z = 1.2\) km).

OBS acquisition geometry contains sparse receivers, which result in artifacts in the migrated images. Stacking over a dense network of shots is an effective way to reduce the artifacts and improve the signal-to-noise ratio. However, the reflector amplitude may decrease after stacking due to the presence of artifacts. These artifacts are generally inconsistent between different shots and across different incidence angles; therefore, they appear as events with anomalous moveout in gathers where they can be attenuated using different techniques, e.g., using Radon transforms (Sava and Guitton, 2005) or plane-wave destruction filters (Fomel, 2002). Such techniques assume that the velocity model used for imaging is accurate, so that the true reflection events are flat as a function of shot number or incidence angle, even when polarity changes exist, which is the case for PS and SP images. This artifact attenuation is necessary, since some artifacts can be significantly stronger than true reflection events. For example, a weak SS reflection can easily be overwhelmed by artifacts caused by fake PS, SP, or even PP reflections.

The depths of fake events are inconsistent with respect to shot location and incidence angle; therefore, in order to remove such artifacts, we must remove non-flat events in CIGs while preserving the amplitudes of flat events. This artifact attenuation can be implemented, for example, in the wavenumber domain using an f-k filter (Stewart and Schieck, 1989) that passes energy around \(k_z = 0\) and attenuates energy at \(k_z \neq 0\). In practice, however, we cannot achieve an ideal filter with
a sharp boundary in the wavenumber domain because such a filter has infinite extent in the original (CIG) domain. To obtain a finite-extent impulse response in the original domain, we window the ideal impulse response using a Hamming window (Oppenheim and Schafer, 2009).

The right panels of Figures 5(a) and 5(b) show the PP and SS shot-domain CIGs, respectively, at $x = 2.5$ km in the 2D synthetic model (Figures 1(a)-1(c)). As seen in the shot-domain CIGs— for example, a PP shot-domain CIG (left panel of Figure 8(a)) — the true reflection is characterized by a flat event as a function of shot number, while the artifacts appear as events with nonzero moveout. The corresponding CIG in the wavenumber domain for PP shot-domain CIGs is shown in the right panel of Figure 8(a). Notice that the energy of flat events is focused around $k_z = 0$. After we apply the f-k filter, only the events focused around $k_z = 0$ are preserved, as shown in Figure 8(b). Figures 9(a) and 9(b) show the PP and SS shot-domain CIGs after artifact attenuation. Choosing a proper filter in the wavenumber domain is tricky. On the one hand, we want the filter to be narrow enough to attenuate even those events with small moveout and still preserve the flat events. On the other hand, we know that the filter will cause smear effect and thus blurs edges of the events which will change the amplitude of the events. Therefore, it is important to choose a proper width for the f-k filter.

Similarly, in PS and SP angle-domain CIGs, the true reflections do not change in depth with incidence angle (right panels of Figures 6(a) and 6(b)). Therefore, we can also employ an f-k filter to attenuate crosstalk artifacts in the angle domain (Figures 9(c) and 9(d)). The artifacts, however, cannot be completely removed as some are nearly flat. Note that although we use shot-domain CIGs to attenuate artifacts in PP and SS images, we could also use angle-domain CIGs and follow the same artifact attenuation procedure used for PS and SP images. In summary, receiver-side water-column multiples can be used to generate elastic images. Similar to the acoustic case, we separate receiver-side water-column multiples from primaries in recorded OBS data, and then use mirror imaging to migrate the multiples. However, additional procedures are required in the elastic case, including reversing the polarity changes in PS and SP images and attenuating artifacts caused by non-physical wave-mode conversions. With these additional procedures, we are able to obtain PP, PS, SP, and SS images of the subsurface.

## 3 Examples

We demonstrate our method on a modified version of the Marmousi model shown in Figure 10. Compared to the original Marmousi model, our model contains a thicker water layer, which allows for wider illumination when we migrate the down-going waves from receiver-side multiples. The P-wave velocity, S-wave velocity, and density models are shown in the top, middle, and bottom panels of Figure 10, respectively. The ocean bottom is not smooth so as to facilitate wave-mode conversions at the ocean bottom to obtain PP, PS, SP, and SS images.

We generate synthetic data using 290 sources evenly distributed along the ocean surface. The source function is a Ricker wavelet with a peak frequency of 25 Hz. There are 30 receivers sparsely located at the ocean bottom from $x = 1.7$ km to $x = 4.0$ km, as indicated by the dots in Figure 10.

A subset of the PP angle-domain CIGs is shown in the middle panel of Figure 11. As we migrate using the correct velocity, the depths of the imaged reflectors do not change with shot location, and the true reflections appear flat. We apply the f-k filter discussed earlier to attenuate the artifacts in the PP angle-domain CIGs shown in the middle panel of Figure 11. As discussed earlier, this procedure can be carried out either in the angle domain or in the shot domain. For PS and SP images, it is convenient to perform artifact attenuation in the angle domain since polarity correction is also performed in this angle domain. For consistency with the PS and SP imaging, we also perform artifact attenuation in the angle domain for PP and SS images. After artifact attenuation, the gathers contain only flat events, indicating that the cross-talk artifacts have been attenuated and that the reflectors are correctly positioned in the subsurface.

We use the same f-k filter to attenuate artifacts in the SS angle-domain CIGs shown in the middle panel of Figure 14. In these CIGs, there are fewer and weaker reflections compared to the PP angle-domain CIGs (middle panel of Figure 11), as SS angle-domain CIGs are computed using the S-modes in the source and receiver wavefields, and the incident S-modes are converted at the water bottom.

The PS and SP angle-domain CIGs are shown in the mid-
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Figure 9. The (a) PP and (b) SS shot-domain CIG after artifact attenuation, and the (c) PS and (d) SP angle-domain CIG after artifact attenuation.

ddle panels of Figures 12 and 13, respectively. Notice the vertical stripes caused by the sparse receivers used for migration. The events with nonzero energy near normal incidence with nonzero energy in the angle-domain CIGs are artifacts resulting from non-physical converted waves at the ocean bottom. The bottom panels of Figures 12 and 13 show the angle-domain CIGs after we reverse the polarity change and apply artifact attenuation. Artifact attenuation eliminates the non-horizontal artifacts but does not remove the artifacts from wave conversions that are flat near zero incidence angle.

By stacking the CIGs (bottom panels of Figures 11, 12, 13, and 14) over angle, we obtain the PP, PS, SP, and SS images shown from top to bottom in Figure 15. Note that the few remaining artifacts seen in the PS image are due to the limited number of shots used in migration. The SS image (bottom panel of Figure 15) is lower amplitude compared to the other three images because the energy of S waves, arising from P-to-S conversions, is much weaker to begin with. However, the events in the middle of the image appear very sharp because S-waves have shorter wavelength compared to P-waves due to their lower propagation velocity. The low wavenumber events in the left and right parts of the image are migrated from S dividing waves, so they contain lower wavenumber content. These events are similar to the low-frequency backscattering artifacts seen in conventional RTM images.

This example demonstrates that elastic migration provides more information about the subsurface compared to acoustic migration. In addition to providing a PP image, which can be obtained using acoustic migration, elastic migration also provides PS, SP, and SS images. These additional images provide information about elastic reflection coefficients, which are related to subsurface material properties that and be used for reservoir characterization and petrophysical analysis. Also, PS, SP, and SS images can provide new geologic information, or they can simply provide added confirmation of structural information obtained from PP images.

Figure 10. The Marmousi model; from top to bottom, the P velocity model, the S velocity model and the density model. The dots represent the locations of the OBS.
Figure 11. PP migration results, depicting from top to bottom the conventional image, and the angle gathers before and after artifact attenuation. The angles are plotted in the $\pm 75^\circ$ range.

4 CONCLUSIONS

We propose a method for elastic imaging using OBS first-order receiver-side multiples. These multiples carry additional information about the subsurface, but more data processing steps are needed in order to place the observed data at correct locations in the subsurface and to attenuate artifacts caused by non-physical conversions at the water bottom.

We use angle-domain CIGs to reverse the polarity change at normal incidence and to attenuate other artifacts due to sparse acquisition and non-physical wave conversions arising during wavefield reconstruction. We apply an f-k filter to attenuate the artifacts in CIGs while preserving amplitudes of the true events.

Our method is subject to several assumptions and limitations. We perform elastic migration by migrating only receiver-side first-order water-column multiples. Higher-order multiples were regarded as noise, as they are weaker than first-order multiples. However, if we could identify the higher-order multiples, they could also be used for elastic migration in order to obtain images with wider illumination. Also, during migration, we use a sharp liquid/solid boundary at the ocean bottom which enables the wave-modes to converse at the ocean bottom. Such a sharp liquid-solid boundary at the ocean bottom does not always exist, and as a result, converted S-modes may be weak; thus reducing our ability to exploit down-going converted modes.

However, the elastic images still provide complementary information about the subsurface, and can be used to infer additional geologic information and to perform more accurate petrophysical analysis compared to more conventional images.
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obtained from acoustic data. The obtained elastic images are based on the physics of wave propagation and thus have clear physical meaning, which offers the potential for elastic velocity analysis.

5 ACKNOWLEDGMENTS

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Figure 14. SS migration results, depicting from top to bottom the conventional image, and the angle gathers before and after artifact attenuation. The angles are plotted in the ±75° range.


Figure 15. Migrated images for the Marmousi model. From top to bottom, the PP, PS, SP, and SS images, respectively.
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Wavelets and warping PS seismic images

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ABSTRACT
The process of warping a PS seismic image to align reflectors with those in a PP seismic image can cause wavelet distortion. This distortion can be reduced by deconvolving the PS wavelet from the PS image, warping to PP time, and then convolving with the PP wavelet. Moreover, this warping-with-wavelets algorithm can be used to estimate the required PS and PP wavelets. The primary difference between this new algorithm and previous methods is deconvolution of the PS wavelet before warping the PS image to PP time.

Key words: seismic image wavelet warping PP-PS registration

1 INTRODUCTION
Multicomponent data provide a different way to image the subsurface. For example, amplitude and phase differences between PP and PS data can be used to estimate subsurface properties (Garotta et al., 2002; Veire and Landro, 2006). Before analyzing these differences, we often first compensate for traveltme differences.

The process of warping (squeezing) a PS image to PP time distorts the seismic wavelet (Bansal and Matheney, 2010; Gaiser et al., 2011, 2013; Ursenbach et al., 2013). If the required amount of squeezing varies in space or time, then the warped PS wavelet will also vary. The resulting nonstationary warped PS wavelet can cause errors in inversions for subsurface properties such as density and P- and S-wave velocities (Jing and Rape, 2004; Veire and Landro, 2006; Khare and Rape, 2007; Bansal and Matheney, 2010).

A proposed solution to this problem of varying PS wavelet distortion has been to design a filter, for each time, that shapes the distorted PS wavelet to a single desired stationary PS wavelet (Bansal and Matheney, 2010). To design the necessary shaping filters, the original wavelet in the PS image must be extracted. Bansal and Matheney (2010) do not describe the wavelet extraction process, but given the original PS wavelet and the amount of squeezing, they compute a squeezed wavelet. The spectrum of the appropriate shaping filter is calculated by dividing the spectrum of the desired stationary PS wavelet by the spectrum of the squeezed wavelet. Applying time- and space-varying shaping filters computed in this way removes wavelet distortion from the squeezed PS image.
Another method of correcting wavelet distortion is proposed by Gaiser et al. (2011). This method uses non-stationary linear filters to modify the S-wave periods of warped PS-waves to match the periods of P-waves. This modification causes the warped PS-waves to resemble S-waves that have been transformed to P-wave time. Gaiser et al. (2011) point out that this method is able to better match the PS-wave data with the P-wave data than if only warping was applied, but wavelet distortions occur because the average $V_p/V_s$ ratios are assumed to be constant over the time of the wavelet. Gaiser et al. (2013) remove this assumption to reduce wavelet distortion and apply filters that compress PS wavelets to yield results similar to those in Gaiser et al. (2011).

The common step to correct wavelet distortion in the previously proposed solutions is to apply a filter after warping the PS image to PP time. We propose a different solution that includes deconvolving the PS wavelet before warping, based on the observation that the problem of wavelet distortion would not exist if the wavelet was an impulse. Moreover, our warping-with-wavelets solution provides a method for estimating the required PS wavelet.

In this paper we first describe the warping-with-wavelets algorithm designed to reduce wavelet distortion. We then describe how wavelet-estimation is possible with this algorithm. Finally, we apply a variant of this algorithm to PP and PS images.

## 2 WARPING WITHOUT DISTORTION

Suppose that the wavelet in a seismogram is a delta function $\delta(t)$. Figures 2a and 2b represent two simple synthetic seismograms:

\[
p(t) = \delta(t - t_1) - \delta(t - t_2),
q(t) = \delta(t - 2t_1) - \delta(t - 2t_2),
\]

such that

\[
p(t) = 2q(2t).
\]

Equation 2 is a special case of a more general relationship,

\[
p(t) = u'(t)q(u(t)),
\]

where $u(t)$ is the mapping from time in $q(t)$ to time in $p(t)$. The amplitude scaling by $u'(t)$ in equations 2 and 3 is necessary because of the squeezing of the impulsive wavelet $\delta(t) = 2\delta(2t)$. Notice that the sequence $2q(2t)$ displayed in Figure 2c exhibits no wavelet distortion.

Let us now consider a non-impulsive wavelet $h(t)$ with $Z$ transform

\[
H(z) = \frac{1 - 0.95z}{(1 - 1.75z + 0.82z^2)(1 - 1.32z^{-1} + 0.82z^{-2})}
\]

and synthetic seismograms computed by convolving this wavelet with the sequences of impulses:

\[
f(t) = h(t) * p(t),
g(t) = h(t) * q(t).
\]

This convolution with the wavelet $h(t)$ complicates the relationship between $f(t)$ (Figure 3a) and $g(t)$ (Figure 3b), so that $f(t) \neq 2g(2t)$ (Figure 3c). Figure 3 illustrates that, where the wavelet is not an impulse, simply warping one trace to align with another trace will distort the wavelet. The examples in Figures 2 and 3 suggest that deconvolution of the wavelet should be performed before warping. That is, letting $a(t)$ denote the inverse of the wavelet $h(t)$ such that $a(t) * h(t) = \delta(t)$, we should (1) convolve with $a(t)$, (2) warp, and (3) convolve with $h(t)$. We call this process warping-with-wavelets.
2.1 Linear Operators

The sampled wavelet \( h(t) \), inverse wavelet \( a(t) \), and sequences \( f(t) \) and \( g(t) \) can be represented by column vectors \( \mathbf{h}, \mathbf{a}, \mathbf{f}, \) and \( \mathbf{g} \), respectively. For illustration, let us suppose that the inverse wavelet \( \mathbf{a} \) has three coefficients \( (a_0, a_1, a_2) \), such that convolution of \( \mathbf{a} \) with \( \mathbf{g} \) is represented by

\[
\mathbf{A} \mathbf{g} = \begin{bmatrix}
a_0 & 0 & 0 & 0 & 0 \\
a_1 & a_0 & 0 & 0 & 0 \\
a_2 & a_1 & a_0 & 0 & 0 \\
0 & a_2 & a_1 & a_0 & 0 \\
0 & 0 & a_2 & a_1 & a_0
\end{bmatrix}
\begin{bmatrix}
g_0 \\
g_1 \\
g_2 \\
g_3 \\
g_4
\end{bmatrix},
\]

or, equivalently,

\[
\mathbf{G} \mathbf{a} = \begin{bmatrix}
g_0 & 0 & 0 \\
g_1 & g_0 & 0 \\
g_2 & g_1 & g_0 \\
g_3 & g_2 & g_1 \\
g_4 & g_3 & g_2
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2
\end{bmatrix}.
\]

Note that columns of \( \mathbf{A} \) contain only delayed copies of \( \mathbf{a} \) and columns of \( \mathbf{G} \) contain only delayed copies of \( \mathbf{g} \), so that both \( \mathbf{A} \) and \( \mathbf{G} \) are Toeplitz matrices. Also recall that convolution is commutative; \( \mathbf{A} \mathbf{g} = \mathbf{G} \mathbf{a} \). Convolution of the inverse wavelet \( \mathbf{a} \) with \( \mathbf{g} \) is equivalent to deconvolution of the wavelet \( \mathbf{h} \) from \( \mathbf{g} \).

Warping is defined by the linear operator \( \mathbf{S} \). If one squeezes \( \mathbf{g} \), the frequency spectrum of \( \mathbf{g} \) is stretched, which could cause aliasing. Therefore, before squeezing, we apply a low-pass filter to \( \mathbf{g} \) to attenuate frequencies that would otherwise be aliased after warping. We include this low-pass anti-alias filter in the warping operator \( \mathbf{S} \). The result of applying \( \mathbf{S} \) to \( \mathbf{Ag} \) is a vector \( \mathbf{SAg} \) containing the warped result of deconvolving wavelet \( \mathbf{h} \) from \( \mathbf{g} \). Because \( \mathbf{S} \) is a time-varying operator, it does not commute with \( \mathbf{A} \); \( \mathbf{SA} \neq \mathbf{AS} \). The final step in our warping-with-wavelets algorithm is to convolve \( \mathbf{h} \) with \( \mathbf{SAg} \) so that

\[
f = \mathbf{HSAg}.
\]

Figure 4 shows that wavelet distortion is reduced by using this algorithm to warp \( g(t) \) to \( f(t) \). Note that, although \( \mathbf{HA} = \mathbf{I} \), \( f = \mathbf{HSAg} \neq \mathbf{HASg} = \mathbf{Sg} \), because \( \mathbf{S} \) and \( \mathbf{A} \) do not commute. In other words, as illustrated in Figure 4, \( f \neq \mathbf{Sg} \); warping alone is inadequate.

3 ESTIMATING THE WAVELET

To warp seismic traces without wavelet distortion, the wavelet \( \mathbf{h} \), or equivalently, its inverse \( \mathbf{a} \), must be known. Let us now consider how we might use equation 8 to estimate \( \mathbf{a} \) and, hence, \( \mathbf{h} \).

Multiplying both sides of equation 8 by \( \mathbf{A} \), we obtain

\[
\mathbf{Af} = \mathbf{SAg}.
\]

Because convolution is commutative, we can rewrite equation 9 as

\[
\mathbf{Fa} = \mathbf{SGa},
\]

or

\[
\mathbf{Fa} = \mathbf{SGa} = \mathbf{0}.
\]

Now define \( \mathbf{D} = \mathbf{F} - \mathbf{SG} \), so that

\[
\mathbf{Da} = \mathbf{0}.
\]

The number of columns in \( \mathbf{D} \) equals the number of unknown coefficients in the inverse wavelet \( \mathbf{a} \), so for coefficients \( (a_0, a_1, a_2) \) we have

\[
\mathbf{D} = \begin{bmatrix} d_0 & d_1 & d_2 \end{bmatrix},
\]

where

\[
\begin{align*}
d_0 &= f_0 - Sg_0, \\
d_1 &= f_1 - Sg_1, \\
d_2 &= f_2 - Sg_2.
\end{align*}
\]

Here, \( f_0 \) is simply \( f \), \( f_1 \) is \( f \) delayed by one sample, and \( f_2 \) is \( f \) delayed by two samples. Likewise, \( g_0 \) is simply \( g \), \( g_1 \) is \( g \) delayed by one sample, and \( g_2 \) is \( g \) delayed by two samples. Notice that in computing the matrix \( \mathbf{D} \), the warping operator \( \mathbf{S} \) is applied multiple times to different delayed versions of \( g \).

The trivial solution to equation 12 is \( \mathbf{a} = \mathbf{0} \). We eliminate this solution by setting the coefficient \( a_0 = 1 \). We could set \( a_0 \) to any value, but this would only scale the coefficients of the wavelet \( h \) by \( a_0 \). Using equation 12 alone, the true amplitudes of \( \mathbf{h} \) and \( \mathbf{a} \) cannot be recovered; only their shapes can be estimated. With \( a_0 = 1 \), equation 12 becomes

\[
\begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} a_1 \\

In equation 15, we have as many equations as time
samples in $f$ and $g$, but only two unknowns $a_1$ and $a_2$, which leads us to use the least-squares method and minimize $\|d_0 + a_1d_1 + a_2d_2\|_2$. To compute $a_1$ and $a_2$, we solve the normal equations obtained by multiplying both sides of equation 15 by $[d_1^\top d_2]^\top$:

$$
\begin{bmatrix}
    d_1^\top d_1 & d_1^\top d_2 \\
    d_2^\top d_1 & d_2^\top d_2
\end{bmatrix}
\begin{bmatrix}
    a_1 \\
    a_2
\end{bmatrix} =
\begin{bmatrix}
    -d_1^\top d_0 \\
    -d_2^\top d_0
\end{bmatrix}.
$$

(16)

The leftmost matrix in equation 16 is not Toeplitz because the matrix $S$ is a time-varying operator that is applied to delayed copies of $g$. This matrix is however symmetric positive semidefinite, which enables us to solve equation 16 by Cholesky decomposition. A similar system of equations can be obtained for any number of coefficients in the inverse wavelet $a$, which need not be causal.

After estimating the inverse wavelet $a$, we easily recover the wavelet $h$ as the filter that shapes the inverse wavelet $a$ to a unit impulse (e.g., Robinson and Treitel, 2000). Then, with the estimated $h$ and $a$, we can apply the warping-with-wavelets algorithm; that is, we can compute $HSAg$.

We tested this wavelet-estimation process using the sequences $f$ and $g$ displayed in Figures 3a and 3b, respectively. We chose to estimate large numbers of coefficients in the inverse wavelet $a$ and the wavelet $h$, which were 81 and 181, respectively, because in this example the known wavelet and its inverse are both infinitely long. (Recall equation 4.) Figure 5 shows that the estimated wavelet is nearly identical to the known wavelet. However, it is important to recall that for this simple example, the wavelet in $f$ is identical to that in $g$.

4 APPLICATION TO PP-PS IMAGES

To estimate wavelets and to reduce wavelet distortion caused by warping a PS image (Figure 6b) to a PP image (Figure 6a), a modification of the warping-with-wavelets algorithm described above is required. Wavelets in PP and PS images are unlikely to be identical, in part, because attenuation often affects S-waves more than it does P-waves (Ursenbach et al., 2013). This difference leads us to use an estimated inverse PS wavelet $a$ to deconvolve the PS wavelet from the PS image, allowing us to warp the resulting image to PP time without wavelet distortion. Although the presence of noise in the PS image will prevent us from completely deconvolving the PS wavelet from the PS image, we show below an example of reduced wavelet distortion in the presence of noise. Then, after we warp the deconvolved PS image, we convolve with the estimated PP wavelet $h$ to obtain a PS image with a PP wavelet. Ideally, remaining differences in the two images should be due to differences in PP and PS reflectivity.

Before estimating the PS inverse wavelet $a$ and the PP wavelet $h$ and applying the warping-with-wavelets algorithm, we first apply a time-varying gain to the PP and PS images displayed in Figures 6a and 6b, respectively, so that PP and PS amplitudes are comparable. Let the PP and PS images be represented by column vectors $f$ and $g$, respectively. We found the time shifts used to warp the PS image $g$ to the PP image $f$ by smooth dynamic warping (Hale and Compton, 2013) and this warping of $g$ to $f$ is again represented by the linear operator $S$.

In equation 8 we have two unknown sequences, $h$ and $a$, and if $HA \neq I$, then we cannot eliminate the unknown wavelet $h$ by simply multiplying both sides of equation 8 by $A$. Instead of solving for both $h$ and $a$ simultaneously, we iteratively compute one and then the other.

We begin this iterative process by letting $a = \delta$. We then compute $q = Sq$, which simplifies equation 8 to

$$
f = Hq,
$$

(17)
or, equivalently,

$$
f = Qh,
$$

(18)

where columns of the matrix $Q$ contain delayed copies of $q$.

In equation 18, we have as many equations as we have samples in the PP and PS images $f$ and $g$, and we have a significantly smaller number of unknown PP wavelet coefficients in $h$. The relatively small number of unknowns leads us to compute $h$ using the least-squares method to minimize $\|f - Qh\|_2$. Specifically, we solve the normal equations obtained by multiplying both sides of the equation 18 by $Q^\top$:

$$
Q^\top Qh = Q^\top f.
$$

(19)

The matrices $Q$, $Q^\top$, and $Q^\top Q$ are Toeplitz because columns of $Q$ contain delayed copies of $q$. Therefore, the resulting PP wavelet $h$ is a filter that shapes the warped...
PS image \( q \) to match the PP image \( f \). In other words, the estimated PP wavelet \( h \) is similar to the shaping filters designed by Bansal and Matheney (2010), but here is time-invariant. At this point, we have completed one iteration.

We then use the estimated PP wavelet \( h \) to compute \( P = HSG \), which reduces equation 8 to

\[
f = Pa.
\]  

In equation 20, we again have as many equations as samples in the PP and PS images \( f \) and \( g \) and a significantly smaller number of unknown inverse PS wavelet coefficients in \( a \). The relatively small number of unknowns compared to equations again leads us to use the least-squares method and minimize \( \|f - Pa\|_2 \). That is, we solve the normal equations obtained by multiplying both sides of equation 20 by \( P^\top \):

\[
P^\top Pa = P^\top f.
\]  

Here the matrices \( P \), \( P^\top \), and \( P^\top P \) are not Toeplitz because the time-varying operator \( S \) in \( P = HSG \) is applied to all delayed copies of \( g \) in the columns of \( G \). Therefore, we compute the PS inverse wavelet \( a \) using Cholesky decomposition of \( P^\top P \), instead of using faster solvers appropriate for Toeplitz matrices. Notice, in equation 20, that if we multiply the PP wavelet \( h \) by a constant and divide the estimated inverse PS wavelet \( a \) by the same constant, then the matrix on the right-hand side will be unchanged, which means that only the shape of the inverse PS wavelet \( a \) can be recovered, not its true amplitudes. This same logic can be applied to equation 18 (where we are solving for the PP wavelet \( h \)), meaning that only the shape of the PP wavelet \( h \) can be recovered, not its true amplitudes.

We next use equation 18 to compute an updated PP wavelet \( h \); but, instead of using the inverse PS wavelet \( a = \delta \), we use the most recent estimate of \( a \). We then compute \( q = SAg \) and solve equation 19 for an updated PP wavelet \( h \). At this point, we have completed two iterations. We can repeat this iterative process for any number of iterations, but in the examples below we use 11 iterations.

The PP wavelet \( h \) and the PS wavelet corresponding to the inverse PS wavelet \( a \) estimated in the first iteration are shown in Figures 7a and 7b, respectively. After only one iteration, the PS wavelet is an impulse because the inverse PS wavelet \( a = \delta \) is initialized to be an impulse \( \delta \), which implies that \( f = HSg \). The PP wavelet \( h \) is a filter that shapes the warped PS image \( Sg \) to the PP image \( f \). If we use the impulse inverse PS wavelet \( a = \delta \) and the estimated PP wavelet \( h \) to implement the modified warping-with-wavelets algorithm, we obtain the warped PS image shown in Figure 9c. The

---

**Figure 6.** A subset of a PP image (a) and a roughly corresponding subset of a PS image (b). Note the differences in PP and PS time scales.
resulting rms (root-mean-square) difference between \( f \) and \( \text{HSAg} \) is 0.672.

After 11 iterations, we obtain the PP wavelet \( h \) and PS wavelet (corresponding to the estimated inverse PS wavelet \( a \)) shown in Figures 8a and 8b, respectively. Figure 9b displays the image \( \text{HSAg} \) computed using the modified warping-with-wavelets algorithm, with the estimated inverse PS wavelet \( a \) and PP wavelet \( h \). The rms difference between \( f \) and \( \text{HSAg} \) is 0.666, which is slightly smaller than the rms difference between \( f \) and \( \text{HSAg} \) after only one iteration. Here, we stopped after 11 iterations in the wavelet-estimation process, when the rate of decrease in the rms difference between \( f \) and \( \text{HSAg} \) was less than 0.00005 per iteration.

5 DISCUSSION

Although the inverse PS wavelet \( a \) and PP wavelet \( h \) obtained after one and 11 iterations differ significantly, the results of using the modified warping-with-wavelets algorithm for one (\( \text{HSg} \)) and 11 (\( \text{HSAg} \)) iterations are similar (Figures 9c and 9b, respectively). \( \text{HSAg} \) and \( \text{HSg} \) are more similar to \( f \) (Figure 9a) than is \( \text{SSg} \) (Figure 9d), due to a reduction in wavelet distortion in both \( \text{HSAg} \) and \( \text{HSg} \). In the shallow portions of \( \text{HSAg} \) and \( \text{HSg} \), highlighted by the black rectangles in Figures 9b and 9c, \( \text{HSAg} \) differs somewhat from \( \text{HSg} \) because the amount of squeezing applied by \( S \) varies most rapidly there. The small reduction (0.672 to 0.666) in rms differences of \( f \) compared to \( \text{HSg} \) and \( f \) compared to \( \text{HSAg} \) may be a result of only this small portion of the warped PS image having varying squeezing.

In any case, a single shaping filter cannot shape \( Sg \) to \( f \). Stewart et al. (2002) observe that PP and PS waves often have different reflection coefficients. This difference in reflection coefficients is part of the reason why the estimated PP wavelet (shaping filter) \( h \) cannot shape \( q = SAg \) to exactly equal \( f \) as shown in equation 18. Indeed, in warping with wavelets we seek to preserve any differences in reflection coefficients.

However, the PP wavelet \( h \) does reduce noise in warped images \( \text{HSAg} \) (Figure 9b) and \( \text{HSg} \) (Figure 9c) because the PP image has less noise than the PS image and the PP wavelet \( h \) will match the noisy \( \text{HSg} \) or \( \text{HSAg} \) to the less noisy PP image \( f \).

In our modified warping-with-wavelets example, we estimated one inverse PS wavelet \( a \) and one PP wavelet \( h \) for the PS and PP images, which implies that we assumed that these wavelets do not vary in time or space. However, we could modify the warping-with-wavelets algorithm to estimate multiple wavelets as a function of time and space. Whether we estimate one wavelet or multiple wavelets, the key is to deconvolve the wavelet in the PS image before warping that image.
6 CONCLUSION

Two warping-with-wavelets algorithms are proposed to minimize wavelet distortion caused by warping. One algorithm assumes a single wavelet for both the PP and PS images and the other algorithm assumes different wavelets in those images. In the case of different wavelets, one filter is designed to deconvolve the PS wavelet from the PS image before warping and another filter is designed to convolve the PP wavelet with the warped deconvolved PS image. This process reduces wavelet distortion caused by warping, while yielding estimates of PP and PS wavelets.

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Horizon volumes with interpreted constraints

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ABSTRACT
We propose two methods for constructing seismic horizons aligned with reflectors in a 3D seismic image. The first method extracts horizons one at a time; the second generates at once an entire volume of horizons. The most significant new aspect of both methods is the ability to specify, perhaps interactively during interpretation, a small number of control points that may be scattered throughout a 3D seismic image. Examples show that control points enable the accurate extraction of horizons from seismic images in which noise, unconformities, and faults are apparent. These points represent constraints that we implement simply as preconditioners in the conjugate gradient method used to construct horizons.

Key words: seismic horizon volume RGT flattening normal vectors

1 INTRODUCTION
In seismic interpretation, by visually tracking or automatically extracting surfaces throughout a 3D seismic image along amplitude peaks or troughs, we are able to identify seismic horizons. These horizons correspond to stratal surfaces which are primary beddings or ancient depositional surfaces that are geologically synchronous (Vail et al., 1977). Color-coding of horizons based on amplitude or other attributes can help reveal ancient depositional environments and geomorphic features (Posamentier et al., 2007). Therefore, extracting horizons from seismic images is a common and important problem for seismic interpretation.

1.1 Horizon volume
Lomask (2010a,b) first presented the concept of a “horizon volume” (Figure 2c), which can be generated from a seismic image (Figure 2a) and used to flatten reflec-
tors (Figure 2d) or to access all horizons at once. A horizon volume \( z(x, y, \tau) \) (Figure 2c) contains horizon depth \( z \) as a function of relative geologic time (RGT) \( \tau \) and horizontal spatial coordinates \( x \) and \( y \). Therefore, horizontally slicing a horizon volume yields the spatial locations \((x, y, z)\) of a horizon corresponding to a constant RGT \( \tau \).

The concept of an “RGT volume” (Figure 2b), first presented by (Stark, 2005), is closely related to the “horizon volume.” An RGT volume \( \tau(x, y, z) \) (Figure 2b) contains RGT \( \tau \) as a function of spatial coordinates \( x, y \) and \( z \). The contours of constant \( \tau \) in an RGT volume correspond to seismic horizons. The only difference between an RGT volume and its corresponding seismic image is that the value of a sample in an RGT volume represents geologic time rather than seismic amplitude (Stark, 2005).

Given an RGT volume \( \tau(x, y, z) \) with \( \tau \) monotonically increasing with its vertical coordinate \( z \), a horizon volume \( z(x, y, \tau) \) can be easily obtained via an inverse linear interpolation method, as discussed by Parks (2010). In practice, we use both volumes to conveniently access horizons. An RGT volume, with axes identical to a seismic image, is first used to look up the RGT value \( \tau \) for a horizon we wish to extract. A horizon volume is then used to directly obtain the spatial coordinates for the horizon by simply horizontally slicing the horizon volume for that \( \tau \).

1.2 Previous methods

Methods for obtaining a horizon volume can be generally classified into three categories. The first is stratal slicing (Zeng et al., 1998a,b), which uses several reference horizons to interpolate a set of horizons that form a horizon volume. With a limited number of horizons for control, the interpolated horizon volume can follow large-scale features but usually cannot resolve local features (Lomask, 2013).

The second category of methods uses seismic reflector dips (Lomask et al., 2006; Parks, 2010) or, equivalently, seismic normal vectors, computed for every image sample to be perpendicular to seismic reflectors (Luo and Hale, 2012). In these methods, a horizon volume is explicitly (Lomask et al., 2006) or implicitly (Parks, 2010; Luo and Hale, 2012) generated to map a seismic image from the depth-space domain to a flattened image in the RGT-space domain.

The third category is similar to the second one in that these methods flatten an entire 3D seismic image, but without use of dips or normal vectors. Instead, they use an RGT volume generated by unwrapping a corresponding seismic instantaneous phase image (Stark, 2005; Wu and Zhong, 2012a). The last two categories of methods automatically identify all horizons in a seismic image at once. Horizon volumes generated by these methods are more accurate for revealing local features than those interpolated from several horizons using the first category of methods.

1.3 This paper

In this paper, we first describe a method for extracting single horizons, one at a time, by using precomputed seismic normal vectors which are normal to seismic reflectors. This method requires at least one control point to indicate the horizon (containing this point) that we want to extract and to initialize a horizontal surface passing through this point. The initial surface is typically inconsistent with the desired horizon, but it is iteratively deformed until vectors normal to the surface are aligned with vectors normal to a reflector in the seis-

**Figure 2.** From a seismic image (a), a RGT volume (b) is computed and then converted to a horizon volume (c) that maps the seismic image to a flattened image (d).
mic image. We extend this method to permit additional control points, which enable reliable extraction of a sequence boundary or a horizon complicated by faults or noise.

We then introduce a second method that generates a complete horizon volume constrained by multiple sets of control points. To generate a horizon volume (Figure 2c), we first use seismic normal vectors to compute an RGT volume (Figure 2b), from which a horizon volume is then interpolated. This process is similar to Park’s (2010) method for flattening a seismic image, but we instead derive the method in a simpler way. Furthermore, similar to the way in which we extract a more accurate horizon using control points, we use multiple sets of control points to generate a more accurate horizon volume from a seismic image complicated by faults or noise. Each set of control points belongs to a single horizon with an unspecified RGT value, and is easily specified in seismic interpretation by simply selecting points that we want to lie on the same horizon. We implement these constraints with simple preconditioners in the conjugate gradient (CG) algorithm we use to compute the RGT and horizon volumes.

2 EXTRACTING A SINGLE HORIZON

To extract or construct a single horizon from a 3D seismic image, one usually first picks a reference point or seed. This seed then grows to a horizon surface by manually or automatically tracking seismic reflectors along seismic amplitude peaks or troughs.

Here, we describe a different method that uses at least one control point to initialize a complete horizontal surface and then updates that surface to conform to seismic normal vectors. We then extend this method to enable use of multiple control points, which improve both the accuracy and efficiency of horizon extraction.

2.1 Horizon extraction without constraints

We first use structure tensors (van Vliet and Verbeek, 1996; Fehmers and Höcker, 2003) to compute, for each image sample, a unit vector $n = [n_x \ n_y \ n_z]^T$ that is perpendicular to the seismic reflector at that sample location (Luo and Hale, 2012). We then assume a single-valued horizon surface $z = f(x, y)$ and define the surface implicitly by

$$U(x, y, z) = z - f(x, y).$$

Then unit vectors normal to the surface are

$$n = \frac{\nabla U(x, y, z)}{||\nabla U(x, y, z)||} = \alpha \begin{bmatrix} -\frac{\partial f}{\partial x} \\ -\frac{\partial f}{\partial y} \\ 1 \end{bmatrix},$$

where $\alpha$ is a scale factor that makes $n$ a unit vector.

The seismic horizon we seek is a surface whose normal vectors $n$ equal the seismic normal vectors $n$ at all positions $(x, y, z)$ on the horizon. However, we initially do not know the positions of the horizon. To solve this problem, we must iteratively update an initial horizontal surface $f^0(x, y)$, by solving

$$\alpha^i \begin{bmatrix} -\frac{\partial f^i}{\partial x} \\ -\frac{\partial f^i}{\partial y} \\ 1 \end{bmatrix} = \begin{bmatrix} n_x^{i-1} \\ n_y^{i-1} \\ n_z^{i-1} \end{bmatrix}. \quad (3)$$

Here, $f^i(x, y)$ is the surface to be updated at the $i$-th iteration; $n_x^{i-1} = n_x(x, y, f^{i-1}(x, y))$, $n_y^{i-1} = n_y(x, y, f^{i-1}(x, y))$, and $n_z^{i-1} = n_z(x, y, f^{i-1}(x, y))$ are the components of seismic normal vectors at positions on the surface obtained in the $(i - 1) - \text{th}$ iteration.

To start this iterative process, we initialize a horizontal surface $f^0(x, y)$ (black lines in Figure 3a) passing through a control point (green circle in Figure 3a) that is located on the seismic horizon we want to extract. This initial surface is then iteratively updated to align with the seismic horizon. In each iteration, we eliminate $\alpha^i$ by setting $\alpha^i = n_z^{i-1}$, and then solve the following inverse-gradient problem (Farnebäck et al., 2007) to update the surface $f^i(x, y)$:

$$\begin{bmatrix} \frac{\partial f^i}{\partial x} \\ \frac{\partial f^i}{\partial y} \end{bmatrix} = \begin{bmatrix} p^{i-1} \\ q^{i-1} \end{bmatrix}, \quad (4)$$

where $p^{i-1} = -n_x^{i-1}/n_z^{i-1}$ and $q^{i-1} = -n_y^{i-1}/n_z^{i-1}$ are reflector slopes in the $x$ and $y$ directions, respectively. These two equations should be satisfied for every sample on the horizon, but it usually helps to weight these equations by some measure $w(x, y, z)$ of the quality of the estimated reflector slopes. For example, $w(x, y, z)$ can be a measure of local planarity in the seismic image, easily computed from structure tensors (Hale, 2009). Then,

$$w^{i-1} \begin{bmatrix} \frac{\partial f^i}{\partial x} \\ \frac{\partial f^i}{\partial y} \end{bmatrix} = w^{i} \begin{bmatrix} p^{i-1} \\ q^{i-1} \end{bmatrix}, \quad (5)$$

where $w^{i-1} = w(x, y, f^{i-1}(x, y))$.

Assuming we have $N$ sampled locations on the horizon surface, we will have $2N$ weighted equations for the $N$ unknowns $f^i(x, y)$. For each iteration, we discretize these equations to obtain the corresponding matrix form

$$WGF = Wv,$$  

where $W$ is a $2N \times 2N$ diagonal matrix containing weights $w(x, y, f^{i-1}(x, y))$, $G$ is a $2N \times N$ sparse matrix obtained by discretizing partial derivatives, $v$ is a $2N \times 1$ vector containing the seismic reflector slopes $p^{i-1}$ and $q^{i-1}$ on the surface $f^{i-1}(x, y)$ obtained in the previous iteration, and $f$ is an $N \times 1$ vector containing surface
depths \( f'(x, y) \) we want to find.

Because this inverse gradient problem has more equations than unknowns, we compute its least-squares solution by solving the normal equations

\[
(WG)^\top WG f = (WG)^\top Wv.
\]  
(7)

To simplify this equation, we let \( A = (WG)^\top WG \) and \( b = (WG)^\top Wv \) to obtain

\[
Af = b.
\]  
(8)

It is convenient that the matrix \( A = G^\top W^\top WG \) is symmetric positive definite (SPD), as this enables use of the CG method to solve this linear system.

2.1.1 Preconditioner

To accelerate the convergence of CG iterations, Harlan (1995) suggests use of a model reparameterization \( f = \tilde{S}f \), where \( S \) is a simplification operator designed to create the desired features in the solution \( f \). Applying this technique to the system of equation 8, we first solve a new system

\[
S^\top AS\tilde{f} = S^\top b
\]  
(9)

for the new unknowns \( \tilde{f} \) and then compute the desired solution \( f = \tilde{S}f \). For an appropriate operator \( S \), the CG method applied to the new system of equation 9 converges much faster than for the original system of equation 8.

In effect, this model reparameterization is equivalent to split preconditioning (Saad, 1996) with left and right preconditioners \( M_L^{-1} = S^\top \) and \( M_R^{-1} = S \). As noted by Saad (1996) and others, this split preconditioning can be implemented with a left preconditioning matrix \( M = M_L M_R \) in a preconditioned CG solution of

\[
M^{-1}Af = M^{-1}b,
\]  
(10)

where \( M^{-1} = SS^\top \).

Recall that \( S \) is a simplification operator used to facilitate desired features in the solution (Harlan, 1995). Here, we implement \( S \) as a smoothing operator \( S = S_x S_y \), where \( S_x \) and \( S_y \) are axis-aligned smoothing filters in the \( x \) and \( y \) directions, respectively. A horizon surface \( f \) is often smooth, except at faults. Therefore, our \( S_x \) and \( S_y \) are spatially variant smoothing filters (Hale, 2009), with the extent of smoothing controlled by a measure of discontinuity of seismic reflectors. This measure could be planarity (Hale, 2009) or fault likelihood (Hale, 2013). Here we use planarity, computed from structure tensors, to control the extent of smoothing in \( S_x \) and \( S_y \).

Now, for each iteration (equation 5) that updates the surface \( f'(x, y) \), we solve equation 8 using the preconditioned CG method with preconditioner \( M^{-1} = S_x S_y S_y^\top S_x^\top \).

In this way, we iteratively update the surface \( z = f(x, y) \) until its normal vectors \( n \) are aligned with the seismic normal vectors \( n(x, y, z = f(x, y)) \).

In summary, given an initially horizontal surface (black curves in Figure 3) that is inconsistent with any seismic reflector, our method iteratively reduces the difference between the normal vectors \( n \) of the surface and the seismic normal vectors \( n(x, y, f(x, y)) \) on the surface to obtain a single seismic horizon surface (blue curves in Figure 3).

2.2 Results without constraints

In Figure 3, using only one control point to indicate which horizon we want to extract, our method updates the initially horizontal surface to the more nearly correct seismic horizon (blue curves in Figure 3) after 9 iterations. The extracted surface is well-aligned with the seismic horizon at conformable areas in the left section of Figure 3a. However, in the sections shown in Figure 3b, this iterative method fails to update the horizon surface to the location of the angular unconformity (green dashed curve in Figure 3b).

Extracting such a sequence boundary or unconformity is an important but difficult problem in seismic interpretation. From structure tensors, we failed to correctly estimate the discontinuous normal vectors at the unconformity and therefore obtained the incorrect horizon surface shown in Figure 3b. In the next section, we will describe a method to correctly extract a sequence boundary using control points.

2.3 Horizon extraction with constraints

Near unconformities, faults, or in areas where an image is noisy, estimated seismic normal vector are not accurate enough to automatically obtain a correct sequence boundary or horizon. Therefore, instead of using a fully automatic method, we might manually interpret the seismic image to obtain a more geologically reasonable surface. However, we need not manually interpret the entire horizon. Using a small number of control points as constraints, we solve a constrained least-squares problem to efficiently and accurately extract a sequence boundary or horizon from a noisy or complex seismic image.

2.3.1 Constrained optimization

As discussed above, in each iteration that updates a horizon surface, we solve a linear system \( Af = b \) for the vector \( f \) that represents the surface. Because the matrix \( A \) is SPD, solving this linear system is equivalent to minimizing the following quadratic function of the vector \( f \):

\[
F(f) = \frac{1}{2} f^\top Af - b^\top f.
\]  
(12)
Suppose we have a set of \( n \) control points \((x_i, y_i, z_i), i = 1, 2, \ldots, n\), and we want to extract a horizon surface that exactly passes through these points. With these constraints, we obtain a constrained optimization problem:

\[
\begin{align*}
\text{minimizer} & \quad F(\mathbf{f}) = \frac{1}{2} \mathbf{f}^\top \mathbf{A} \mathbf{f} - \mathbf{b}^\top \mathbf{f}, \\
\text{subject to} & \quad \mathbf{C} \mathbf{f} = \mathbf{z},
\end{align*}
\]

where \( \mathbf{z} = [z_1, z_2, \ldots, z_n]^\top \) is an \( n \times 1 \) column vector, and \( \mathbf{C} \) is an \( n \times N \) (where, again, \( N \) is the number of depths sampled on the surface) sparse matrix with ones at the positions corresponding to control points and zeros elsewhere. Assuming we have found some solution \( \mathbf{f}_0 \) to the constraint equation \( \mathbf{C} \mathbf{f}_0 = \mathbf{z} \), and a matrix \( \mathbf{Z} \) whose columns form a basis for the null space of \( \mathbf{C} \) so that \( \mathbf{C} \mathbf{Z} = \mathbf{0} \), then any solution \( \mathbf{f} \) of the constraint equation \( \mathbf{C} \mathbf{f} = \mathbf{z} \) can be written as

\[
\mathbf{f} = \mathbf{f}_0 + \mathbf{Z} \mathbf{p},
\]

where \( \mathbf{p} \) is a reduced \((N - n) \times 1\) column vector, and again \( n \) is the number of control points. The control points must be unique to ensure that the matrix \( \mathbf{C} \) has \( n \) linearly independent rows and \( \mathbf{Z} \) has \( N - n \) linearly independent columns.

Substituting equation 14 into equation 12, we obtain a quadratic function \( F(\mathbf{p}) \) with the reduced vector \( \mathbf{p} \):

\[
F(\mathbf{p}) = \frac{1}{2} (\mathbf{f}_0 + \mathbf{Z} \mathbf{p})^\top \mathbf{A} (\mathbf{f}_0 + \mathbf{Z} \mathbf{p}) - \mathbf{b}^\top (\mathbf{f}_0 + \mathbf{Z} \mathbf{p}).
\]

Minimizing this quadratic function for the reduced solution \( \mathbf{p} \), is equivalent to, solving the following reduced linear system

\[
\mathbf{Z}^\top \mathbf{A} \mathbf{Z} \mathbf{p} = \mathbf{Z}^\top (\mathbf{b} - \mathbf{A} \mathbf{f}_0).
\]

We can now solve this reduced system to get \( \mathbf{p} \), and then recover the desired solution \( \mathbf{f} \) by using equation 14.

### 2.3.2 Constrained preconditioner

Before we can solve equation 16, we must find matrix \( \mathbf{Z} \) and vector \( \mathbf{f}_0 \). Fortunately, these subproblems are simple. For example, assume we have three control points: \( f_0 = z_0, f_2 = z_2, \) and \( f_3 = z_3 \), then \( \mathbf{z} = [z_0, z_2, z_3]^\top \) and the matrix \( \mathbf{C} \) is

\[
\mathbf{C} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\
\end{bmatrix}_{3 \times N}.
\]

We can immediately find a solution \( \mathbf{f}_0 = [z_0, z_2, z_3, 0, \cdots, 0]^\top \) to the constraint equation

\[
\mathbf{C} \mathbf{f}_0 = \mathbf{z}.
\]

The columns of matrix \( \mathbf{Z} \) form a basis of the null space of matrix \( \mathbf{C} \), so that \( \mathbf{C} \mathbf{Z} = \mathbf{0} \). We generate such a matrix \( \mathbf{Z} \) from an \( N \times N \) identity matrix, by simply removing any columns that are identical to rows in the matrix \( \mathbf{C} \):

\[
\mathbf{Z} = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
\end{bmatrix}_{N \times (N-3)}.
\]

Given \( \mathbf{Z} \) and the solution \( \mathbf{f}_0 \), we are ready to solve the reduced system shown in equation 16. Because the matrix \( \mathbf{Z}^\top \mathbf{A} \mathbf{Z} \) is SPD, we can use the CG method to solve this reduced system. Many authors (e.g., Nash and Sofer, 1996; Gould et al., 2001; Dollar, 2005) have dis-
Figure 4. Seismic sections (a) intersect sequence boundaries extracted using one control point (blue curve) and 19 control points (green curve). (b) and (c) show a 3D view of the extracted surfaces using one control point and 19 control points, respectively.

cussed the solution of this system using the preconditioned CG method. In this paper, we use a simple preconditioner $P_z$ described in Nash and Sofer (1996):

$$P_z = Z^T M^{-1} Z \approx (Z^T A Z)^{-1}, \quad (19)$$

where $M^{-1} = S_x S_y S_y^T S_x^T$ as in equation 11, and $Z^T Z = I$ since the columns of $Z$ form a basis. Therefore, our preconditioner for the reduced system is

$$P_z = Z^T S_x S_y S_y^T S_x^T Z. \quad (20)$$

In the preconditioned CG method for the reduced system, one would compute the initial residual $r_z = Z^T (b - A f_0) - Z^T A Z p$, and the preconditioned residual $g_z = P_z r_z$.

Instead of solving the reduced system to obtain $p$, and then recovering the desired solution $f$, we can instead directly solve for $f$ because we have a relationship between the reduced and full solutions $f = f_0 + Z p$. As discussed by Gould et al. (2001), to explicitly perform the multiplication by $Z$ and the addition of the term $f_0$ in the CG method, we may choose $f = f_0 + Z p$, $Z^T r = r_z$, and $g = Z g_z$, so that $g = Z P_z Z^T r$. This process is equivalent to applying the preconditioned CG method to the unconstrained linear system $A f = b$, with a preconditioner

$$P = Z P_z Z^T = Z Z^T M^{-1} Z Z^T = Z Z^T S_x S_y S_y^T S_x^T Z Z^T. \quad (21)$$

In practice, we do not explicitly form the matrices $A$ and $Z Z^T$ because the preconditioned CG method requires only the computation of the residual vector $r = b - A f$ and gradient vector $g = P r$.

It is trivial to compute vector $Z Z^T x$ for any $N \times 1$
vector \( \mathbf{x} \) because \( \mathbf{Z} \mathbf{Z}^\top \) has the form

\[
\mathbf{Z} \mathbf{Z}^\top = \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{bmatrix}_{N \times N},
\]

(22)

Computation of \( \mathbf{Z} \mathbf{Z}^\top \mathbf{x} \) simply zeros all the elements of \( \mathbf{x} \) with indices corresponding to the locations of control points.

With the preconditioner \( \mathbf{P} \) denoted by equation 21, the preconditioned gradient \( \mathbf{g} = \mathbf{Z} \mathbf{g}_0 = \mathbf{Z} \mathbf{P} \mathbf{Z}^\top \mathbf{r} \) is projected to be in the null space of \( \mathbf{C} \). As a result, all updates to the solution \( \mathbf{f} \) in this preconditioned CG method will also lie in the null space of \( \mathbf{C} \). Therefore, because the initial solution \( \mathbf{f}_0 \) satisfies the constraints \( \mathbf{C} \mathbf{f}_0 = \mathbf{z} \), the solution \( \mathbf{f} \) after each CG iteration also satisfies \( \mathbf{C} \mathbf{f} = \mathbf{z} \).

### 2.4 Results with constraints

Where seismic normal vectors estimated from structure tensors are inaccurate (e.g., near unconformities, faults and noisy data), the use of control points helps to extract a more reliable horizon or sequence boundary. As shown in Figure 3, when we extract a sequence boundary constrained by only one control point (green circle in Figure 3a), the surface we extract (blue curves in Figure 3) is well aligned with a seismic reflector in the conformable areas (the left-side section and the left part of the right-side section in Figure 3a), where seismic normal vectors can be estimated accurately. However, the surface (blue curves) extracted at the unconformity (Figure 3b) deviates from the correct surface (dashed green curve in Figure 3b) because the normal vectors estimated there are inaccurate.

Using 19 control points (green points in Figure 4c), we obtain a more accurate sequence boundary. Figure 4a shows crossline and inline seismic sections that intersect the sequence boundaries extracted using (1) only one control point (blue curves), and (2) 19 control points (green curves). We observe that the sequence boundary extracted using 19 control points correctly represents the unconformity, while the one extracted using only one control point does not. Figures 4b and 4c show the same extracted sequence-boundary surfaces colored with seismic amplitudes. Amplitude values for 19 control points (Figure 4c) are more uniform than those for one control point (Figure 4b).

Note that this sequence boundary is also complicated by faults, highlighted by red ellipses in Figure 4. The surface extracted using only 1 control point (blue curves in Figure 4a and the surface in Figure 4b) is inaccurate near faults. However, the surface with 19 control points (green curves in Figure 4a and the surface in Figure 4c) correctly follows the faults. This example demonstrates that constraints facilitate extraction of a horizon surface complicated by faults.

Moreover, with more control points, an initially horizontal surface converges more quickly to the final extracted horizon. We can use more control points to interpolate a better initial surface \( f^0(x, y) \) that is closer to the seismic horizon \( f(x, y) \) we want to extract, which therefore enables the CG method to more quickly converge to that horizon. For example, it takes nine iterations to converge using one control point (blue curves in Figure 4a), but only five iterations to converge using 19 control points (green curves in Figure 4c) correctly follows the faults. This example demonstrates that constraints facilitate extraction of a horizon surface complicated by faults.

### 3 Generating a Horizon Volume

Using the method discussed above, we can extract a single seismic horizon or sequence boundary with one or more control points that represent interpreted constraints. With similar constraints, we can also extract all seismic horizons from a seismic image at once, and thereby generate a complete horizon volume. In a horizon volume \( z(x, y, \tau) \), as shown in Figure 2c, the vertical axis is RGT \( \tau \) and color denotes depth \( z \). Horizontally slicing a horizon volume at any single RGT value \( \tau \) yields a seismic horizon.

Here, we first describe a method for using seismic normal vectors to automatically generate a horizon volume without constraints, which is usually accurate for seismic images with simple structures. To better handle images complicated by faults or noise, we then extend this method, by incorporating scattered sets of points that correspond to multiple seismic horizons, to generate a more reliable horizon volume that honors those interpreted constraints.

#### 3.1 Horizon Volume without Constraints

As discussed by Parks (2010), a horizon volume \( z(x, y, \tau) \) can be generated from an RGT volume \( \tau(x, y, z) \) by inverse linear interpolation if we assume that \( \tau \) in the RGT volume increases monotonically with depth \( z \). Some authors have described methods to generate such an RGT volume using phase unwrapping (e.g., Stark, 2005; Wu and Zhong, 2012b) or reflector dips (Parks, 2010). Here we rederive the latter method in a simpler way to compute an RGT volume.

In an RGT volume \( \tau(x, y, z) \) like that shown in Figure 5a or 2b, contours (Figure 5b) of constant \( \tau \) represent seismic horizons, which means these contours should have the same structures as seismic reflectors in the seismic image (Figure 5b). Therefore, gradient vectors for an RGT volume \( \tau(x, y, z) \), that are perpendicular to RGT contours, should be parallel to seismic normal vectors \( \mathbf{n} = [n_x, n_y, n_z]^\top \), that are perpendicular to seismic reflectors. If we assume that these vectors...
Substituting equation 25 into equation 24, we obtain where the function \( f(x, y, z) \) represents vertical shifts. Substituting equation 25 into equation 24, we obtain

\[
\begin{bmatrix}
\frac{\partial s}{\partial x} - \frac{\partial z}{\partial x} \\
\frac{\partial s}{\partial y} - \frac{\partial z}{\partial y} \\
\frac{\partial s}{\partial z} - \frac{\partial z}{\partial z}
\end{bmatrix}
\approx
\begin{bmatrix}
n_x \\
n_y \\
n_z
\end{bmatrix}
\]  

where again \( p = -n_x/n_z \) and \( q = -n_y/n_z \) are estimated inline and crossline slopes of seismic reflectors. Equation 27 is what Parks (2010) solved to obtain shifts that flatten a seismic image.

As suggested by Lomask et al. (2006), we add a third equation \( \epsilon \ddot{s_z} \approx 0 \) to reduce vertical variations in the shifts. We also weight the equations above by a measure \( w(x, y, z) \) of the quality of the estimated reflector slopes \( p(x, y, z) \) and \( q(x, y, z) \). We then compute the shifts by solving the following equations:

\[
\begin{bmatrix}
w(-\frac{\partial s}{\partial x} - p\frac{\partial s}{\partial x}) \\
w(-\frac{\partial s}{\partial y} - p\frac{\partial s}{\partial y}) \\
\epsilon \frac{\partial s}{\partial z}
\end{bmatrix}
\approx
\begin{bmatrix}
w
w
0
\end{bmatrix}
\]

If we have \( N \) image samples, then equation 28 represents \( 3N \) equations for \( N \) unknown shifts, and these equations can be expressed in matrix form as

\[
\mathbf{WGS} = \mathbf{Wv}.
\]

The matrix \( \mathbf{A} \) is both SPD and sparse. In practice, we do not explicitly form the matrices \( \mathbf{A} \), \( \mathbf{W} \), and \( \mathbf{G} \). Instead, we solve this linear system using the CG method, which requires only the computation of matrix-vector products like \( \mathbf{As} = (\mathbf{WG})^\top \mathbf{WG} \mathbf{s} \) and \( \mathbf{b} = (\mathbf{WG})^\top \mathbf{Wv} \).

As when extracting a single seismic horizon, we solve equation 31 using the preconditioned CG method with a preconditioner defined by

\[
\mathbf{M}^{-1} = \mathbf{S}_x \mathbf{S}_y \mathbf{S}_z \mathbf{T}_x \mathbf{T}_y \mathbf{T}_z,
\]

where, again, \( \mathbf{S}_x \), \( \mathbf{S}_y \), and \( \mathbf{S}_z \) are filters that smooth in the \( x \), \( y \), and \( z \) directions, respectively. We again expect the solution to be laterally smooth except at faults, as shown in Figure 7b. Therefore, the lateral smoothing filters \( \mathbf{S}_x \) and \( \mathbf{S}_y \) are spatially variant filters (Hale, 2009), and the extent of smoothing is proportional to a measure of reflector continuity, so that these filters smooth less at faults. In this example, we expect the shifts to be smooth vertically because we do not have unconformities. Therefore, our vertical smoothing filter \( \mathbf{S}_z \) in this example is spatially invariant.

Note that we derive all of the equations above for

---

**Figure 5.** The same RGT volume (a) as shown in Figure 2b, contours (b) of the RGT volume are horizons in the corresponding seismic image.
3D images, but they can be easily modified to work for 2D images, by simply omitting the second equation for the $y$ direction from equation 28. For the 2D example shown in Figure 2, we first solved equation 31 to get shifts $s(x, z)$. We then computed an RGT volume $\tau(x, z) = z + s(x, z)$ (Figure 2b), where $\tau$ increases monotonically with depth $z$. Finally we computed a horizon volume $z(x, \tau)$ (Figure 2c) from the RGT volume $\tau(x, z)$ by inverse linear interpolation (Parks, 2010). This horizon volume $z(x, \tau)$ maps the seismic image (Figure 2a) to a flattened image (Figure 2d).

For seismic images with simple geologic structures and little noise, as in Figure 2a, we can use the method discussed above to compute an accurate RGT volume (Figure 2b) and then interpolate a horizon volume (Figure 2c) that well flattens (Figure 2d) a seismic image. However, for seismic images complicated by faults, as in Figure 6a, the generated RGT volume (Figure 6b) is inaccurate, so that seismic reflectors are not flattened correctly (Figure 6c). Therefore, we extend this method to compute more accurate RGT and horizon volumes by incorporating scattered sets of control points that may correspond to multiple horizons.

### 3.2 Horizon volume with constraints

For specified sets of control points, we solve a constrained optimization problem similar to that we solve when extracting a single seismic horizon:

\[
\min s \quad F(s) = \frac{1}{2}s^T As - b^T s,
\]

subject to $Cs = d$. (33)

As when extracting a single horizon, solving the constrained problem above is equivalent to solving a corresponding unconstrained problem $As = b$ using a preconditioned CG method with an initial solution $s_0$ to the constraint equation $Cs_0 = d$ and a constrained preconditioner $P = ZZ^T M^{-1} ZZ^T$, where $M^{-1} = S_x S_y S_z$. Therefore, to solve this problem, we need only an initial solution $s_0$ and the matrix $ZZ^T$ for the preconditioner $P$.

Let us use a tiny 3D seismic image with only $N = 2 \times 2 \times 2$ samples to explain how to implement multiplication by the matrix $ZZ^T$ and to find an initial solution $s_0$. As in equation 25, we want to compute a 3D RGT volume $\tau(x, y, z)$ with shifts $s(x, y, z)$. In this simple example, both $\tau$ and $s$ have only $N = 2 \times 2 \times 2$ samples, and we can express equation 25 in vector form as

\[
\tau = z + s,
\]
where

\[ \tau = [\tau_0 \tau_1 \tau_2 \tau_3 \tau_4 \tau_5 \tau_6 \tau_7]^T, \]
\[ z = [z_0 z_1 z_2 z_3 z_4 z_5 z_6 z_7]^T, \] (35)
\[ s = [s_0 s_1 s_2 s_3 s_4 s_5 s_6 s_7]^T. \]

Assume that we have 2 sets of constraints: the first set has 3 control points with sample indices \( \{3, 5, 7\} \), and the second set has 2 control points with sample indices \( \{1, 6\} \). Within each set of constraints, all control points are interpreted to be on a single seismic horizon. Therefore, we have \( \tau_3 = \tau_5 = \tau_7 \) and \( \tau_1 = \tau_6 \). According to equation 34, this means that \( s_5 - s_3 = z_3 - z_5 \) and \( s_7 - s_3 = z_3 - z_7 \), and \( s_6 - s_1 = z_1 - z_6 \). We can therefore write the constraint equation \( Cs = d \) as follows:

\[
\begin{bmatrix}
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
z_3 - z_5 \\
z_3 - z_7 \\
z_1 - z_6
\end{bmatrix},
\] (36)

where again, \( s = [s_0 s_1 s_2 s_3 s_4 s_5 s_6 s_7]^T \).

In this example, matrix \( C \) has 3 linearly independent rows so that matrix \( Z \) must have \( N - 3 \) linearly independent columns, such that \( CZ = 0 \), because the columns of matrix \( Z \) form a basis for the null space of \( C \). Construction of matrix \( Z \) is only slightly more complicated than for the single-horizon case. Specifically,

\[
Z = \begin{bmatrix}
e_{c1} & e_{c2} & e_0 & e_2 & e_i
\end{bmatrix},
\]

\[
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}_{8 \times 5},
\] (37)

where \( e_{c1} = e_3 + e_5 + e_7, e_{c2} = e_1 + e_6, \) and \( e_i, \) for \( i = 0, 1, \ldots, N - 1, \) is an \( N \times 1 \) unit vector with \( 1 \) at the \( i \)th index. In other words, we begin with the identity matrix and simply sum the unit vectors \( e_i \) with indices \( i \) in \( \{3, 5, 7\} \), corresponding to the first set of control points, to obtain the first column of \( Z \); similarly, we obtain the second column of \( Z \), corresponding to the second set of control points with indices \( \{1, 6\} \); and finally, we use all of the remaining unit vectors \( e_i \) that do not correspond to any control point for remaining columns of \( Z \). In the same way, we can easily construct matrix \( Z \) for any number of sets of control points.

We can normalize the columns of matrix \( Z \) to ob-

\[ \text{Figure 7. A seismic image (a) with 3 pairs of interactively interpreted control points (green circles, pluses and squares), generated RGT volume (b) and flattened image (c).} \]
Figure 8. Input seismic image (a) and a corresponding RGT volume (b) computed with three sets of control points.
Figure 9. The flattened seismic image is sliced at $\tau = 1.664$ (a) and $\tau = 1.751$ (b). Horizontal slices in a flattened image correspond to seismic horizon surfaces (upper-right panels in (a) and (b), for which color denotes depth) in an unflattened image.
conditioned CG method. In practice, we find that this alternative reference point. In practice, we find that this alternative approach reduces the number of iterations required by the preconditioned CG method.

We first compute the average of each set of control points, and then choose any point among them as a reference point with zero shift (e.g., $s_2 = 0$ for the first set of control points, and $s_1 = 0$ for the second set of control points), and use the depth difference between the reference point and other control points for the remaining initial shifts in $s_0$.

We instead construct the initial shifts $s_0$ in a slightly different way. We first compute the average depth for each set of control points, and then choose the point with depth nearest to that average as the reference point. In practice, we find that this alternative approach reduces the number of iterations required by the preconditioned CG method.

We can also easily find an initial solution $s_0$ to the constraint equation $Cs_0 = d$:

$$s_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_5^{s_0},$$

(38)

with columns that form an orthonormal basis for the null space of the constraint matrix $C$.

We then find that

$$ZZ^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{5}^{s_0}.$$  (39)

For any vector $x = [x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7]^T$, it is easy to compute the product

$$ZZ^T x = [x_0, x_{c2}, x_2, x_{c1}, x_4, x_{c1}, x_6, x_{c1}]^T,$$  (40)

where $x_{c1} = (x_3 + x_5 + x_7)/3$ and $x_{c2} = (x_1 + x_6)/2$. In other words, we compute $ZZ^T x$ by simply gathering and averaging all elements of $x$ with indices corresponding to each set of control points, and then scattering the averages back into those same elements. In each CG iteration, when we apply the constrained preconditioner $P = ZZ^T M^{-1} ZZ^T$ to a vector, we need only compute averages and apply smoothing filters.

We can also easily find an initial solution $s_0$ to the constraint equation $Cs_0 = d$:

$$s_0 = \begin{bmatrix} s_0 \ 0 \ s_3 \ 0 \ s_5 \ s_6 \ s_7 \end{bmatrix},$$

(41)

in which elements with indices corresponding to the first set of control points are $s_2 = 0$, $s_3 = s_5 = s_7$; elements corresponding to the second set of control points are $s_1 = 0$ and $s_6 = z_1 - z_0$. Therefore, to construct an initial set of shifts $s_0$, we use zeros for elements that do not correspond to any control points; for each set of control points, we choose any point among them as a reference point with zero shift (e.g., $s_3 = 0$ for the first set of control points, and $s_1 = 0$ for the second set of control points), and use the depth differences between the reference point and other control points for the remaining initial shifts in $s_0$.

With an initial solution $s_0$ and the constrained preconditioner $P = ZZ^T M^{-1} ZZ^T$, we can apply the preconditioned CG method to the unconstrained system $As = b$ to obtain a solution $s$ that satisfies the constrained problem of equation 33. In each CG iteration, we compute a residual as $r = b - As$. Using the constrained preconditioner $P$, we compute a constrained residual $r_P = ZZ^T M^{-1} ZZ^T r$ that is in the null space of the constraint matrix $C$. This means that all of the updates to the initial solution $s_0$ in this preconditioned CG method will also be in the null space of $C$. Therefore, because the initial solution $s_0$ satisfies the constraint equation $Cs_0 = d$, the final solution $s$ obtained after any number of CG iterations will also satisfy the constraints.

Figure 7 is a 2D example that shows how constraints help to generate a more accurate horizon volume and better flatten a seismic image. In this example, we use the same input seismic image (Figure 7a) complicated by faults that is displayed in Figure 6a, but now we have 3 sets of constraints. For each set of constraints, we interpret 2 control points (green circles, pluses, and squares in Figure 7a) to lie on a seismic horizon. Figure 7b is the computed RGT volume, with which we interpolate a horizon volume that correctly flattens (Figure 7c) seismic reflectors across faults.

### 3.3 3D results with constraints

Figure 8a shows a 3D seismic image that is also complicated by faults. To flatten this 3D image or generate a horizon volume, we choose weights $w(x, y, z)$ corresponding to faults in equation 28. Specifically, we use the method developed by Hale (2013) to first compute an image of fault likelihoods $f(x, y, z) \in [0, 1]$ in which values near 1 indicate fault locations. We then use $w = (1 - f)^8$ as weights in equation 28.

For this example, we use three sets of constraints, corresponding to three horizons in the 3D seismic image, to compute an accurate horizon volume and correctly flatten the seismic image. The first set contains 5 control points, the second one contains 7 control points (green points in Figure 9a), and the third one contains 11 control points (green points in Figure 9b). Using these three sets of constraints, we first compute an RGT volume as shown in Figure 8b, from which we then interpolate a horizon volume that correctly flattens (Figure 9a or 9b) seismic reflectors across faults. Note that the constraints help to flatten not only reflectors passing through the control points, but also other reflectors in the 3D seismic image as well.

Figure 1 displays a 3D view of 6 seismic horizons extracted from the horizon volume computed with constraints. In Figure 1a, different colors denote different seismic horizons, but deeper horizons are obscured by the top one. We therefore, in Figure 1b, display cutaway views of each of the horizons. We observe that the
horizons with control points (the cyan and yellow surfaces) and others without control points coincide well with seismic reflectors.

4 CONCLUSION

We propose methods to (1) extract one seismic horizon at a time and (2) to compute at once a complete horizon volume. We designed these two methods to compute horizons that honor interpreted constraints, specified as sets of control points. We incorporate the control points with simple constraint preconditioners in the CG method used to compute horizons.

The first method is useful, even though we can extract all horizons at once using the second method, because it can more quickly extract a single horizon. Using multiple control points, this method can reliably extract complicated geologic surfaces such as sequence boundaries and horizons with faults. Furthermore, this first method might be used to efficiently extract horizons that might serve as control surfaces (large sets of control points) for the second method.

The second method generates a complete horizon volume at once. With a small number of interpreted constraints, this method works well for seismic images complicated by faults.

Interpreted constraints are necessary, because completely automatic interpretation cannot yet handle complicated seismic horizons. The proposed methods provide an especially simple way to specify such constraints by simply interactively picking points in a 3D seismic image that belong to the same seismic horizon. These methods can be implemented to interactively add or move control points, while quickly updating a single seismic horizon or complete horizon volume.

These methods might be further improved if we could predict areas where control points are required to generate more reliable results so that the interpretation of constraints could be more straightforward and efficient.

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3D seismic image processing for unconformities

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ABSTRACT
We propose a 3D seismic unconformity attribute to detect complete unconformities, highlighting both their termination areas and correlative conformities. These detected unconformities are further used as constraints to more accurately estimate seismic normal vectors at unconformities. Then, using seismic normal vectors and detected unconformities as constraints, we can better flatten seismic images containing unconformities.

Key words: unconformity seismic normal vectors flattening

1 INTRODUCTION
An unconformity is a non-depositional or erosional surface separating older strata below from younger strata above, and thus represents a significant gap in geologic time (Vail et al., 1977). In seismic images, an unconformity can be first identified by seismic reflector terminations (i.e., truncation, toplap, onlap or downlap) and then be traced to its corresponding correlative conformity.

Unconformity detection is a significant aspect of seismic stratigraphic interpretation, because unconformities represent discontinuities in otherwise continuous depositions and hence serve as boundaries when interpreting seismic sequences that represent successively deposited layers.

Moreover, unconformities pose challenges for other automatic techniques used in seismic interpretation. First, it is difficult to accurately estimate normal vectors or slopes of seismic reflectors at an unconformity with multiply-oriented structures due to seismic reflector terminations. Second, automatic seismic flattening methods cannot correctly flatten reflectors at unconformities with geologic time gaps. To address these challenges, we first detect unconformities and then use them as constraints for seismic normal vector estimation and image flattening.

1.1 Unconformity detection
Seismic coherence (Bahorich and Farmer, 1995), which highlights reflector discontinuities, is used as a seismic attribute to detect faults and termination areas of unconformities. However, this coherence attribute is better suited for detecting faults than unconformities, because reflector discontinuities across a fault are usually more apparent than discontinuities across an unconformity. Hoek et al. (2010) propose a better unconformity attribute that measures the degree of seismic reflector convergence (or divergence), and thereby highlights the termination areas of an unconformity. Both of these methods process a seismic image locally (Ringdal, 2012) to
compute unconformity attributes that can highlight an unconformity within its termination area, but cannot detect its correlative conformity.

Ringdal (2012) proposes a global method that first extracts a 2D flow field that is everywhere tangent to reflectors in a 2D seismic image. Then the flow field is used to compute an unconformity probability image by repeating the following processing for each sample: four seeds are first placed at the four neighbors of the sample in the 2D flow field; the four seeds then move along the flow field to produce trajectories; the separation rate of the trajectories is finally calculated and used as unconformity probability for that sample. The advantage of this method is that it can use long trajectories to detect the correlative conformity of an unconformity. The disadvantage is that, to detect such a correlative conformity, the trajectories are required to start from the parallel area (correlative conformity) and end in the non-parallel area (termination). For 3D seismic images, this method processes inline and crossline slices separately throughout the volume to compute an unconformity probability volume.

1.2 Seismic normal vector estimation
Orientation vector fields, such as vectors normal to or slopes of seismic reflectors, are useful for seismic interpretation. For example, estimated orientation information is used to control slope-based (Fomel, 2002) and structure-oriented (Fehmers and Höcker, 2003; Hale, 2009) filters so that they smooth along reflectors to enhance their coherencies. Seismic normal vectors or slopes are also used to flatten (Lomask et al., 2006; Parks, 2010) or unfold (Luo and Hale, 2013) seismic images, or to generate horizon volumes (Wu and Hale, 2014).

Structure tensors (van Vliet and Verbeeck, 1995; Fehmers and Höcker, 2003) or plane-wave destruction filter (Fomel, 2002) have been proposed to estimate seismic normal vectors or slopes. These methods can accurately estimate orientation vectors for structures with only one locally dominant orientation. This means that they can correctly estimate the normal vectors (or slopes) of the reflectors in conformable areas of a seismic image, but for an angular unconformity where two different structures meet, these methods yield smoothed vectors that represent averages of orientations across the unconformity.

1.3 Seismic image flattening
Seismic image flattening (Lomask et al., 2006; Parks, 2010; Wu and Hale, 2014) or unfolding (Luo and Hale, 2013) methods are applied to a seismic image to obtain a flattened image in which all seismic reflectors are horizontal. From such a flattened seismic image, all seismic horizons can be identified by simply slicing horizontally.

Extracting horizons terminated by faults or unconformities is generally difficult for these methods. Luo and Hale (2013) extract horizons across faults by first unfaulting a seismic image; Wu and Hale (2014) do the same by placing control points on opposite sides of faults. However, none of these methods correctly flattens a seismic image with unconformities, which should produce gaps in the flattened image (e.g., Figure 1b).

1.4 This paper
In this paper, we first propose a method to automatically detect an unconformity, complete with its termination area and correlative conformity, as shown in Figure 1a. We then describe how to more accurately estimate seismic normal vectors at unconformities by using the detected unconformities as constraints. Finally, we discuss how to better flatten (Figure 1b) seismic images containing unconformities by using these estimated seismic normal vectors and again using constraints derived from detected unconformities.

2 UNCONFORMITY DETECTION
In manual 3D seismic stratigraphic interpretations, an unconformity is first recognized as a border at which seismic reflectors terminate (i.e., truncation, toplap, onlap or downlap), and then is traced to its correlative conformities where reflectors are parallel. Therefore, to obtain a complete unconformity, an automatic method should be able to detect both the termination areas (green ellipse in Figure 2a) and correlative conformities (dashed blue ellipse in Figure 2a) within the unconformity.

We propose an unconformity attribute that measures differences between two seismic normal vector fields computed from two structure-tensor fields, one is computed using a vertically causal smoothing filter, and the other using a vertically anti-causal smoothing filter. This attribute can highlight both the termination areas and correlative conformities of an unconformity.

2.1 Structure tensor
The structure tensor (van Vliet and Verbeeck, 1995; Fehmers and Höcker, 2003) can be used to estimate seismic normal vectors that are perpendicular to seismic reflectors. For a 2D image, the structure tensor $T$ for each sample is a $2 \times 2$ symmetric positive-semidefinite matrix constructed as the smoothed outer product of image gradients:

$$
T = \langle gg \rangle_{h,v} = \begin{bmatrix}
\langle g_1 g_1 \rangle_{h,v} & \langle g_1 g_2 \rangle_{h,v} \\
\langle g_1 g_2 \rangle_{h,v} & \langle g_2 g_2 \rangle_{h,v}
\end{bmatrix}
$$

(1)

where $g = [g_1 \ g_2]^T$ represents the image gradient vector computed for each image sample; $\langle \rangle_{h,v}$ represents
2.2.1 Vertical smoothing

To compute two structure-tensor fields that differ significantly at an unconformity, the causal smoothing filter that averages from above should smooth along the direction perpendicular to the structures above the unconformity, while the anti-causal filter should smooth along the direction perpendicular to the structures below the unconformity. Here, we simply use vertically causal and anti-causal filters because unconformities are usually tendency to be horizontal in seismic images. We implement these two filters with one-sided exponential smoothing filters, which are efficient and trivial to implement.

A one-sided causal exponential filter for input and output sequences $x[i]$ and $y[i]$ with lengths $n$ can be implemented in C++ (or Java) as follows:

```c
float b = 1.0f-a;
y[i] = y[0] = x[0];
for (int i=1; i<n; ++i)
    y[i] = yi = a*yi+b*x[i];
```

Similarly, a one-sided anti-causal exponential filter can be implemented as follows:

```c
float b = 1.0f-a;
y[i] = y[n-1] = x[n-1];
for (int i=n-2; i>-1; --i)
    y[i] = yi = a*yi+b*x[i];
```

The parameter $a$ in these two one-sided exponential fil-

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ters controls the extent of smoothing.

From structure-tensor fields $T_s$ and $T_a$ computed for the same seismic image using vertically causal and anti-causal smoothing filters, respectively, we estimate two seismic normal vector fields $\mathbf{u}_s$ and $\mathbf{u}_a$. As shown in Figure 3a, the two seismic normal vector fields $\mathbf{u}_s$ (green segments in Figure 3a) and $\mathbf{u}_a$ (yellow segments in Figure 3a) are identical in conformable areas with parallel seismic reflectors, because orientation of structures locally averaged from above (used to compute $T_s$) are identical to orientation of structures averaged from below (used to compute $T_a$). However, at the termination area of an unconformity, the two vector fields are different, because the structure tensors $T_s$ computed with structures locally averaged from above, should be different from $T_a$ computed with structures locally averaged from below.

Therefore, as shown in Figure 3a, the difference between estimated normal vector fields $\mathbf{u}_s$ and $\mathbf{u}_a$ provides a good indication of the termination area of an unconformity. However, a complete unconformity, that is, a curve (in 2D) or surface (in 3D) with geologic time gaps, extends from its termination area to its correlative conformity with. Thus we should extend normal vector differences from the termination area, where these differences originate, to the correlative conformity.

### 2.2.2 Structure-oriented smoothing

To detect a correlative conformity, we extend vector differences (between $\mathbf{u}_s$ and $\mathbf{u}_a$) at an unconformity from its termination area to its correlative conformity, by replacing the horizontal Gaussian smoothing filter in equations 3 and 4 with a structure-oriented smoothing filter (Hale, 2009) when computing structure tensors.

Then, the structure tensors $T_{s,c}$ and $T_{s,a}$, computed with a laterally structure-oriented filter and vertically causal and anti-causal filters, are defined by

$$
T_{s,c} = \begin{bmatrix}
\langle g_1 g_1 \rangle_{s,cc} & \langle g_1 g_2 \rangle_{s,sc} \\
\langle g_1 g_2 \rangle_{s,sc} & \langle g_2 g_2 \rangle_{s,cc}
\end{bmatrix},
$$

and

$$
T_{s,a} = \begin{bmatrix}
\langle g_1 g_1 \rangle_{s,va} & \langle g_1 g_2 \rangle_{s,va} \\
\langle g_1 g_2 \rangle_{s,va} & \langle g_2 g_2 \rangle_{s,va}
\end{bmatrix},
$$

where the subscript $s$ represents a structure-oriented filter that smoothes along reflectors in a seismic image. Note that the structure-oriented smoothing is generally more expensive than the vertically causal and anti-causal smoothing. We therefore first apply the structure-oriented smoothing filter to each element of $\mathbf{gg}^T$ to obtain $T_s = (\mathbf{gg}^T)_s$, which then is smoothed separately by vertically causal and anti-causal filters to obtain $T_{s,c}$ and $T_{s,a}$, respectively. By doing this, we apply the relatively expensive structure-oriented smoothing only once. However, if we first apply the vertically causal and anti-causal smoothing to compute two differently smoothed outer products $\langle \mathbf{gg} \rangle_c$ and $\langle \mathbf{gg} \rangle_a$, we then need to apply the structure-oriented smoothing twice to obtain two structure-tensor fields $T_{s,c}$ and $T_{s,a}$.

As discussed by Hale (2009, 2011), to obtain a smoothed output image $q(x)$ from an input $p(x)$, the structure-oriented smoothing method solves a finite-difference approximation to the following partial differential equation:

$$
q(x) - \frac{\sigma^2}{2} \nabla \cdot \mathbf{D}(x) \cdot \nabla q(x) = p(x),
$$

where $\mathbf{D}(x)$ is a diffusion-tensor field that shares the eigenvectors of the structure tensor computed from an image, and therefore orients the smoothing along image structures. Similar to the half-width $\sigma$ in a Gaussian smoothing filter, the parameter $\sigma$ controls the extent of
smoothing.

In 2D, we use the eigenvectors $u(x)$ and $v(x)$, estimated using the structure tensors shown in equation 1, to construct our diffusion-tensor field

$$D(x) = \lambda_u(x)u(x)u^T(x) + \lambda_v(x)v(x)v^T(x).$$

(8)

Then, because eigenvectors $u(x)$ and $v(x)$ are perpendicular and parallel to seismic reflectors, respectively, we can control the structure-orient filter to smooth along reflectors by setting the corresponding eigenvalues $\lambda_u(x) = 0$ and $\lambda_v(x) = 1$ for all tensors in $D(x)$.

As indicated by the seismic normal vectors shown in Figure 2b, normal vectors (magenta segments) estimated using the structure tensors computed in equation 1 are inaccurate at unconformities. However, they are accurate in conformable areas, including the area near correlative conformity of the unconformity. Thus the structure tensors in equation 1 are adequate for constructing diffusion-tensors $D(x)$ for structure-oriented smoothing along seismic reflectors, including those near the correlative conformity of an unconformity. By applying such a structure-oriented filter to the elements of the structure tensors $T_{s,c}$ and $T_{s,a}$, we extend structural differences, originating within the termination area of an unconformity, to the corresponding correlative conformity.

As shown in Figure 3, using structure tensors $T_a$ and $T_c$ computed with a horizontal Gaussian filter and vertically causal and anti-causal filters, the estimated seismic normal vectors $u_c$ (green segments in Figure 3a) and $u_a$ (yellow segments in Figure 3a) differ only within the termination area of the unconformity. Using structure tensors $T_{s,c}$ and $T_{s,a}$ computed with a structure-oriented smoothing filter instead of a horizontal Gaussian filter, the differences between the estimated seismic normal vectors $u_{s,c}$ (green segments in Figure 3b) and $u_{s,a}$ (yellow segments in Figure 3b) are extended from the termination area to the correlative area.

In summary, by first applying a structure-oriented filter to each structure-tensor element of $gg^T$, we extend any structure differences, which originate within the termination area of an unconformity, to its correlative conformity. Then, applying vertically causal and anti-causal filters for each structure-tensor element, we compute two different structure-tensor fields $T_{s,c}$ and $T_{s,a}$ with seismic normal vector fields $u_{s,c}$ and $u_{s,a}$ that differ within both the termination area and correlative conformity of an unconformity. Finally, the differences between the two estimated vector fields $u_{s,c}$ and $u_{s,a}$ can be used as an unconformity attribute that highlights the complete unconformity.

Figure 5. Applying our method to the synthetic image cut from Hoek et al. (2010), we obtain unconformity likelihoods before (a) and after (b) thinning.

Figure 6. Unconformity likelihoods before (a) and after (b) thinning.
Figure 7. Unconformity likelihoods before (a) and after (b) thinning. Thinned unconformity likelihoods form unconformity surfaces (b).
2.3 Unconformity likelihood

As shown in Figure 3b, the vectors $u_{s,c}$ (green segments) and $u_{s,a}$ (yellow segments) are identical everywhere except at the unconformity, including its termination area and correlative conformity. Therefore, we define an unconformity likelihood attribute $g$, that evaluates the differences between $u_{s,c}$ and $u_{s,a}$, to highlight unconformities:

$$g = 1 - (u_{s,c} \cdot u_{s,a})^p.$$  \hfill (9)

A large power $p$ ($p \gg 1$) increases the contrast between samples with low and high unconformity likelihoods. For the example shown in Figure 4a, the unconformity likelihoods are computed with $p = 200$.

Using a process similar to that used by Hale (2012) for extracting ridges of fault likelihoods, we extract ridges of unconformity likelihood by simply scanning each vertical column of the unconformity likelihood image (Figure 4a), preserving only local maxima, and setting unconformity likelihoods elsewhere to zero. Figure 4b shows that ridges of unconformity likelihood coincide with the unconformity that appears in the synthetic seismic image.

Figure 5 shows a more complicated 2D synthetic image used by Hoek et al. (2010). The geometric attributes they compute highlight only the termination areas of unconformities apparent in this synthetic image. In comparison, unconformity likelihoods before (Figure 5a) and after (Figure 5b) thinning, computed using our method, highlight the complete unconformities, including their termination areas as well as correlative conformities.

Figure 6 shows an example of a real 2D seismic image, in which generated unconformity likelihoods before (Figure 6a) and after (Figure 6b) thinning correctly highlight two unconformities apparent in the seismic image.

For a 3D seismic image, following the same process as above, we compute an unconformity-likelihood volume as shown in Figure 7, which correctly highlights two apparent unconformities. In the time slices of unconformity likelihoods before and after thinning, we observe that samples in the lower-left and upper-right areas, separated by high unconformity likelihoods, suggest different seismic facies. This indicates that they belong to two different depositional sequences that have different geologic times.

From ridges of unconformity likelihoods (Figure 7b), we connect adjacent samples with high unconformity likelihoods to form unconformity surfaces as shown in upper-right panel of Figure 7b.

3 APPLICATIONS

We first use unconformity likelihoods as constraints to more accurately estimate seismic normal vectors at unconformities. Then, using more accurate normal vectors and unconformity likelihoods as constraints in our seismic image flattening method, we are able to better flatten an image containing unconformities.

3.1 Estimation of seismic normal vectors at unconformities

Using structure tensors computed with horizontal and vertical Gaussian filters as shown in equation 1, we find smoothed seismic normal vectors (magenta segments in Figure 2b) in the termination area, because discontinuous structures across the unconformity are smoothed
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Figure 9. RGT (a) and flattened (c) images generated with inaccurate seismic normal vectors (Figures 8a and 8c) and without unconformity constraints. Improved RGT (b) and flattened (d) images with more accurate seismic normal vectors (Figures 8b and 8d) and constraints from unconformity likelihoods (Figure 6).

by symmetric Gaussian filters. Therefore, to obtain correct normal vectors (cyan segments in Figure 2b) that are discontinuous in the termination area, we must use more appropriate filters to compute structure tensors.

To preserve structure discontinuities, we compute the structure tensors using horizontal and vertical filters that do not smooth across unconformities:

\[
T = \begin{bmatrix}
    \langle g_1 g_1 \rangle_{sh,sv} & \langle g_1 g_2 \rangle_{sh,sv} \\
    \langle g_1 g_2 \rangle_{sh,sv} & \langle g_2 g_2 \rangle_{sh,sv}
\end{bmatrix},
\]

where the \( \langle \cdot \rangle_{sh,sv} \) represent horizontal (subscript \( sh \)) and vertical (subscript \( sv \)) filters that vary spatially, and for which the extent of smoothing is controlled by the thinned unconformity likelihoods.

The horizontal and vertical filters are similar to the edge-preserving smoothing filter discussed in Hale (2011):

\[
q(x) - \frac{\sigma^2}{2} \nabla \cdot c^2(x) \cdot \nabla q(x) = p(x).
\]

We compute \( c(x) = 1 - g_1(x) \) to prevent this filter from smoothing across unconformities. \( g_1(x) \) is a thinned unconformity likelihood image as shown in Figure 6b, which has large values (close to 1) only at unconformities and zeros elsewhere.

Figure 8 shows seismic normal vectors estimated for the image with two unconformities shown in Figure 6. Both the vertical (Figure 8a) and horizontal (Figure 8c) components of seismic normal vectors, estimated from structure tensors computed as in equation 1, are smooth at the unconformities; those estimated from structure tensors computed as in equation 10 preserve discontinuities at unconformities (Figures 8b and 8d).

3.2 Seismic image flattening at unconformities

Seismic normal vectors or dips can be used to flatten (Lomask et al., 2006; Parks, 2010) or unfold (Luo and Hale, 2013) a seismic image to generate a horizon volume (Wu and Hale, 2014), that allows for the extraction of all seismic horizons in the image. Faults and unconformities, which represent discontinuities of reflectors in a seismic image, present challenges for these methods. Luo and Hale (2013) and Wu and Hale (2014) have extended their methods to handle faults by first unfaulting the seismic image or by placing control points on opposite sides of a fault. However, neither of these methods correctly handles seismic images with unconformities, because estimated seismic normal vectors or dips are inaccurate at unconformities, and because unconformities are not automatically detected and then used as constraints in these methods.

In this paper we have proposed methods to automatically detect unconformities and more accurately estimate seismic normal vectors at unconformities. Therefore, we can easily extend the flattening method described in Wu and Hale (2014), to better flatten a seismic image at unconformities, by using seismic normal vectors estimated from structure tensors computed with equation 10, and by incorporating constraints derived from unconformity likelihoods into the flattening method. We incorporate unconformity constraints in our flattening method by weighting the equations for flattening using unconformity likelihoods, and then using the unconformity likelihoods to construct preconditioner in the conjugate gradient method used to solve those equations.
Figure 10. Generated RGT volume (a) and flattened (b) 3D seismic image. Discontinuities in the RGT volume correspond to vertical gaps in the flattened image at unconformities.
3.2.1 Weighting

To generate a horizon volume or to flatten a seismic image, we first solve for vertical shifts \(s(x, y, z)\) as discussed in Wu and Hale (2014):

\[
\begin{align*}
\begin{bmatrix}
w(-\frac{\partial}{\partial x} - p^{\frac{\partial}{\partial z}}) \\
w(-\frac{\partial}{\partial y} - p^{\frac{\partial}{\partial z}}) \\
\epsilon \frac{\partial}{\partial z}
\end{bmatrix} & \approx \\
\begin{bmatrix}
w_p \\
w_q \\
0
\end{bmatrix},
\end{align*}
\]

where \(p(x, y, z)\) and \(q(x, y, z)\) are inline and crossline reflector slopes computed from seismic normal vectors; \(w(x, y, z)\) represent weights for the equations; and the third equation \(\epsilon \frac{\partial}{\partial z}\approx 0\), scaled by a small constant \(\epsilon\), is used to reduce rapid vertical variations in the shifts.

For a seismic image with unconformities, we incorporate constraints derived from unconformity likelihoods into the equations 12 by setting \(w(x, y, z) = 1 - g_l(x, y, z)\) and we use a spatially variant \(\epsilon(x, y, z)\) instead of a constant value:

\[
\epsilon(x, y, z) = \epsilon_0 (1 - g_l(x, y, z)),
\]

where \(\epsilon_0\) is a small constant number (we use \(\epsilon_0 = 0.01\) for all examples in this paper), and \(g_l(x, y, z)\) denotes the thinned unconformity likelihoods, such as those shown in Figure 7b.

This spatially variant \(\epsilon(x, y, z)\), with smaller values (nearly 0) at unconformities, helps to generate more reasonable shifts with gradual variations everywhere except at unconformities.

3.2.2 Preconditioner

As discussed in Wu and Hale (2014), to obtain the shifts \(s(x, y, z)\) in equation 12 for a 3D seismic image with \(N\) samples, we solve its corresponding least-squares problem expressed in a matrix form:

\[
(WG)^T W G s = (WG)^T W v,
\]

where \(s\) is an \(N \times 1\) vector containing the unknown shifts \(s(x, y, z)\), \(G\) is a \(3N \times N\) sparse matrix representing finite-difference approximations of partial derivatives, \(W\) is a \(3N \times 3N\) diagonal matrix containing weights \(w(x, y, z)\) and \(\epsilon(x, y, z)\), and \(v\) is a \(3N \times 1\) vector with 2\(N\) slopes \(p\) and \(q\), and \(N\) zeros.

Because the matrix \((WG)^T W G\) is symmetric positive-semidefinite, we can solve the linear system of equation 14 using the preconditioned conjugate gradient method, with a preconditioner \(M^{-1}\) as in Wu and Hale (2014):

\[
M^{-1} = S_x S_y S_z S_x^T S_y^T S_z^T,
\]

where \(S_x\), \(S_y\) and \(S_z\) are filters that smooth in the \(x\), \(y\) and \(z\) directions, respectively.

For a seismic image with unconformities, the filters \(S_x\), \(S_y\) and \(S_z\) are spatially variant filters designed as in equation 11, to preserve discontinuities in shifts \(s(x, y, z)\) at unconformities.

3.2.3 Results

With the computed shifts \(s(x, y, z)\), we first generate a relative geologic time (RGT) volume \(\tau(x, y, z) = z + s(x, y, z)\) (Figures 9a and 10a). We then use the RGT volume to map a seismic image \(f(x, y, z)\) (Figures 6 or 7) in the depth-space domain to a flattened image \(\tilde{f}(x, y, \tau)\) (Figures 9b or 10b) in the RGT-space domain.

From the 2D example shown in Figure 9, the RGT (Figure 9a) and flattened (Figure 9c) images, generated with inaccurate seismic normal vectors (Figure 8a and 8c) and without unconformity constraints, are incorrect at unconformities, where we expect discontinuities in the RGT image and corresponding gaps in the flattened image. With more accurate seismic normal vectors (Figure 8b and 8d) and with constraints derived from unconformity likelihoods (Figure 6), we obtain an improved RGT image (Figure 9b) with discontinuities at unconformities. Using this RGT image, we obtain an improved flattened image (Figure 9c), in which seismic reflectors are horizontally flattened and unconformities appear as vertical gaps.

Figure 10 shows a 3D example with two unconformity surfaces, highlighted by unconformity likelihoods in Figure 7. We observe obvious discontinuities in RGT at unconformities in our generated RGT volume (Figure 10a). These RGT discontinuities result in vertical gaps in the corresponding flattened seismic image (Figure 10b). The time slice of an RGT image shows large RGT variations between samples in the lower-left and upper-right areas that are separated by an unconformity. This indicates that the sediments, represented by the samples in the two different areas, are deposited in two different sedimentary sequences occurring at different geologic times.

4 CONCLUSION

We have proposed a method to obtain an unconformity likelihood attribute from the differences between two seismic normal vector fields estimated from two structure-tensor fields, one is computed using a vertically causal smoothing filter, and the other using a vertically anti-causal filter. From a seismic image, we first compute smoothed outer products of image gradients by applying a structure-oriented smoothing filter to each element of these outer products. These smoothed outer products are then smoothed using vertically causal and anti-causal filters to compute two different structure-tensor fields, and their corresponding normal vector fields.

Using structure-oriented smoothing filters for the
outer products, we extend structure variations from a termination area to the corresponding correlative conformity. In doing this, we assume that the correlative conformity is not dislocated by faults. If faults appear in the seismic image, we could perform unfaulting (Luo and Hale, 2013) before attempting to detect unconformities.

We use separate vertically causal and anti-causal filters to obtain structure tensors that differ at unconformities. Unconformity likelihoods might be further improved by instead using causal and anti-causal filters that smooth in directions orthogonal to unconformities.

As examples of applications, we have shown how to estimate more accurate seismic normal vectors and better flatten seismic reflectors at unconformities by using unconformity likelihoods as constraints.

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Simultaneous correlation of multiple well logs

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ABSTRACT
Well log correlation is an important step in geophysical interpretation, but as the number of wells increases, so does the complexity of the correlation process. We propose a new method for automatic and simultaneous well log correlation that provides a globally optimal alignment of all logs, and in addition, is relatively insensitive to large measurement errors common in well logs. First, for any number of well logs, we use a new variant of the dynamic warping method, requiring no prior geologic information, to find for each pair of logs a set of corresponding depths. Depths in one log may have one or more corresponding depths in another log, and many such pairs of corresponding depths can be found for any pair of well logs. Requiring consistency among all such pair-wise correlations gives rise to an overdetermined system of linear equations, with unknown depth shifts to be computed for each log sample. A least-squares solution of these equations, found using the conjugate gradient method, yields for every well log a sequence of depth shifts that optimally align all well logs.

Key words: well logs correlation alignment dynamic warping least-squares

1 INTRODUCTION
Correlation of well logs is an important step in geophysical and geological interpretation tasks, such as building geologic models and time-depth conversion. Well log correlation is the process of determining corresponding depths among well logs. A single set of such corresponding depths often represents a single geologic time in which sediments with similar properties were deposited over large areas. Today these properties are measured in well logging.

We can therefore view well log correlation as the task of mapping each well log from depth to geologic time. An example of this mapping is illustrated in Figure 1, where we use the phrase “relative geologic time” to indicate that the geologic time scale used here is arbitrary. We know only that sediments with larger relative geologic time (RGT) were deposited before those with smaller RGT. For each RGT in Figure 1, we have a set of six corresponding depths, one for each well log.

While a small number of well logs can be manu-
wise correlations of logs we find that a depth problem in well log correlation, suppose that in pair-wise correlations can be made, as indicated by the 15 pair-wise correlations shown in Figure 1, which correspond to 342 distinct walks along three or more edges, beginning and ending at the same vertex, without repetition.

To see why this large number of cycles poses a problem in well log correlation, suppose that in pair-wise correlations of logs we find that a depth \( z_1 \) in log 1 corresponds to another depth \( z_2 \) in log 2, which in turn corresponds to a third depth \( z_3 \) in log 3. Then, in the pair-wise correlation of logs 1 and 3, we should find that depth \( z_1 \) corresponds to depth \( z_3 \). Imposing this sort of constraint for all 342 cycles among three or more well logs is an impossible task for human interpreters to perform in pair-wise correlation of six logs. Moreover, because the number of cycles (constraints) grows factorially with the number of logs, this task is usually infeasible even for computers.

Several methods for automatic correlation have been developed to solve two main problems: correlation of a single pair of well logs and correlation of many well logs. Cross correlation methods, such as that described by Rudman and Lankston (1973), have been used to perform log correlation on a single pair of logs, but these methods fall short when logs contain gaps in geologic time due to faults or erosion. With this shortcoming in mind, Smith and Waterman (1980) developed a dynamic waveform matching approach that finds an optimal alignment between two well logs and is not affected by gaps in geologic sequences.

Also using dynamic warping, Wu and Nyland (1987) perform pair-wise well log correlation in two main steps: contact recognition and interval identification. Contact recognition uses a statistical approach to partition a log into distinct segments that highlight contacts between beds. Interval identification uses dynamic warping to align corresponding segments among two well logs. Any of the aforementioned methods could be used to correlate any of the 15 log pairs represented by line segments in Figure 2.

For correlating many well logs, many different approaches exist. The method developed by Le Nir et al. (1998) uses the most geologically complete well as a key well to which all other well logs are aligned. Alternatively, Fang and Chen (1992) and Kovalevsky et al. (2007) propose divide-and-conquer methods for correlating multiple well logs. In these methods, correlations are first performed for the three pairs that can be formed from only three logs. After an acceptable alignment is found, a larger number of logs are correlated using the alignment of the previous well logs as constraints for the new correlations. This process of increasing the number of logs in the correlation is continued until all logs have been aligned. Due to the propagation of error in such methods, the final correlation of all logs depends on the order in which the wells are correlated.

In this paper, we propose a method for automatically and simultaneously correlating any number of well logs. Our method is relatively insensitive to large measurement errors that are common in well logs and provides a globally optimal (least-squares best) alignment of all logs.

Using the six deepest velocity logs from Teapot Dome as an example, we first explain our modifications to the dynamic warping algorithm and show how it can be used to find corresponding depths for any pair of logs. We then show how to use these corresponding depths to find shifts for all logs that optimally align them.

2 CORRELATING TWO LOGS

In general, the range of depths sampled within a set of well logs varies. Therefore, before log correlation, we resample all logs such that each log has \( N_x = (z_{\max} - z_{\min})/\Delta z + 1 \) samples, where \( z_{\min} \) and \( z_{\max} \) denote the minimum and maximum depths sampled in all logs and \( \Delta z \) is a specified sampling interval. In practice, missing data commonly occur within well logs due to gaps in acquisition, and null values serve as placeholders for any missing data in a log. For the Teapot Dome example shown in Figure 1, we chose a depth sampling interval \( \Delta z = 1 \text{ m} \).

2.1 Alignment errors

Dynamic warping is commonly used in automatic well log correlation to estimate a sequence of pairs of indices \((i, j)\), called a path, that optimally aligns a pair of well logs (Smith and Waterman, 1980; Lineman et al., 1987; Le Nir et al., 1998). The first step in finding this optimal path is to compute alignment errors for each sample \( f_j[i] \).
Simultaneous correlation of multiple well logs

of log \( I \) and each sample \( f_j[j] \) of log \( J \) as follows:

\[
e_{IJ}[i,j] = |f_i[i] - f_j[j]|^p \tag{1}
\]

for \( i = 0, 1, \ldots, N_i - 1 \) and \( j = 0, 1, \ldots, N_j - 1 \), where \( I \) and \( J \) are indices of two logs, \( i \) and \( j \) are indices of depths sampled in those two logs, and \( p \) is any positive value. Beginning at \((i = 0, j = 0)\), alignment errors are accumulated along all possible paths across the \( ij \)-grid to find the accumulated alignment errors at the end of each path. The accumulated alignment error \( D \) at the end of the optimal path is therefore defined as

\[
D = \min_{\text{path}} \sum_{(i,j) \in \text{path}} e_{IJ}[i,j]. \tag{2}
\]

Figure 3a shows two possible warping paths \( a \) and \( b \) for logs \( I \) and \( J \) on the \( ij \)-coordinate system. Each point on a path represents a pair of corresponding depths in logs \( I \) and \( J \).

A path along the diagonal of the \( ij \)-grid, from \((0, 0)\) to \((N_i - 1, N_j - 1)\), implies that log values for indices \( i = 0, 1, 2, \ldots, N_i - 1 \) in log \( I \) are already aligned with those for indices \( j = 0, 1, 2, \ldots, N_j - 1 \) in log \( J \), respectively. Path \( b \) lies relatively close to the diagonal of the \( ij \)-grid, which implies that the pairs of log values in path \( b \) are reasonably well-aligned without warping. However, for path \( b \), depths in log \( I \) are greater than corresponding depths in log \( J \), because this path lies below the diagonal. The opposite is true for path \( a \); depths in log \( I \) are shallower than corresponding depths in log \( J \).

Additionally, path \( a \) has fewer \((i,j)\) pairs than path \( b \) and is therefore likely to have less accumulated alignment error, simply because it is shorter. For this reason, some authors force the optimal path to pass through points \((0, 0)\) and \((N_i - 1, N_j - 1)\), but this requires manual interpretation of the first and last corresponding depths (Le Nir et al., 1998; Wu and Nyland, 1987).

To address this problem, we instead compute alignment errors in a rotated \( kl \)-coordinate system with

\[
k = j + i \quad \iff \quad i = \frac{k - l}{2}, \tag{3}
\]

\[
l = j - i \quad \iff \quad j = \frac{k + l}{2}.
\]

where \( k \) indexes depth and \( l \) indexes lag, a difference between two depths. In the new \( kl \)-coordinate system, alignment errors are now

\[
e_{IJ}[k,l] = \left| f_i \left[ \frac{k - l}{2} \right] - f_j \left[ \frac{k + l}{2} \right] \right|^p \tag{4}
\]

for \( k = 0, 1, \ldots, N_k - 1 \), \( l = 1_{\text{min}}, \ldots, l_{\text{max}} \), where \( N_k = 2N_i - 1 \), \( l_{\text{min}} \) and \( l_{\text{max}} \) are specified bounds on lag (depth difference), and \( p \) is any positive value. We chose \( p = 0.25 \) to reduce the influence of large measurement errors in well logs. As the value of \( p \) decreases, the alignment error \( e_{IJ}[k,l] \) becomes less sensitive to such outliers. Because \( i \) and \( j \) must be integers, equation 4 implies that alignment errors \( e_{IJ}[k,l] \) are calculated only for values of \( k \) and \( l \) for which \( k + l \) (and \( k - l \)) is even.

In dynamic warping, we now seek a path, a sequence of \((k,l)\) pairs, that minimizes the accumulated alignment errors:

\[
D = \min_{\text{path}} \sum_{(k,l) \in \text{path}} e_{IJ}[k,l]. \tag{5}
\]

Figure 3b shows paths \( a \) and \( b \) on the \( kl \)-coordinate system with path \( a \) extending into areas where either \( f_i[i] \) or \( f_j[j] \) are null. Both paths begin at \( k = 0 \) and end at \( k = N_k - 1 \), so that accumulated alignment errors for path \( a \) will not be less than those for path \( b \), simply because path \( a \) is shorter, as in Figure 3a.

### 2.2 Replacing missing data

In practice, it is common for the optimal path to pass through \((i,j)\) pairs where either \( f_i[i] \) or \( f_j[j] \) is null, as for path \( a \) in Figure 3b. When calculating alignment errors, we temporarily replace null values in \( f_i[i] \) and \( f_j[j] \) with randomly chosen non-null values from these respective logs. Replacing missing data in this way yields alignment errors comparable to those computed from non-null values.

Suppose, instead, that null values are replaced with zeros. An alignment error \( e_{IJ}[k,l] \) calculated for non-null \( f_i[i] \) and \( f_j[j] \) would be equal to a power \( p \) of the non-null value of \( f_i[i] \). Paths passing through several \((i,j)\) pairs for which either \( f_i[i] \) or \( f_j[j] \) are null would therefore have relatively large accumulated alignment errors and would not be optimal. Alternatively, if both \( f_i[i] \) and \( f_j[j] \) are null, the alignment error \( e_{IJ}[k,l] \) would equal zero and the optimal path would tend to pass through the point \((i,j)\).
2.3 Accumulation and backtracking

From the alignment errors $e_{I,J}[k,l]$, we recursively compute accumulated alignment errors $d_{I,J}[k,l]$ as follows:

$$d_{I,J}[0,l] = e_{I,J}[0,l], \quad (6)$$

$$d_{I,J}[1,l] = e_{I,J}[1,l] + \min \begin{cases} d_{I,J}[0,l-1] \\ d_{I,J}[0,l+1] \end{cases}, \quad (7)$$

$$d_{I,J}[k,l] = e_{I,J}[k,l] + \min \begin{cases} d_{I,J}[k-1,l-1] \\ d_{I,J}[k-1,l+1] \\ d_{I,J}[k-2,l] \\ d_{I,J}[k-1,l+1] \end{cases}, \quad (8)$$

for $k = 2, 3, \ldots, N_k - 1$.

All lags $l$ in the specified range $[l_{\min}, l_{\max}]$ must be considered in the calculation of accumulated alignment errors $d_{I,J}[k,l]$ because we do not yet know which lags lie on the optimal path. The computational stencil for equation 8 is shown in Figure 4 for one sample of $d_{I,J}[k,l]$. Like $e_{I,J}[k,l], d_{I,J}[k,l]$ is computed only where $k + l$ (and hence $k - l$) is even.

After accumulation, we scan the accumulated alignment errors $d_{I,J}[k,l]$ at index $k = N_k - 1$ to find the indices $(k,l)$ for the minimum accumulated alignment error $D$ at the end of the optimal path. From equation 8, it is clear that the previous sample on the path must be $(k-1, l-1), (k-2, l),$ or $(k-1, l+1)$, depending upon which of the three accumulated alignment errors there is smallest. This backtracking is performed for $k = N_k - 1, N_k - 2, \ldots, 0$ to find the optimal sequence of $(k,l)$ pairs, which in turn represent pairs of corresponding depths. The solid white curve in Figure 5b represents the optimal sequence of $(k,l)$ pairs (the optimal path) for velocity logs $I = 1$ and $J = 3$ from Teapot Dome. The diagonal white line of large alignment errors near index $i = 1401$ is due to a large measurement error (velocity = 6880 m/s) in log 1 at index $i = 1391$.

By warping each of the 15 log pairs (denoted by line segments in Figure 2), we obtain for each log pair a sequence of corresponding depths. Figure 6 displays three pairs of velocity logs before and after alignment using our modified dynamic warping algorithm. Figures 6b, 6d, and 6f show log values for only the computed corresponding log depths. Although several measurement errors are apparent in log 4, they have little effect on the alignment of the logs in Figure 6f.

3 CORRELATING ALL LOGS

Although optimal correlations have been found for each pair of well logs, we require that pair-wise correlations be consistent for all logs. For example, suppose we choose a depth $z_I$ in log $I$ and find corresponding depths in all other logs. From Figure 7 we can see that corresponding depths $z_I$ and $z_J$ in logs $I$ and $J$ are offset by some depth shift $s$. If we sum depth shifts like this one, for each pair of corresponding depths along any of the 342 cycles in Figure 2, we expect the sum of all of the shifts to be zero when we return to depth $z_I$ in log $I$. This consistency cannot be achieved with dynamic warping alone.

Therefore, after computing many pairs of corresponding depths like that illustrated in Figure 7, we must find depth shifts $s$ so that all pairs of corresponding depths are consistent. For every depth $z_J$ in log $J$, let $\tau_I$ denote a corresponding relative geologic time defined by

$$\tau_I(z_I) \equiv z_I + s_I(z_I), \quad (9)$$

where $s_I(z_I)$ represents the depth shift for log $I$ at depth $z_I$, $I = 1, \ldots, L$, and $L$ is the number of logs. Depths $z_I$ in log $I$ and $z_J$ in log $J$ correspond if they have the same relative geologic time, $\tau_I = \tau_J$, so that

$$z_I + s_I(z_I) = z_J + s_J(z_J). \quad (10)$$
Simultaneous correlation of multiple well logs

Figure 6. Three velocity log pairs before (a, c, e) and after (b, d, f) warping. Log indices \((I, J)\) are denoted in black and red corresponding to the colors of the logs.

Figure 7. Each depth \(z_I\) in log \(I\) has one or more corresponding depths \(z_J\) in log \(J\). For such pairs of depths, log values \(f_I(z_I)\) and \(f_J(z_J)\) should be similar.

Rearranging equation 10 so that corresponding depths are on the right and unknown shifts are on the left, we have

\[
s_I(z_I) - s_J(z_J) = z_J - z_I. \tag{11}
\]

Every depth in every log will have a corresponding shift, yielding \(N_x \times L\) unknown shifts \(s\). The number of pairs of corresponding depths we can find in all logs depends on the number of non-null values in the logs, but typically is much greater than the number of shifts. Therefore, equation 11 gives rise to a system of linear equations with many more equations than unknowns. We use the conjugate gradient method to find a least-squares solution to these equations.

The solution is not unique, because correlated well logs, such as those in Figure 1b, will remain aligned if all logs are shifted or squeezed or stretched vertically. To prevent such unnecessary shifting while correlating, a preconditioner for the conjugate gradient method is applied to constrain the average shift over all logs to be zero for all depths.

After shifts \(s\) have been computed in this way, equation 9 can be used to compute a time \(\tau_I\) for each sampled depth \(z_I\) and thereby find a log value \(f_I(\tau_I)\) for every time \(\tau_I\). Specifically, we first convert the function \(\tau_I(z_I)\) to \(z_I(\tau_I)\) and then compute \(f_I(\tau_I) = f_I(z_I(\tau_I))\). Figure 1a shows velocity logs \(f_I(z_I)\) for \(I = 1, 2, \ldots, 6\); and Figure 1b shows the same logs \(f_I(\tau_I)\) after correlation.

4 DISCUSSION

In our least-squares solution for shifts \(s_I(z_I)\), weights can be assigned to each equation 11 to improve the accuracy of well log correlations. For example, other information, perhaps from human interpreters, can be used to determine pairs of corresponding depths for which higher weights might be assigned. When such information is not available, weights might be assigned in other ways. For instance, we might assign weights that decrease with distance between wells. In all examples shown in this paper, all corresponding depths in all wells were weighted equally. Even so, resulting correlations reveal consistent thin beds in the velocity logs displayed in Figure 8a.

Consider the 11 gamma ray logs from Teapot Dome shown after correlation in Figure 8b. In this example, there are 55 possible pair-wise correlations and over 30 million cycles among the well locations. Consistent manual correlation of these 11 logs is therefore infeasible. Note also that, although many large measurement errors are apparent in the gamma ray logs, our automatic method yields consistent correlations.

5 CONCLUSION

Our method for automatic and simultaneous correlation of well logs is twofold. First, we use our modified dynamic warping algorithm to find for each log pair a
Figure 8. Close up of (a) deepest six velocity logs and (b) deepest 11 gamma ray logs from Teapot Dome after correlation.

sequence of corresponding depths. Our procedure for pair-wise log correlation is similar to existing automatic well log correlation methods, but differs primarily in the calculation of alignment errors. Using a transformed coordinate system, we ensure that all possible warping paths have equal length, so that a path will not be optimal solely due to its short length. At the end of this first step, we’ve done nothing yet to guarantee consistency of pair-wise correlations over all wells.

Using a least-squares method, we then find depth shifts for every depth in every log that maximize consistency among all pairs of corresponding depths. By applying these depth shifts, we are able to find relative geologic times for every log depth, and thereby map the well logs from depth to relative geologic time. With examples from Teapot Dome, we have shown that our method yields consistent correlations and is robust in the presence of large measurement errors frequently found in well logs.

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REFERENCES

Implementing an anisotropic and spatially varying Matérn model covariance with smoothing filters

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Figure 1. A horizon slice of a 3D seismic image (a) provides a model of spatial correlation (b) for an anisotropic and spatially varying Matérn model covariance used here in a geostatistical simulation of porosities (c). The model covariance is implemented with smoothing filters.

ABSTRACT
While known to be an important aspect of geostatistical simulations and inverse problems, an a priori model covariance can be difficult to specify and implement, especially where that model covariance is both anisotropic and spatially varying. The popular Matérn covariance function is extended to handle such complications, and is implemented as a cascade of numerical solutions to partial differential equations. In effect, each solution is equivalent to application of an anisotropic and spatially varying smoothing filter. Suitable filter coefficients can be obtained from auxiliary data, such as seismic images. An example with simulated porosities demonstrates the effective use of a Matérn model covariance implemented in this way.

Key words: inversion model covariance

1 INTRODUCTION
The solution of many inverse problems in geophysics is facilitated by a priori information about the desired solution. In least-squares inverse theory this information is provided in the form of an initial model estimate \( \mathbf{m}_0 \) and a covariance matrix \( \mathbf{C}_M \), which can be used to compute a better (a posteriori) model estimate \( \tilde{\mathbf{m}} \) as follows (Tarantola, 2005):

\[
\tilde{\mathbf{m}} = \mathbf{m}_0 + \mathbf{C}_M \mathbf{G}^\top (\mathbf{G} \mathbf{C}_M \mathbf{G}^\top + \mathbf{C}_D)^{-1} (\mathbf{d} - \mathbf{G} \mathbf{m}_0). \tag{1}
\]

Here, \( \mathbf{d} \) denotes observed data, which are assumed to be approximately related to the true model \( \mathbf{m} \) by \( \mathbf{d} \approx \mathbf{G} \mathbf{m} \), for some linear operator \( \mathbf{G} \). The data covariance matrix \( \mathbf{C}_D \) quantifies uncertainties due to errors (e.g., measurement errors or ambient noise) in this approximation, while the model covariance \( \mathbf{C}_M \) quantifies spatial correlation of the model \( \mathbf{m} \).

The matrices in equation 1 can be viewed more generally as linear operators, and this view is especially useful for the model covariance matrix \( \mathbf{C}_M \). For a model \( \mathbf{m} \) with \( M \) parameters, the matrix \( \mathbf{C}_M \) would contain \( M^2 \).
elements, which for large $M$ cannot be stored in computer memory. Moreover, analytical expressions for the elements of $C_M$ may be unavailable, as when $m$ is a sampled function of space with covariance that is both anisotropic and spatially varying. We are therefore motivated to implement multiplication by $C_M$ in equation 1 as an algorithm that applies the linear operator $C_M$ without explicitly constructing and storing a matrix.

Although we might avoid computing and storing the matrix $C_M$, we must still specify parameters that describe this linear operator. This task can be especially difficult for anisotropic and spatially varying models of spatial correlation.

Figure 1 illustrates one way to parameterize the linear operator $C_M$ using additional information. The spatial correlation of seismic amplitudes displayed in Figure 1a is clearly anisotropic and spatially varying. A quantitative measure of this spatial correlation is obtained from structure tensors (Weickert, 1999; Fehmers and Höcker, 2003) computed for every sample in the seismic image. In Figure 1b, a small subset of these structure tensors are represented by ellipses. Spatial correlation of seismic amplitudes is high at locations where ellipses are large and in directions in which they are elongated. In this paper I show how this tensor field can almost completely parameterize an anisotropic and spatially varying model covariance operator $C_M$.

Moreover, if the model covariance operator can be factored such that $C_M = FF^\top$, then we can easily simulate models $m$ with covariance $C_M$ by applying the operator $F$ (or $F^\top$) to an image of random numbers (Cressie, 1993). Figure 1c displays a simulated model $m$ of porosities computed in this way. In this example, porosity is not directly correlated with seismic amplitude. In other words, we cannot accurately predict porosity at some location from the seismic amplitude at that location. However, the spatial correlation of porosities mimics that of seismic amplitudes, because the model covariance operator $C_M$ used in this simulation was derived from the seismic image.

In this paper I describe a method for using tensor-guided smoothing filters to implement a linear operator $C_M$ that approximates the Matérn covariance, which is widely used in geostatistics (Stein, 1999). My implementation of $C_M$ is an approximation that extends the Matérn covariance to be both anisotropic and spatially varying. I illustrate the use of this $C_M$ in a tensor-guided kriging method for gridding data sampled at scattered locations.

2 THE MATÉRN COVARIANCE

In its simplest form, the Matérn covariance function is defined by

$$c(r) = \frac{2^{1-\nu} \Gamma(\nu)}{\Gamma(\nu)} r^\nu K_\nu(r),$$  

where $r$ is the Euclidean distance between two points in space, $\nu$ is a positive real number that controls the function’s shape, and $K_\nu(r)$ denotes the modified Bessel function of the second kind with order $\nu$. The function $c(r)$ is normalized to have unit variance $c(0) = 1$, but may easily be scaled to have any variance $\sigma^2$.

Figure 2 displays the Matérn covariance function $c(r)$ for four different choices of the shape parameter $\nu$. For any value of $\nu$, this function decays smoothly and monotonically with increasing distance $r$. For $\nu = 0.5$, the Matérn covariance is simply the exponential function $c(r) = e^{-r}$.

Despite its somewhat complex definition in terms of special functions, the Matérn covariance function is widely used in spatial statistics (Stein, 1999), partly because of the flexibility provided by the shape parameter $\nu$. Indeed, equation 2 is sometimes described as defining the Matérn family of covariance functions, because any $\nu > 0$ yields a valid (positive definite) covariance function, for any number of spatial dimensions.

For $d$ spatial dimensions, the Fourier transform of $c(r)$ is

$$C(k) = \frac{\Gamma\left(\frac{d}{2} + \nu\right)}{\Gamma(\nu)} \frac{(2\sqrt{\pi})^d}{(1 + k^2)^{\frac{d}{2} + \nu}},$$

where $k$ is the magnitude of the wavenumber vector $k$. Because the covariance function $c(r)$ is real and symmetric about the origin, its Fourier transform $C(k)$ is real and symmetric as well. Because $C(k)$ decays monotonically with increasing $k$, we may view the Matérn covariance function $c(r)$ as the impulse response of a smoothing filter that attenuates high spatial frequencies.

2.1 Range scaling

Figure 2 illustrates that the effective width or range of the Matérn covariance function increases as the shape parameter $\nu$ increases. In practice, we wish to specify both the shape and the range of this function, indepen-
Anisotropic covariance functions can be obtained by using a spatially varying tensor field \( D(x) \). In practice, where \( D \) varies spatially, such that \( D = D(x) \), it is most convenient to keep these factors separate, so that we can adjust the shape \( \nu \) or effective range \( a \) without modifying the tensor field \( D(x) \).

Where the tensor \( D \) is spatially invariant, we can apply the Matérn model covariance operator \( C_M \) by convolution with the function \( c(r) \). Let \( p(x) \) denote the input to the function that applies the operator \( C_M \) to obtain an output \( q(x) \). Then

\[
q(x) = \int p(y) c \left( \sqrt{(x - y)^\top D^{-1}(x - y)} \right) \, dy.
\]

Equivalently, and perhaps more efficiently, we can

(i) Fourier transform \( p(x) \) to obtain \( P(k) \),
(ii) compute \( Q(k) = C \left( \sqrt{k\top Dk} \right) P(k) \), and
(iii) inverse Fourier transform \( Q(k) \) to obtain \( q(x) \).

Note that, for either convolution with \( c(r) \) or multiplication by \( C(k) \), we need not construct and store a matrix representing \( C_M \).

The problem addressed in this paper is that neither convolution in the space domain nor multiplication in the wavenumber domain is valid when the tensors \( D \) vary spatially. In this case, the output \( q(x) \) should be computed as the solution to a partial differential equation with spatially varying coefficients.

### 2.2 Partial differential equations

To simplify the discussion below, let us consider only the 2D case for which \( d = 2 \), although the methods proposed in this paper can be extended to any number of spatial dimensions. In 2D, the (unscaled) Fourier transform of the Matérn covariance is simply

\[
C(k) = \frac{4\pi\nu}{(1 + k^2)^{1+\nu}}.
\]

This simple form for \( C(k) \) follows from equation 3 and the identity \( \Gamma(1 + \nu) = \pi\Gamma(\nu) \).

Multiplication by the Fourier transform \( C(k) \) of the 2D Matérn covariance function has been shown (Whittle, 1954; Guttorp and Gneiting, 2006) to be equivalent to solving the following partial differential equation (PDE):

\[
(1 - \nabla \cdot \nabla)^{1+\nu} q(x) = 4\pi\nu p(x).
\]

This equivalence results from the fact that multiplication by \( k^2 \) in the wavenumber domain is equivalent to applying the differential operator \( -\nabla \cdot \nabla \) in the space domain.

Our reason for considering solution of partial differential equations like equation 10 is the need to apply the Matérn model covariance operator in contexts where the direction and extent of correlation are described by a spatially varying tensor field \( D(x) \). In such contexts equation 10 should be rewritten as

\[
|D|^{-\frac{1}{2}}(1 - \nabla \cdot D(x) \cdot \nabla)^{1+\nu} |D|^{-\frac{1}{2}}(x)q(x) = 4\pi\nu p(x),
\]
This equation is analogous to equation 10, with the anisotropic range scaling of equation 7. Note that we must move the factor \(|D|^{\frac{1}{2}}\) in equation 7 to the left-hand side of equation 11 and split it into two parts, to ensure that the product of symmetric positive definite (SPD) operators on the left-hand side remains symmetric. That product is proportional to the inverse of the desired Matérn model covariance operator \(C_M\), and so must be SPD.

For any tensor field \(D(x)\), solution of equation 11 is straightforward when the shape parameter \(\nu\) is an integer. This is one reason that Whittle (1954) considered the integer shape \(\nu = 1\) to be most natural for 2D problems. In a similar but more recent context, Fuglstad (2011) and Lindgren et al. (2011) have likewise assumed integer \(\nu\) and thereby avoided the complexities of fractional PDEs.

In practical applications with spatially invariant 2D Matérn model covariances, commonly used values for the shape parameter \(\nu\) lie in the interval \([0.5, 1.5]\), which includes only the one integer value \(\nu = 1\). In practice, permitting only integer \(\nu\) may reduce the Matérn family of covariance functions to just one function.

3 A SMOOTHING COVARIANCE

To facilitate more general (non-integer \(\nu\)) shapes of covariance functions in the Matérn family, let us consider approximations to the fractional partial differential equation 11. For simplicity in developing these approximations, I temporarily omit the tensor field \(D(x)\) and use the Fourier transform \(\hat{C}(k)\) in equation 9 as a convenient shorthand for equations 10 and 11.

The approximation to \(\hat{C}(k)\) proposed here is of the form

\[
\hat{C}(k) = \frac{\gamma}{(1 + \alpha k^2)^\frac{\nu}{2}(1 + \beta k^2)^\gamma}, \tag{12}
\]

where \(\alpha, \beta, \gamma, r\), and \(l\) are constants computed from the shape \(\nu\) of the desired Matérn covariance function. Computation of the constant integer \(l\) is easy: \(l = \lfloor 1 + \nu \rfloor\). For example, if \(\nu = \frac{1}{2}\), then \(l = 1\); if \(\nu = 1\), then \(l = 2\).

If \(\nu\) is an integer, then \(l = 1 + \nu, \alpha = 1, \beta = 0, \) and \(\gamma = 4\pi \nu r\) yields an exact match to the Matérn covariance function. In this case \(\hat{C}(k)\) in equation 12 exactly equals \(C(k)\) in equation 9, and no approximation is required.

Otherwise, after computing \(l\), I compute the three non-negative constants \(\alpha, \beta, \gamma\) so that the approximate covariance function \(\hat{c}(r)\) corresponding to \(\hat{C}(k)\) matches exactly three values of the Matérn covariance function \(c(r)\) given by equation 2. To obtain an approximation that is accurate for both large and small distances \(r\), I choose to match the values 0.1, 0.9 and 1.0.

I first express the scale factor \(\gamma\) in terms of \(\alpha\) and \(\beta\) so that \(\hat{c}(0) = 1\), which is one of the three values to be matched. Recall that \(\hat{c}(0)\) is just the 2D inverse Fourier transform of \(\hat{C}(k)\) evaluated at \(r = 0\) which, in turn, equals the 2D integral of \(\hat{C}(k)\) divided by \(4\pi^2\). By analytically performing this integration I obtained the expressions for \(\gamma\) listed in Table 1.

I then use bisection to find distances \(r_1\) and \(r_9\) such that \(c(r_1) = 0.1\) and \(c(r_9) = 0.9\). Finally, I use the iterative Newton-Raphson method to compute the constants \(\alpha\) and \(\beta\) such that \(\hat{c}(r_1) = 0.1\) and \(\hat{c}(r_9) = 0.9\). The Newton-Raphson iterations require values and derivatives (with respect to \(\alpha\) and \(\beta\)) of the approximate covariance function \(\hat{c}(r)\).

Using symbolic mathematical software, it is possible to obtain \(\hat{c}(r)\) as a function of \(\alpha\) and \(\beta\) by analytical 2D inverse Fourier transform of \(\hat{C}(k)\) in equation 12. The value of \(\hat{c}(r)\) is given by one of the expressions in Table 2. Though straightforward to compute, these expressions are somewhat complicated, so I use finite differences to approximate derivatives of \(\hat{c}(r)\) with respect to \(\alpha\) and \(\beta\), as required by the Newton-Raphson method. I begin the Newton-Raphson iterations with initial values \(\alpha = 1\) and \(\beta = 0\).

The results of the fitting process are listed in Table 3. In practical applications we might use these tabulated values directly to obtain adequate approximations to \(c(r)\) or \(\hat{C}(k)\). Note that for integer shapes \(\nu\), no approximation is required.

Table 1. The scale factor \(\gamma\) in \(\hat{C}(k)\), as a function of \(\alpha\) and \(\beta\), chosen so that \(\hat{c}(0) = 1\). See equation 12.

<table>
<thead>
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<th>(l)</th>
<th>(\gamma)</th>
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</thead>
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<tr>
<td>1</td>
<td>(\frac{4\pi(\alpha-\beta)}{\log(\alpha/\beta)})</td>
</tr>
<tr>
<td>2</td>
<td>(\frac{4\pi(\alpha-\beta)^2}{\alpha-\beta-\beta\log(\alpha/\beta)})</td>
</tr>
<tr>
<td>3</td>
<td>(\frac{8\pi(\alpha-\beta)^3}{(\alpha-\beta)(\alpha-\beta)+2\beta^2\log(\alpha/\beta)})</td>
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</table>

Figure 4 displays approximations to Matérn covariance functions computed in this way. All of these approximations match the Matérn covariance for at least three values of distance \(r\), those with covariance values 0.1, 0.9 and 1.0. For shape \(\nu = 1\), no approximation is necessary, since for this case we have simply \(l = 2\), \(\alpha = 1, \beta = 0, \) and \(\gamma = 4\pi\). For shape \(\nu = 1.5, \) the approximate covariance is almost indistinguishable from the Matérn covariance. The approximations are worse for \(\nu < 1\), but even for \(\nu = 0.5\) may be adequate.

Although I derived the covariance functions \(\hat{c}(r)\) displayed in Figure 4 as approximations, we may consider them to be a practical alternative to the Matérn family of covariance functions \(c(r)\) displayed in Figure 2. We can control the shape of a function in this alternative family using a single parameter \(\nu\), just as we might do for the Matérn family. A key advantage of the alternative covariance functions \(\hat{c}(r)\) is that corresponding model covariance operators can be easily applied by solving partial differential equations.
Table 3. Parameters for covariance functions $\tilde{c}(r)$.

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$l$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
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3.1 PDE implementations

Recall that the motive for an alternative family of covariance functions $\tilde{c}(r)$ is that factors in their Fourier transforms $C(k)$ have only integer exponents, which greatly simplifies PDE implementations of the corresponding model covariance operators $C_M$. The PDE corresponding to $C(k)$ in equation 12 is

\[
(1 - \alpha \nabla \cdot \nabla) \left(1 - \beta \nabla \cdot \nabla\right) q(x) = \gamma p(x). \quad (13)
\]

For anisotropic and spatially varying tensor fields $D(x)$, the corresponding PDE is

\[
|D|^{-\frac{1}{2}}(x) (1 - \alpha \nabla \cdot D(x) \cdot \nabla)^l \left(1 - \beta \nabla \cdot D(x) \cdot \nabla\right) |D|^{-\frac{1}{2}}(x) q(x) = \gamma p(x). \quad (14)
\]

I have again constructed the product of SPD operators on the left-hand side of equation 14 to be SPD. To see this, note that the differential operators $1 - \alpha \nabla \cdot D(x) \cdot \nabla$ and $1 - \beta \nabla \cdot D(x) \cdot \nabla$ share the same eigenvectors, which implies that they commute. Therefore, the composite left-hand-side operator is SPD.

To solve the partial differential equation 14 numerically, we could approximate the differential operators with finite differences, and then use the method of conjugate gradients to compute the output $q(x)$. However, the condition number for the complete left-hand-side operator grows exponentially with the number of differential operators in parentheses. As an example, for $\nu = 1$ $(l = 2, \alpha = 1, \beta = 0)$, the condition number for the complete operator is the square of that for each of the two differential-operator factors, so that a large number of iterations may be required for convergence of the conjugate-gradient method.

A more efficient approach is to use the method of conjugate gradients multiple times, once for each of the differential operator factors on the left-hand side of equation 14. In this approach we solve the following
sequence of equations:

\[ q_0(x) = |\gamma^2 D(x)|^\frac{1}{2} p(x) \]

\[ (1 - \alpha \nabla \cdot D(x) \cdot \nabla) q_1(x) = q_0(x) \]

\[ (1 - \alpha \nabla \cdot D(x) \cdot \nabla) q_2(x) = q_1(x) \]

\[ \ldots \]

\[ (1 - \alpha \nabla \cdot D(x) \cdot \nabla) q_i(x) = q_{i-1}(x) \]

\[ (1 - \beta \nabla \cdot D(x) \cdot \nabla) q_L(x) = q_L(x) \]

\[ q(x) = |\gamma^2 D(x)|^\frac{1}{2} q_L(x). \quad (15) \]

In the special case where \( \beta = 0 \) (\( \nu \) is an integer), solution of the last PDE for \( q_L(x) \) is unnecessary. In any case, the total number of conjugate-gradient iterations grows only linearly, not exponentially, with the number of equations 15.

The solution of each PDE in the sequence of equations 15 is equivalent to applying a smoothing filter to a function \( q_i(x) \) on the right-hand side. Tensor coefficients \( D(x) \) can be specified so that this smoothing is both anisotropic and spatially varying. At each location \( x \) the extent of smoothing is greatest in directions of eigenvectors corresponding to the largest eigenvalues of \( D(x) \). The cascade of tensor-guided smoothing filters in equations 15 approximates the application of a Matérn model covariance operator \( C_M \) in which covariance is both anisotropic and spatial varying.

Because this approximation is implemented with smoothing filters, it is henceforth referred to as a smoothing covariance.

4 TENSOR-GUIDED KRIGING

A simple and common use of a model covariance operator \( C_M \) is in the interpolation of measurements acquired at locations scattered in space. In this application, the linear operator \( C \) in equation 1 is simply a model-sampling operator \( K \) that extracts values of the model \( \mathbf{m} \) at scattered locations to obtain data \( \mathbf{d} \approx K \mathbf{m} \). Here, the approximation is due to measurement errors that may be non-zero. Substituting the model-sampling operator \( K \) for \( C \) in equation 1, we obtain

\[ \tilde{\mathbf{m}} = \mathbf{m}_0 + C_M K\top (KC_M K\top + C_D\top)^{-1} (\mathbf{d} - K \mathbf{m}_0). \quad (16) \]

As noted by Hansen et al. (2006), the process of computing a model estimate \( \tilde{\mathbf{m}} \) with equation 16 is equivalent to gridding with simple kriging, a process well known in geostatistics. In this process, we estimate the model \( \mathbf{m} \) at locations on a uniform and dense sampling grid. The number of gridded-model samples in \( \mathbf{m} \) is typically much larger than the number of scattered-data samples in \( \mathbf{d} \). The operator \( K \) gathers values from a small subset of locations in the uniform grid, the scattered locations where data are available, and the operator \( K\top \) scatters values into those same locations.

Tarantola (2005) shows that simple kriging can also be performed in a different but equivalent way:

\[ \tilde{\mathbf{m}} = \mathbf{m}_0 + (C_M^{-1} + K\top C_D\top K)^{-1} K\top C_D\top (\mathbf{d} - K \mathbf{m}_0). \quad (17) \]

This alternative is appealing because finite-difference approximations to \( C_M^{-1} \) are compact and can be applied more efficiently than those for \( C_M \), which requires solution of partial differential equations 15. However, equations 16 and 17 are equivalent only when the inverses of matrices in these equations exist. If measurement errors are negligible, so that \( C_D \approx 0 \), then \( C_D \) is nearly singular and equation 17 is ill-conditioned. The fact that equation 17 is invalid without measurement error is just one reason to favor gridding with equation 16.

Another reason is that the size of the matrix \( K C_M K\top + C_D \) equals the number of scattered data samples, which is often small enough to enable efficient direct solution of the kriging equations 16. In contrast, the size of the matrix \( C_M^{-1} + K\top C_D\top K \) equals the (typically) much larger number of gridded model samples, so that iterative solution of equation 17 is required. When \( C_D \approx 0 \), and without a good preconditioner, convergence of iterative methods is slow.

For these reasons I choose to use equation 16 to implement gridding with an anisotropic and spatially varying model covariance. However, even with this choice, an iterative solution is required, because we lack analytic expressions for elements of the smoothing covariance matrix \( C_M \) and, hence, the composite matrix \( A_M \equiv KC_M K\top + C_D \). The smoothing equations 15 provide only a method for applying (performing multiplication by) \( C_M \). Nevertheless, that method is sufficient for iterative conjugate-gradient solution of equation 16.

4.1 Paciorek’s approximation

Convergence of conjugate-gradient iterations is greatly accelerated by a good preconditioning operator \( P \approx A_M^{-1} \), one that can be computed and applied more quickly than \( A_M \) itself. Recalling that \( A_M \equiv KC_M K\top + C_D \), one way to obtain such a preconditioner is to find a good approximation to the smoothing covariance operator \( C_M \).

Paciorek (2003) and Paciorek and Schervish (2006) propose an approximation \( C_P \approx C_M \) whose elements can be computed quickly by the following modification of equation 6:

\[ c(r) \rightarrow a(x, y) c \left( \sqrt{(x - y)^\top D^{-1}(x, y)(x - y)} \right), \quad (18) \]

where

\[ D(x, y) \equiv \frac{D(x) + D(y)}{2}, \quad (19) \]

and

\[ a(x, y) \equiv \left| D(x) \right|^\frac{1}{2} \left| D(y) \right|^\frac{1}{2} \left| D(x, y) \right|^{-\frac{1}{2}}. \quad (20) \]
In effect, these expressions approximate the covariance of the model at location \( x \) and the model at location \( y \) using averages of tensors \( D(x) \) and \( D(y) \) at only those two locations. The approximation is best where \( D(x) \) varies slowly within the effective range of that function.

Assuming that the number of scattered measurements in \( d \) is sufficiently small, say, less than 1000, we can use Paciorek’s approximation to quickly compute and store the elements of the approximate composite matrix \( A \equiv KC_kK^\top + CD \). We can then use Cholesky decomposition to compute the preconditioner \( P = A^{-1} \), for use in a conjugate-gradient solution of equation 16. The resulting process is a method for performing simple kriging with an anisotropic and spatially varying Matérn model covariance or, more simply, tensor-guided kriging.

4.2 Simulated model and data

To test the process, I synthesized data by sampling a known gridded model at 256 random locations. Figure 5a shows the known model porosities \( m \), while Figure 5b shows the sampled data porosities \( d \). The data were sampled without error; i.e., the data covariance \( C_D = 0 \).

The model has a known covariance \( C_M \) that I derived from seismic amplitudes, also shown in Figure 5. (The images in Figures 1a and 1c are identical to those in Figures 5c and 5a, respectively.) These amplitudes were extracted from a 3D seismic image at depths corresponding to a seismic horizon. Stratigraphic features apparent in this image suggest an anisotropic and spatially varying model covariance \( C_M \).

I computed the known model \( m \equiv m(x) \) displayed in Figure 5a by smoothing an image \( r \equiv r(x) \) of pseudo-random values independently generated for a normal distribution \( N(0.25, 0.02) \). The smoothing was performed by solving a finite-difference approximation to the following partial differential equation:

\[
(1 - \nabla \cdot D(x) \cdot \nabla) m(x) = |\gamma^2D(x)|^{\frac{1}{4}} r(x),
\]

(21)

where \( D(x) \) are structure tensors computed from the seismic image displayed in Figure 5c.

Equation 21 is comparable to the “stochastic Laplace equation” described by Whittle (1954), but here with anisotropic and spatially varying coefficients \( D(x) \). Equation 21 is also equivalent to the first two equations in the sequence of equations 15, for the special case where the Matérn shape is \( \nu = 1 \) (\( l = 2, \alpha = 1, \beta = 0 \)). In this case, if we define this first half of equations 15 as a linear operator \( F \), then the second half of those same equations is its transpose \( F^\top \). In summary, I used the factor \( F \) in \( C_M = FF^\top \) to generate a known porosity model \( m \) with model covariance \( C_M \) (Cressie, 1993).

It is important to emphasize that I did not assume any direct correlation between seismic amplitudes and porosities. Instead, I used the seismic image only to construct an anisotropic and spatially varying model covariance for porosity. Structure tensors \( D(x) \) computed from the seismic image enable the effective range of the model covariance \( C_M \) to be both anisotropic and...
spatially varying. For this example I set the maximum range to be 1 km.

4.3 Gridding with kriging

Figure 6 displays model estimates \( \tilde{m} \) obtained by kriging via equation 16 the simulated data \( d \) for three different model covariances. In all three cases, I used the correct model mean \( m_0 \equiv m_0(x) = 0.25 \).

For Figure 6a, I used the correct smoothing covariance \( C_M \) (and \( C_D = 0 \)) in tensor-guided kriging with equation 16 to estimate the model \( m \). This model estimate \( \tilde{m} \) is most accurate because it is most consistent with the method (described above) used to synthesize the data \( d \). In the iterative conjugate-gradient solution of equation 16 I used a preconditioner based on Paciorek's approximation. The model estimate shown here was obtained after 16 iterations.

For Figure 6b, I replaced \( C_M \) in equation 16 with Paciorek's approximate model covariance \( C_P \), for which the matrix \( A_P = KCPK^T + C_D \) can be efficiently computed and factored directly. In this example, the resulting model estimate \( \tilde{m} \) differs significantly from that obtained using the correct model covariance \( C_M \) (Figure 6a), because the tensors \( D(x) \) vary significantly within the maximum range (1 km) of the covariance function. This result indicates that, while Paciorek's approximation may provide a useful preconditioner for tensor-guided kriging, it may not be an adequate substitute for the correct smoothing covariance \( C_M \).

To estimate the model displayed in Figure 6c, I replaced \( C_M \) with a simple isotropic and spatially invariant model covariance. The resulting model estimate \( \tilde{m} \) is least accurate of those displayed in Figure 6, because the correct model covariance \( C_M \) is both anisotropic and spatially varying.

The implementation of tensor-guided kriging demonstrated with this simple 2D example has a significant shortcoming not addressed in this paper. Recall that the number of scattered measurements in the data vector \( d \) in this example is 256. As that number increases to, say, 10,000 or more, the computational cost of the Paciorek preconditioner becomes prohibitive; this cost is mostly that of Cholesky decomposition of the matrix \( A_P \), which grows with the cube of the number of data samples.

This high cost makes the implementation of tensor-guided kriging proposed in this paper infeasible for 3D gridding of scattered borehole data. For such large 3D subsurface gridding problems, we might exploit the fact that the range for vertical correlation of subsurface properties is typically much smaller than that for lateral correlation, and solve the large problem with a sequence of solutions to overlapping smaller problems.

5 CONCLUSION

By solving a cascade of partial differential equations with tensor coefficients, we effectively implement an anisotropic and spatially varying Matérn model covariance \( C_M \). In addition to the tensor coefficients \( D(x) \), this model covariance requires only a few additional pa-
rameters: variance $\sigma^2$, maximum effective range $a$, and shape $\nu$. The latter parameters are well-known to those familiar with the popular Matérn covariance function.

For some inverse problems, the necessary tensor field $D(x)$ can be obtained directly from auxiliary data, such as seismic images. For other problems, a necessary first step might be for experts to specify this tensor field, perhaps using new interactive software tools developed for this purpose. In any case, it is difficult to imagine a geophysical inverse problem for which an isotropic or spatially invariant model covariance is appropriate.

ACKNOWLEDGMENT

This work was inspired by conversations with Paul Sava about model covariance operators in the context of least-squares inverse theory. The horizon slice of a seismic image shown in Figures 1 and 5 was extracted from the Parihaka 3D seismic image provided courtesy of New Zealand Petroleum and Minerals.

REFERENCES

Numerical computation of the sensitivity kernel for time-lapse monitoring with multiply scattered waves

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ABSTRACT
In time-lapse monitoring of weak localized changes within a medium using coda waves, we can either use the resulting decorrelation and/or the phase shift of the coda waves. The formulation for both the decorrelation and the phase-shift of the coda waves resulting from the weak time-lapse changes contains a common sensitivity kernel that is needed to image the weak localized changes. We provide a novel approach to compute the sensitivity kernel which uses numerical modeling (finite difference) of the wavefields from the source and the receiver with an a priori scattering model. These wavefields give the intensities needed to compute the sensitivity kernels. This approach is different from common methods of computing the sensitivity kernel that use analytical approximations of the scattered intensity. The numerical solution of the sensitivity kernel allows us to use arbitrary earth model that may include a free surface without simplifying the property of the scattering model. We demonstrate the behavior of the numerical solution of the sensitivity kernel with end members of heterogeneous scattering models.

1 INTRODUCTION
Multiple scattering of seismic waves along their travel paths potentially provides information about the subsurface that can be used to increase the resolution of an imaged subsurface (with multiple reflected waves) (Belkebir et al., 2006), increase illumination especially within a poorly illuminated subsurface (Gabarro et al., 2007; Fleury, 2013) and monitor weak time-lapse changes within the earth’s subsurface (Poupinet et al., 1984; Snieder et al., 2002; Schaff and Beroza, 2004). Using scattered waves, especially for monitoring temporal weak changes within the subsurface, we can increase the illumination and resolution of time-lapse changes within a subsurface (Malcolm et al., 2009; Berkhout et al., 2012). Multiply scattered waves are used for monitoring temporal weak changes such as velocity changes as weak as 0.1% (Snieder et al., 2002) and monitoring defects within mechanical structures (Masera et al., 2011). However, the complexity in the travel paths of the multiply scattered waves, which depends on the scattering properties of the medium of interest, makes it challenging to accurately describe the origin, travel paths, and distribution of the scattered waves within the medium. In a strongly scattering medium, the multiply scattered waves for late lapse time can be described as a diffusive process (Wesley, 1965; Shapiro and Knobloch, 1993; Page et al., 1995). The diffusion model has been used successfully in imaging algorithms that use multiply scattered waves in medical imaging (Yodh and Chance, 1995) or in imaging of missing scatterers (Rossetto et al., 2011). But, the validity of using the diffusion intensity model in explaining the multiple scattering of waves depends on the strength of the scattering process. Even in a strongly scattering medium, the diffusion intensity model is only accurate at large lapse times, i.e. for \( t >> r/c \), where \( t \) is the travel-time, \( r \) is the source-receiver distance and \( c \) is the average velocity of the medium. Alternatively, the scattered intensity can be modeled using the radiative transfer intensity model, which more accurately predicts the scattered intensity for all scattering regimes (Paasschens, 1997; Turner and Weaver, 1994). The diffusion and radiative transfer intensity models are analytical models developed under the assumption of a stochastic wave equation. These analytical intensity models, as well as most other intensity models for the scattered waves, are based on homogeneous or simple scattering media. The scattered intensity can easily become complicated if the statistical properties of the scattering medium are heterogeneous. We show that for more realistic media, a more accurate scattered intensity can be modeled numerically using, for example, finite-difference modeling rather than using the analytical intensity models.

Time-lapse changes within the earth’s subsurface (both natural and induced) are usually either gradual, weak, or localized both in space and time. Detecting these changes in many cases requires data that are highly sensitive to the changes. Vlastos et al. (2006)
show that the sensitivity of multiply scattered waves to the weak changes is significantly higher than the sensitivity of the ballistic or direct part of the seismic waves to the weak changes. This increase in the sensitivity is due to the repeated sampling of the weak velocity changes by the scattered waves (Rossetto, 2013). The high sensitivity of the multiply scattered waves has led to successful detection of time-lapse velocity changes within the earth’s subsurface. Poupinet et al. (1984) use coda waves generated by repeating earthquakes to observe an average S wave velocity change of 0.2% after the 1979 Mw 5.9 Coyote Lake earthquake on the Calaveras Fault, California. Using correlation functions generated from seismic noise, Wegler and Sens-Schnfelder (2007) detect a sudden decrease in the seismic velocity of the region surrounding the 2004 Mw 6.6 Mid-Niigata earthquake rupture. Using controlled-source monitoring Nishimura et al. (2000) detected a velocity decrease of 0.3-1.0 % due to the 1998 Mw 6.1 Mount Iwate earthquake. However, Pacheco and Snieder (2005) show that the spatial sensitivity of the multiply scattered waves to the time lapse changes is not uniformly distributed but is dependent on the source and receiver locations.

Most time-lapse monitoring of weak changes within the earth’s subsurface using multiply scattered waves has been limited to identifying weak changes rather than localizing these changes. Except in very densely distributed source and receiver set-ups, the velocity changes detected by coda wave interferometry are spatially averaged velocity changes. Recently, successful efforts have been made not only in identifying the weak changes using multiply scattered waves, but also in localizing the changes in a statistically homogeneous scattering medium (Rossetto et al., 2011). However, to localize changes within the earth’s subsurface - a scattering medium which is most likely to be inhomogeneous - we will need to appropriately handle the inhomogeneities of the earth’s subsurface. Because of the prominence of surface waves, we also need to account for the presence of a free surface.

In this study, we explore the capability of computing the sensitivity kernel that we need for time-lapse monitoring and localizing the weak changes within a medium. In the next section, we describe the theoretical connection of the sensitivity kernel to the estimated time lapse time-shifts or the decorrelations in the time lapse scattered waves resulting from the weak changes. In section 3, we develop a novel approach for computing the kernel numerically for any scattering model and explore the behavior of the kernel using various endmember scattering models. In section 4, we discuss the practicality of the kernel computation.

2 SENSITIVITY KERNEL

Pacheco and Snieder (2005) use the intensity of multiply scattered waves to develop a sensitivity kernel $K(s, x_o, r, t)$ which relates the mean travel time changes ($\tau$) to the localized relative velocity change within the subsurface $\delta v(x_o)$:

$$\langle \tau(t) \rangle = -\int V K(s, x_o, r, t) \frac{\delta v}{v}(x_o) dV(x_o), \quad (1)$$

where $t$ is the travel-time of the scattered wave, $V$ is the scattering volume, and $s$ and $r$ are the source and the receiver locations, respectively. The sensitivity kernel $K(s, x_o, r, t)$ depends on the source and receiver locations, the scattering property of the medium, and the travel-time of the scattered wave.

Rossetto et al. (2011) consider a different problem where the local scattering strength changes. This change is accounted for by a change in scattering cross-section $\delta \sigma(x_o)$. Using the correlation function $C(s, r, t)$ of the multiply scattered waves in a medium with this time-lapse change, they relate the decorrelation $1 - C(s, r, t)$ of the time lapse scattered waves to the time lapse change in the total scattering cross-section of the medium:

$$1 - C(s, r, t) = \int V \frac{v(x_o) \delta \sigma(x_o)}{2} K(s, x_o, r, t) dV(x_o). \quad (2)$$

Using either the time-shifts or the decorrelation values from the time-lapse multiply scattered waves for resolving localized weak changes, the sensitivity kernel $K(s, x_o, r, t)$ forms the building block for the Fréchet derivatives needed to resolve the weak changes. The sensitivity kernel $K(s, x_o, r, t)$ is given by

$$K(s, x_o, r, t) = \frac{\int_0^1 P(s, x_o, t') P(x_o, r, t - t') dt'}{P(s, r, t)}, \quad (3)$$

where $P$ is the normalized intensity of the multiply scattered waves (Pacheco and Snieder, 2005).

The normalized intensity in a homogeneous scattering medium in the diffusion approximation is given by (Paasschens, 1997):

$$P(s, r, t) = \frac{1}{(4 \pi D t)^{d/2}} \exp \left( - \frac{R_{sr}^2}{4 D t} \right), \quad (4)$$

where $d$ is the dimension of the scattering medium, $R_{sr} = |r - s|$ is the source-receiver distance and $D$ is the diffusion coefficient. The normalized intensity can also be described by the radiative transfer model. The 2D radiative transfer intensity (Paasschens, 1997) is

$$P(s, r, t) = \frac{\exp(-vt/l)}{2 \pi R_{sr}} \delta(vt - R_{sr}) + \frac{1}{2 \pi v l t} (1 - R_{sr}^2/v^2 t^2)^{-1/2} \times \exp \left( \sqrt{\nu^2 t^2 - R_{sr}^2 / l - vt} \right) \Theta(vt - R_{sr}), \quad (5)$$

where $l$ is the scattering mean free path; $\delta$ and $\Theta$ are the Dirac delta and the Heaviside step functions, respectively. Complex heterogeneous models require a more complex mathematical intensity model to describe
the intensity of the scattered waves (Margerin and Sato, 2011). These complex heterogeneous media include heterogeneously layered media (Margerin et al., 1998; Haney et al., 2005) and media with nondiffusive regions (Ripoll et al., 2001). In addition, one may also need to account for the presence of a free surface.

3 NUMERICAL COMPUTATION

For most scattering media, especially for complex heterogeneous scattering media, one might need to compute the sensitivity kernel $K(s, x_o, r, t)$ numerically. For these scattering media, one might not have an exact analytical formulation for either the intensity of the scattered waves or the corresponding sensitivity kernel for imaging the weak changes. Using equation 3 and a model of the scattering medium, we can numerically compute the sensitivity kernel by simulating the scattered wavefield with the scattering model, then compute the intensity field from the simulated wavefield. However, the numerical computation of the sensitivity kernel depends on how well one knows the statistical properties of the scattering medium. The characteristics of the heterogeneous medium, such as the scattering mean free path length and the average velocity, can be estimated from the analysis of the coda waves in the recorded data or using additional information such as velocity values from well log measurements when available or a velocity model obtained from other geophysical methods.

In the following section, we compute the sensitivity kernel using equation 3. We generate the source and receiver wavefields by numerical computation of waves excited at the source and receiver locations, respectively. Here, we use acoustic modeling. We also use absorbing boundary conditions at the boundaries of our models. We do not account of effects due free-surface boundary conditions except in the model with variable topography (section 3.3). The respective normalized intensities are the square of the envelope of the generated wavefields normalized by the spatial integral of the intensity (section 3.3). The respective normalized intensities are the square of the envelope of the generated wavefields normalized by the spatial integral of the intensity.

To compute the sensitivity kernel $K(s, x_o, r, t)$, we convolve the source and the receiver intensity fields and normalize with the denominator in equation 3. We simulate the wavefields with finite difference modeling using a realization of the random velocity fluctuation for the von-Karman PSDF in equation 6 (Sato et al., 2012).

3.1 Numerical vs. analytical computation

The sensitivity kernel can be estimated from analytical models of the scattered intensity (the diffusion and radiative transfer approximation of the scattered intensity). To compare the analytical and numerical solution of the sensitivity kernel, we use a von-Karman scattering model defined by the following parameters: $\lambda_o = 0.01 \text{ km}$, $\epsilon = 0.1$, $\kappa = 0.5$, and $f = 15 \text{ Hz}$ ($\lambda = 0.23 \text{ km}$). This scattering model and the dominant scattering wavenumber correspond to $ka = 0.27$. The scattering velocity model is given in Figure 1 with an average velocity of 3.5 km/s. The theoretical scattering mean free path ($l_{VK}$) and transport mean free path ($l_{VK}^T$) for these von-Karman parameters and the scattered waves are 5.7 km and 6.9 km, respectively (Sato et al., 2012).

We compute the analytical solution of the sensitivity kernel by using the corresponding theoretical mean free path lengths and the approximations of the scattered intensity using the diffusion (equation 4) and the radiative transfer (equation 5) models. We also convolve the analytical intensities with the intensity of the source wavelet used for the computation of the numerical kernel. Figures 2, 3 and 4 show time snapshots of the temporal evolution of the sensitivity kernel for both the numerical and the analytical solutions, respectively. In Figures 2, 3 and 4, the time-shots are taken at 0.95 s, 1.20 s, 2.00 s, and 4.00 s. The numerical solution of the sensitivity kernel is bound by the a kernel front which is defined as the edge of the kernel that is dominantly composed of single scattering and bounds the $x$- and $z$- directions, respectively. The correlation length is $a = \sqrt{a_x^2 + a_z^2}$. A wide range of values have been associated with the von-Karman parameters for the earth subsurface. The values of the von-Karman parameters depend on the subsurface lithology and the depth of the subsurface. Using the autocorrelation function of surface rock samples from Westerly and Oshima granite, Spetzler et al. (2002) estimate for the Westerly granite $\epsilon = 8.5 \%$ and $a = 0.22 \text{ mm}$ while $\epsilon = 9.3 \%$ and $a = 0.46 \text{ mm}$ for the Oshima granite. Yoshimoto and Sato (1997), using 149 waveforms in the frequency band of 8 - 16 Hz from 10 earthquakes occurring at depths shallower than 10 km, estimates the range of $\epsilon$ and $a$ values to be 5 - 8 % and 0.3 - 0.8 km, respectively, in the Nilko area of Japan.

The scattering model we use for kernel computation consists of the random velocity fluctuations defined by the von-Karman parameters and constant density. The velocity model is given as

$$v(x) = v_0(x)[1 + \zeta(x)],$$

where $v_0(x)$ is the background velocity and $\zeta(x)$ is a realization of the random velocity fluctuation for the von-Karman PSDF in equation 6 (Sato et al., 2012).
Numerical solution of sensitivity kernel

Figure 2. Temporal and spatial evolution of the sensitivity kernel (numerical solution).

multiple scattering contributions. The numerical sensitivity kernel shows similar features represented in the radiative transfer model (Figures 3) of the sensitivity kernel including the direct line-of-sight characteristic of the ballistic kernel (0.95 s) and the elliptical shape of the kernel front at lapse times after the first arrival time. The major difference between the numerical kernel and the radiative transfer kernel is the fluctuations in the numerical kernel. The numerical kernel in Figure 2 is computed with one realization of the scattering model. These kernel fluctuations are suppressed by averaging the numerical kernel over many realizations of the scattering model. The diffusion approximation of the sensitivity kernel (Figures 4), as expected, only reproduces the spatial features of the kernel at long lapse times \((ct/R_s \gg 1)\) (Paasschens, 1997) with no clear ellipsoidal edge. Figure 5 shows a cross-section of the kernel along the source-receiver line. The diffusion kernel fails to explain the zero or near-zero sensitivity of the kernel beyond the kernel front, evident in both the numerical and the radiative transfer kernels.

The spatial and temporal behavior of the kernel implies that to resolve time-lapse changes within a scattering medium with homogeneous statistical properties, the radiative transfer kernel can closely substitute for the numerical kernel solution. Figure 5 shows that the numerical kernel has more fluctuations compared to the analytical solutions. This numerical kernel is computed with one realization of the scattering model. The kernel fluctuations which are due to isolated scatterers within the scattering model, are present in the multiply scattering contribution to the kernel. We can suppress the kernel fluctuations by averaging the kernel over a number of the realizations of the scattering model with the same statistical properties. Figure 6 shows the averaged kernel at travel-time 2.00 s over 1, 5, 10 and 20 realizations of the scattering model. Increasing the number of realizations of the model used for the kernel computation reduces the fluctuations both at the singly scattered and multiply scattered part of the kernel. With few realizations of the random model (5-10 realizations) we can achieve stability in the sensitivity kernel (Figure 7).
3.2 Scattering Velocity models

3.2.1 Random isotropic scattering model

Figure 2 shows the kernel for a given source-receiver pair (S-R) at the following time snapshots: 0.95 s, 1.20 s, 2.00 s, and 4.00 s. The direct wave excited by a source S arrives at the receiver R at travel-time $t = 0.90$ s with an average velocity of 3.5 km/s. The time snapshots of the kernel shown in Figure 2 show the sensitivity to the changes in the scattering model for the scattered phases arriving at a specific travel-time $t$. The kernel at $t = 0.95$ s corresponds to the sensitivity of scattered waves dominated by direct and forward scattered waves. With increasing time, the area covered by the sensitivity kernel progressively increases. The spatial broadening of the kernel with time increases the detectability of any change in the scattering property of the medium due to multiple interaction between the scattered waves and the change with time, especially changes away from the path of the direct wave (which is along the source-receiver line in this case). However, the resolving power of the kernel is expected to decrease with increasing time because of the spatial broadening of the kernel. The shape of the kernel with increasing time depends on the source and receiver locations, the corresponding travel-time, and the properties of the scattering medium. At times $t > 0.95$ s, the kernel assumes an elliptical shape with the major axes along the source-receiver line and the minor axes perpendicular to the source-receiver line. The edge of the kernel is dominated by contributions from single scattering. The kernel for the singly scattered waves is given by (Pacheco and Snieder, 2006)

$$K(x_o, t) = \frac{1}{2\pi h \sqrt{(ct/R_{sr} - 1)}} \left[ \frac{r_s}{s} + \frac{r_r}{r} \right],$$

where $s$ and $r$ are the distances from the point $x_o$ to the source and receiver, respectively; $r_s$ and $r_r$ are the distances from any point on the kernel front to the source and receiver, respectively.

The single-scattering-dominated part of the kernel spatially bounds the multiple scattering part of the kernel. The inner part of the kernel accounts for multiple scattering, which has lower amplitude compared to the kernel contribution from the single scattering. Within the multiple-scattering-dominated part of the kernel,
there are high sensitivities at the source and receiver locations which are predicted accurately by the analytical solutions in Figures 5. These high sensitivities at or near the source and receiver locations suggest that the dominant contribution to the multiply scattered waves recorded at receiver R due to a source S originates from scattering near the source and receiver locations.

3.2.2 Random non-isotropic scattering model

The scattering properties within the earth’s subsurface are generally complex and inhomogeneous. The scattering characteristics of the subsurface can vary from place to place depending on both the underlying lithology and overlaying stress conditions of the local and regional subsurface. The scattering properties of the earth’s subsurface also vary with depth (Shearer and Earle, 2008). The stress- and depth- dependent scattering properties of the subsurface controls the scattering process of the seismic wave traveling through the subsurface. The effective scattering of the subsurface are defined by both the scattering properties of the subsurface and the characteristics of the incident seismic phase that is scattered. The characteristics of the incident phase include the incidence angle of the wave, the spectral properties of the incident wave, and the wave mode of the incident wave (Levander, 1990).

To explore the dependence of the sensitivity kernel on the scattering medium, we test two scattering models whose background velocity is a 3-layered velocity model. In both models (Figures 8 and 11), the top and bottom layers have the same scattering properties given by the von-Karman PSDF. For the top layer, $\kappa = 0.5$, $\epsilon = 0.5$, and $a_z = a_x = 0.05$ km; for the bottom layer, $\kappa = 0.8$, $\epsilon = 0.1$, and $a_z = a_x = 0.1$ km. However, in the middle layer, one model consists of vertical velocity perturbations while the other model is composed of horizontal velocity perturbations. The model with vertical velocity perturbations mimics a highly vertically fractured reservoir while the model with horizontal velocity perturbations represents a shale-like reservoir with a thin laminated layering. The middle layer for both models is defined by the following von-Karman parameters. For the vertically fractured model, $a_z = 0.5$ km.
and \( a_x = 0.0001 \ km \); for the shale-like model, \( a_x = 0.0001 \ km \) and \( a_z = 0.5 \ km \). In both models, \( \kappa = 0.1 \) and \( \epsilon = 0.5 \) for the middle layer. In both scattering models, we compute the sensitivity kernels using two source-receiver configurations. In one configuration, the source-receiver line is vertical while in the second configuration the source-receiver line is horizontal and is embedded within the middle layer. These source-receiver configurations resemble a source such as a microseismic event or an earthquake embedded within the subsurface with either a receiver at the near surface (for the vertical source-receiver line) or a receiver within a borehole (for the horizontal source-receiver line).

Figures 9 and 10 give time snapshots of the sensitivity kernel in the vertically fractured model for the vertical and horizontal source-receiver configurations, respectively. In both source-receiver configurations, the kernels show many of the features present in the kernel of the random isotropic model (Figure 2), which include the spatial broadening of the kernel with increasing time, the high sensitivity at the source and receiver locations, and the presence of the single scattering contributions to the kernel at the kernel front. However, the heterogeneity in the scattering model introduces extra features to the kernel of the vertically fractured model which are not present in the random isotropic kernel (Figure 2). In both source-receiver configurations, the width of the kernel (along the minor axes of the kernel) at each layer of the model depends on the effective velocity in that layer (see Figures 9 and 10 at \( t = 2.50 \ s \)). In the vertical source-receiver configuration at \( t = 2.50 \ s \) and \( 5.00 \ s \), there are extra scattered fronts marked as \( S'' \) within the kernel; these fronts are secondary scattered intensity fronts due to reflections from the layer interfaces in the model. These reflected fronts lag behind the singly scattered front. In the horizontal source-receiver configuration (Figure 10), the direct wave refracts through the top interface of the bottom layer because of the higher velocity of the bottom layer (Figure 10 at \( t = 1.38 \ s \)). A few milliseconds later, many of the forward scattered waves are confined within the middle layer (Figure 10 at \( t = 1.40 \ s \)). At later lapse times, the singly scattered kernel front propagates out from the middle layer into the top and the bottom lay-
The extent of the front propagation depends on the average velocity of the layer. The reflected wavefronts $S''$ seen in the vertical source-receiver configuration are absent in the kernel with the horizontal source-receiver configuration. This is because the reflectors are parallel to the source-receiver line in the horizontal configuration. The sensitivity is dominant within the middle layer because the vertical velocity perturbations within the middle layer persistently reflects back and forth waves traveling horizontally between the source and the receiver. The orientations of the velocity perturbations are perpendicular to the dominant forward propagation of the waves which is along the source-receiver line (Figure 5). Also the scattered waves recorded at the receiver has a higher probability of exploring the whole model space without being trapped within middle layer due to scattering. With the horizontal source-receiver setup, much of the recorded scattered waves are generated within the middle layer.

Figures 12 and 13 give time snapshots of the sensitivity kernel in the shale-like model using the vertical and horizontal source-receiver configurations, respectively. The kernel for the shale-like model, which uses a vertical source-receiver configuration, exhibits similar features present in the vertically fractured model with a similar source-receiver setup. Differences in the kernels are in the multiply scattered part of the kernel in the middle layer of the vertically fractured model where there are more scattering close to the source. However, with the horizontal source-receiver configuration (Figure 13), the kernel at late lapse time ($t = 2.50$ s and $5.00$ s) shows features different from those present in the kernel with the vertically fractured model using the horizontal source-receiver configuration. The high sensitivity present in the vertically fractured model using the horizontal source-receiver setup is absent from the
model with the horizontal velocity perturbation in the middle layer because the direction of wave propagation between the source and the receiver is parallel to the velocity layering in the middle layer, which results in relatively less scattering of the seismic wave. The effect of the source-receiver setup on the kernel behavior implies that the setup of the source and receiver pairs relative to the location of a particular time-lapse change within a scattering medium has large implications for the capability of detecting and resolving the time-lapse change. For example, if there is a change in the middle layer, the horizontal source-receiver setup provides a better scenario for detecting and resolving the time-lapse change in the middle layer than will the vertical source-receiver setup. This connection between the source-receiver setup and the behavior of the sensitivity kernel suggests the need for the source and receiver to be in close proximity to the location of the time-lapse change. The relative direction of the forward scattering between the source and the receiver to the orientation of the length of the scatterers impacts the relative magnitude of the sensitivity of the scattered intensity to the time-lapse change. This impact is evident in a comparison of the kernel strength in the middle layers of the vertically fractured model to the kernel strength in the middle layer of the shale-like model. The relative strength of the kernel is higher in the vertically fractured model where the perpendicular alignment of the scatterers allows for a stronger generation of multiply scattered waves.

The kernels at early lapse times are almost identical in both models (Figures 9, 10, 12 and 13 at $t = 1.38 \text{s}$ and $1.40 \text{s}$). The kernels at these times consist of mainly direct, refracted, and forward scattered waves.

3.3 Topography-induced Scattering

Seismic waves are not only scattered by heterogeneities within the earth’s subsurface, but also by near-surface heterogeneities such as variable topography or low velocity unconsolidated lithology in the near-surface layers. Due to the high impedance contrast across the free surface and the higher heterogeneities within the near-surface compared to the heterogeneities deeper in the subsurface (Shearer and Earle, 2008), the multiple scattering from variable topography and near-surface scattering effects can dominate bulk scattering. Rough or variable topography plus the presence of the free surface can focus or defocus seismic waves and can convert seismic waves from one wave mode to another such as conversion of body waves to surface waves and vice versa (Levander, 1990). Bouchon et al. (1996) show that a simple symmetric ridge can induce amplification of a monochromatic $SH$ wave by up to a factor of 1.5 at the crest of the ridge. The amplification factor of the incident wave depends on the incident angle of the wave and the height to width ratio of the topography perturbation.

To explore the effect of topography on the sensitivity kernel, we simulate an acoustic wavefield using a homogeneous velocity model with a variable topography. The velocity model consists of 3 homogenous layers: a top air layer with a velocity of 330$m/s$, a thin layer under the topography with a velocity of 2000 $m/s$, and the rest of the model with a velocity of 3000 $m/s$ (Figure 14). The free-surface is approximated using the air-subsurface interface based on the velocity discontinuity assuming a constant density (Taillander et al., 2009). We model the variable topography using a 1D von-Karman PSDF in order to create a random vari-
Numerical solution of sensitivity kernel

Figure 9. Temporal and spatial evolution of the sensitivity kernel (numerical solution) in a reservoir with vertical-fractured-like velocity perturbation with a near-surface receiver. $S'$ corresponds to the reflected scattered phase.

able topography with a correlation across the topography defined by a correlation distance of 0.5 km.

We compute the sensitivity kernel with the scattered waves induced by the variable topography using a vertical source-receiver setup (Figure 15) and a horizontal source-receiver setup (Figure 16). At time $t = 1.40$ s, the kernel accounts for the direct wave between source S and receiver R. The direct wave kernel using the vertical source-receiver setup is shown in Figure 15 at $t = 1.40$ s. At later lapse time (Figure 15; $t > 1.40$ s), the kernel expands into a singly scattered front which broadens with time. The kernel has a relatively large magnitude within the tiny low velocity layer underneath the topography (Figure 15, $t = 2.50$ s and 5.00 s). This part of the kernel is due to scattering contributions from the topography-induced scattering which are trapped in the near-surface layer. This topography-induced sensitivity is contained predominantly within the near-surface layer by the velocity interface on top of the high velocity halfspace. The topography-induced sensitivity, however, increases and broadens away from the receiver location within the near-surface layer with an increase in time.

When the source and receiver are embedded within the near-surface layer underneath the variable topography, the kernel of the first arrival consists of refracted waves off of the higher velocity halfspace underneath the near-surface low velocity layer (Figure 16, $t = 1.6$ s). Similar to the vertical source and receiver setup, the direct/refracted kernel with the horizontal source and receiver setup splits for $t = 1.8$ s into the singly scattered kernel. In Figure 16, only the downward section of the singly scattered kernel is present because the high velocity contrast across the topography prevents propagation of scattered waves into the air. At later lapse time ($t = 2.50$ s and 5.00 s, Figure 16), the dominant part of the kernel lies within the low velocity layer which results from the topography-induced scattering and the trapped waves within the low velocity layer. The presence of the thin low velocity layer underneath the topography induces kernels with similar behavior at large lapse time for both source-receiver setups.

The behavior of the kernel in the presence of variable topography and a thin low velocity layer in the above results demonstrates the need to accurately char-
Figure 10. Temporal and spatial evolution of the sensitivity kernel (numerical solution) in a reservoir with vertical-fractured-like velocity perturbation with a receiver within the reservoir.

cacterize the scattering properties of the medium being monitored. Incorporating both the topography and the appropriate velocity (scattering) model in the kernel computation provides the distribution and origin of the scattered waves via the kernel that can be used for time-lapse monitoring. Due to the effect of variable topography and of a heterogeneous scattering medium on the kernel, we will need to use an appropriate source-receiver array setup to image a time-lapse change within the subsurface. For example, in a case of monitoring velocity changes within a reservoir that might be due to fluid production or injection within the reservoir, a borehole array in close proximity to the change might be a more effective source-receiver setup for monitoring than using a surface-receiver array. In this case, the borehole array records more of the scattered waves generated within a given layer. This results in higher sensitivity to a change in that layer. Also, the borehole array, depending on its relative depth to the free-surface, will have less of the topography- or near-surface-induced scattering.

4 DISCUSSION AND CONCLUSION

We propose a novel approach to compute the sensitivity kernel that can be used to resolve weak changes within the earth’s subsurface or any other medium using multiply scattered waves. These are changes which are usually irresolvable with singly scattered waves. Our approach does not rely on analytical models of the scattered intensity such as the diffusion and radiative transfer models. To compute the sensitivity kernel, we compute the scattered intensity field needed for the kernel computation using an a priori model of the scattering medium from numerically generated scattered wavefield. In this paper, we use the finite-difference modeling for the computation of the seismic wavefield. The numerical modeling of the scattered intensity can take advantage of various numerical methods for seismic wavefield computation. Using our approach we can incorporate any complexities of the scattering medium and any boundary conditions of the medium. With an appropriate a priori scattering model, we can obtain a more accurate estimate of the sensitivity kernel which accurately describes
the intensity of the scattered wave recorded by a given source-receiver pair. Our kernel computation approach is suitable for a medium such as the earth’s subsurface where in most cases the scattering properties are heterogeneous and whose scattered intensity may not be described analytically.

The caveat to the computation of the scattered intensity and in extension the sensitivity kernel for the time lapse monitoring, are the computation cost of both the scattered intensity and the corresponding kernel and the need for an accurate a prior model of the statistical properties of the scattering medium. The cost of the kernel computation mostly depends on the traveltime of the scattered phase for the kernel, the sum of number of sources and receivers, the number of the scattering model realizations needed, the cost of the forward modeling of the scattered intensity for both the source and receiver intensity fields, and the cost for the convolution between the source and receiver intensity fields. The cost for the forward modeling of the scattered intensity depends on the spatial dimensions of the scattering model used for modeling the intensity field which makes a significant difference if the medium is 2D or 3D and the temporal or the frequency dimension of the scattered waves. All numerical examples we show in this paper uses 2D models but the computation of the numerical kernel in 3D follows exactly the same procedure.

Another limitation of the numerical computation of the sensitivity kernel is the need to know the statistical property of the a prior model. This limitation is also inherent to the analytical computation of the kernel where the scattering parameter such the diffusion coefficient and the mean free path length are needed for the diffusion and the radiative transfer models, respectively. The details of the a prior scattering model needed for the kernel computation can be relaxed, with the requirement that the a prior model used should have similar statistical property to that of the true model and that the corresponding scattered intensity should explain significantly the envelope of the scattered wave rather than the individual phases of the multiply scattered waves. The effect of the fluctuations in the microscopic properties between the a prior scattering model and the true scattering medium can be suppressed while computing
the kernel with multiple realizations of the scattering model. A few realizations of the scattering model (for example, 5-10 realizations) can significantly stabilize the fluctuations in the sensitivity kernel.

The numerical examples in this study are based on acoustic wave propagation with isotropic source radiation. This acoustic wave assumption can be considered valid at the equipartition regime of multiply scattered waves at which S-wave energy dominates the scattered waves (Weaver, 1982; Hennino et al., 2001). Equipartitioning of the scattered waves is achieved late in the coda (traveltimes at which diffusion approximation is usually considered). However, our recipe for computing the numerical sensitivity kernel remains valid for elastic wave propagation. For elastic waves, we expect to get the sensitivities to the change in P- and S-wave velocities. The computation of the elastic sensitivity kernels will involve additional considerations. Some of these considerations include, separating the P- and S-wave modes in order to get the sensitivities to P- and S-wave velocity changes and using the appropriate radiation pattern for the receiver intensity field that depends on the propagation directions of either the recorded P-wave or S-wave. The details on the elastic sensitivity kernel will be in a forthcoming publication.

Figure 13. Temporal and spatial evolution of the sensitivity kernel (numerical solution) in a reservoir with shale-like velocity perturbation.

Figure 14. Velocity model with variable topography.
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Figure 16. Temporal and spatial evolution of the sensitivity kernel (numerical solution) showing topography-induced scattering using horizontal source-receiver line.
Numerical solution of sensitivity kernel


Time-lapse imaging of localized weak changes with multiply scattered waves

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ABSTRACT

Multiply scattered seismic waves, due to their long path length in a limited volume, provide information that can be used to detect and image weak time-lapse changes within a medium. Such weak changes are usually not resolved with singly scattered waves. Previous use of multiply scattered waves for time-lapse monitoring assume some level of homogeneity about the scattering medium. This homogeneity is usually characterized either by a constant mean free path or diffusion coefficient. In a complex medium, however, this assumption of homogeneity likely breaks down. We demonstrate the capability of resolving a localized time-lapse velocity change within a 3-layer scattering model using multiply scattered waves. The layers within the model have different scattering properties. We localize the weak velocity change but the resolution of the imaged change degrades with increasing coda traveltime. Also we use multiply scattered waves to resolve the weak changes in a laboratory experiment with a 3D concrete block due to stress loadings to the surface of the block.

1 INTRODUCTION

Various monitoring tools have been developed to monitor and characterize weak changes within a wide spectrum of media which includes the earth’s surface, mechanical structures such as buildings, and medical specimens. These monitoring tools range from single-scattering wave methods (such as sonic/seismic tomography (Lumley, 2001; Nolet, 1987), reflection/transmission methods (Daley et al., 2008), and time-lapse microscopy (Landra, 2001)) to multiply-scattering wave methods (such as diffusion (acoustic) wave spectroscopy (D(A)WS) (Pine et al., 1988; Cowan et al., 2002) and coda wave interferometry (CWI) (Snieder et al., 2002; Snieder, 2006)). The strengths and limitations of these monitoring tools depend on both the sensitivity of the monitoring data to the change in the medium and how accessible the imaging data is for the monitoring process. It has been demonstrated that multiply scattered waves are more likely to identify weak temporal changes than do single-scattered waves (Grêt et al., 2006; Larose and Hall, 2009). Multiply scattered waves, due to increased scattering, provide information about the monitored medium that can be used to increase the resolution of the imaged medium (for example, using reflected multiples) (Belkebir et al., 2006), increase illumination especially within poorly illuminated subsurface (Gaburro et al., 2007) and increase the detection of weak time-lapse changes within the medium (Poupinet et al., 1984). Increased scattering of seismic waves potentially improves the detectability of the weak changes especially in cases where these weak changes are not resolvable with singly scattered waves. However, because of the averaging effect of multiply scattered waves and the complexity in the travel paths of the multiply scattered waves, the desired spatial resolution of the weak changes are usually not achieved.

Initial efforts in using multiply scattered waves for time-lapse monitoring were successfully used for detecting changes within a monitored medium (Poupinet et al., 1984; Nishimura et al., 2000; Snieder et al., 2002; Wegler and Sens-Schnefelder, 2007). Techniques like D(A)WS and CWI are geared toward measuring global change within a medium thereby averaging the changes present across the monitored medium regardless of whether the changes are localized or not. However, recently focus have been given to imaging the locations of weak time-lapse changes. The first known recipe for inverting a localized change using multiply scattered waves is given by Pacheco and Snieder (2005) where they connect the mean traveltime changes to a localized slowness change. Rossetto et al. (2011) introduce a technique they call LOCADIFF which uses the decorrelation of the time-lapse scattered waves to image a localized change in the scattering medium. These efforts at localizing changes within a scattering medium assume homogeneous random scattering models. How well these methods resolve weak changes in complexly...
2 THEORY

Pacheco and Snieder (2005) using the intensity of multiply scattered waves developed a sensitivity kernel $K(s, x_o, r, t)$ which relates mean traveltime changes $\langle \tau(t) \rangle$ to a localized relative velocity change within a medium $\delta v/v(x_o)$ at $x_o$:

$$\langle \tau(t) \rangle = - \int_V K(s, x_o, r, t) \frac{\delta v}{v} (x_o) dV(x_o),$$

where $V$ is the scattering volume, and $s$ and $r$ are the source and the receiver locations, respectively. We estimate the traveltime changes $\langle \tau(t) \rangle = \epsilon t$ using the stretching method (Hadziioannou et al., 2009) on a given time window of the scattered waves, where $\epsilon$ is the stretching factor. The stretching factor is equal to the estimated fractional velocity change within the time window. To obtain the optimal stretch factor, we minimize

$$\min \ f(\epsilon) = ||\hat{u}(t+\epsilon t) - u(t)||_2^{t_w},$$

where $u(t)$ and $\hat{u}(t)$ are the original and the time-lapse coda signals, respectively.

Rossetto et al. (2011) relate the decorrelation $D(s, r, t)$ between time-lapse multiply scattered waves to the time-lapse change in the total scattering cross-section $\delta \sigma(x_o)$ of a medium:

$$D(s, x_o, r, t) = \frac{\nu(x_o) \delta \sigma(x_o)}{2} K(s, x_o, r, t).$$

The change in the total scattering cross-section describes the scattering potential of the change within the medium. Practically, the time-windowed decorrelation is obtained using

$$D(s, r, t) = 1 - C(s, r, t)$$

$$= 1 - \frac{\int_{t-t_w}^{t+t_w} u(s, r, t') \hat{u}(s, r, t' + t_w) dt'}{[\int_{t-t_w}^{t+t_w} u(s, r, t')^2]^{1/2} [\int_{t-t_w}^{t+t_w} \hat{u}(s, r, t')^2]^{1/2}},$$

where $t$ is the traveltime and $t_w$ is the half-length of the time-window use to compute the cross-correlation $C(s, r, t)$. Equations 1 and 3 contain the same sensitivity kernel $K(s, x_o, r, t)$ (Planès et al., 2014), which depends on the source and receiver locations, the scattering property of the medium, and the traveltime of the scattered waves. The sensitivity kernel is given by

$$K(s, x_o, r, t) = \frac{\int_0^t P(s, x_o, t') P(x_o, r, t-t') dt'}{P(s, r, t)},$$

where $P$ is the normalized intensity of the multiply scattered waves (Pacheco and Snieder, 2005). The normalized intensity or the corresponding sensitivity kernel can be computed either by using analytical intensity models such as the diffusion intensity model or the radiative transfer intensity model (Paasschens, 1997); or by using a numerical approach where the sensitivity kernel is computed by convolving numerically generated source and receiver intensity fields. The intensity fields are computed from numerically simulated scattered wavefields excited at the source and the receiver locations. For details about the numerical computation of the sensitivity kernel we refer the reader to Kanu and Snieder (2014). If we have an accurate statistical properties of the scattering model, the numerical solution of the kernel provides a more accurate kernel for imaging the time-lapse changes present in a medium, especially in a heterogeneously complex medium.

3 IMAGING TIME-LAPSE WEAK VELOCITY CHANGES: NUMERICAL EXAMPLE

3.1 Model setup:

To explore the capability of resolving a localized velocity change using either the estimated travel-time change or the decorrelation of the time-lapse data, we setup an imaging problem given in Figure 1, which shows both the velocity (scattering) model and the time-lapse velocity change for our imaging problem. The velocity model is a 3 layer model, with each layer having scattering properties of different statistical characteristics. The statistical characteristics of the top and the bottom scattering layers are homogeneous and structurally isotropic, while the middle layer is heterogeneous characterizing a highly fractured reservoir. The time-lapse change is a 0.5 % velocity change in the rectangle embedded within the middle layer shown in Figure 1 (bottom). To monitor and resolve this localized change, we use two vertical receiver arrays representing two borehole lying on opposite sides of the localized change. These boreholes record scattered waves generated by the 9 sources that are located along a horizontal line. This source-receiver setup depicts time-lapse monitoring with repeating microseismic events whose scattered waves (codies) are recorded by the 2 boreholes. We assume acoustic wave propagation hence we are not accounting for the effect of the source radiation and elastic seismic wave modes.
3.2 Time-lapse inversion

To localize the weak velocity change, we minimize the objective function \( \phi \):

\[
\min \phi = ||\tilde{d} - KS^{-1}Sm||^2.
\] (6)

This objective function \( \phi \) (equation 6) corresponds to solving the following weighted least square problem:

\[
K^T \tilde{d} = (K^T K)S^{-1} S \tilde{m} = (K^T K)S^{-1} \tilde{m}_S,
\] (7)

where \( K \) is the discretized numerical solution of the sensitivity kernel, \( \tilde{d} \) is the estimated traveltime change \( \delta t \), and \( \tilde{m} = S^{-1} \tilde{m}_S \) is the fractional velocity change. The sensitivity weighting matrix \( S \) is a diagonal matrix with the elements \( S_{ij} = \delta_{ij}w_j \) that are weighting functions \( w_j \) given by (Li and Oldenburg, 2000):

\[
w_j = \left( \sum_{k=1}^{N} K_{kj}^2 \right)^{\beta/2}, \quad j = 1, ..., M; \quad \beta = 0 - 2.
\] (8)

where \( N \) is the number of data and \( M \) is the number of the discretized model space. We apply \( S \) to suppress elevated sensitivity at the source and receiver locations (Kanu and Snieder, 2014). The sensitivity weighting matrix acts as a preconditioner for the inversion problem. To solve equation 7, we use linear conjugate-gradient method. All the imaged velocity changes in this section are obtained after 5 iterations.

To estimate the traveltime changes, we use 0.6 s time windows with each window overlapping 0.1 s with the previous time window. Each time window contains about 10 cycles of the signal and the windowed signals are tapered with a Tukey time window. Using a time window which is 10 times the dominate period helps stabilize the estimated traveltime changes (Snieder et al., 2002). To get more data points for the inversion, we interpolate estimated traveltime changes from 0.5 s to 0.02 s. Figure 2 shows the inverted time-lapse velocity changes using estimated traveltime changes across different coda time windows (the rectangular boxes in Figure 2 top). Each of the coda time window starts 0.3 s before the traveltime of the first arrival event. Figures 2A, 2B, 2C, and 2D show inverted velocity changes using estimated traveltime changes extending to 0.55 s, 0.80 s, 1.05 s, and 1.30 s, respectively after the traveltime of the first arrival. The inverted velocity images correctly recovers the location of the velocity change. However, with these time windows, the resolution of the inverted velocity change varies with different coda time windows. The image of the first time window (Figure 2A) shows the best resolution of the velocity change within the middle layer but has the weakest amplitude of the resolved velocity change. The weak amplitude of Figure 2A velocity change results from the low sensitivity of the early scattered waves to the velocity change. Also there is a smearing of the change onto the locations of the receivers above the true location of the velocity change. With longer time windows the inverted velocity changes become closer to the true magnitude of the velocity change \((\Delta v/v = 0.5 \%)\) and the resolved change lie within the middle layer of the velocity model. However, the inverted velocity change become progressively smeared horizontally within the middle layer. This smearing of the inverted velocity change is due to the spatial broadening of the sensitivity kernel with traveltime and to the strong scattering in the reservoir that might bias the localization of the velocity change. The sensitivity kernel for the velocity model (Figure 1) is given in Kanu and Snieder (2014).

We invert these velocity changes in Figure 2 using \( \beta = 0.5 \) to construct the sensitivity weighting matrix. Practically, the optimal \( \beta \) value depends on the amplitude of the kernel front and kernel at the source/receiver location relative to the rest of the kernel. The effect of the sensitivity weighting matrix is seen in Figure 3 showing the inverted velocity change with and without the weighting matrix \( S \). Using the weighting matrix \( S \), we
Figure 2. Inverted fractional velocity change using various coda time windows. Top inset: a typical recorded coda signal with time windows used to invert the velocity changes. Inverted velocity change in A: with black time window, B: with red time window, C: with blue time window, and D: with green time window. 

improve the resolution of the imaged velocity change. The weighting matrix $S$ helps to project the velocity change onto the correct location which is within the middle layer. The higher amplitudes at the kernel fronts and the source/receiver locations hinder the localization of the velocity change in the inverse problem when the weighting matrix $S$ is not used.

4 IMAGING TIME-LAPSE WEAK CHANGES IN A 3D CONCRETE BLOCK

4.1 Laboratory experiment in a 3D concrete block

To demonstrate the capability of resolving weak changes within a physical scattering medium, we invert for a change induced by a localized stress loading on the surface of a 3D concrete block with dimensions 1.5 x 1.5 x 0.5 $m^3$. The concrete block, with an average P-wave velocity of 4 $km/s$, consists of aggregates (which act as scatterers) of size approximately 32 $mm$ and reinforcements at the lower half of the block (Figure 4).
The reinforcements are three 150 mm grid 8 mm diameter rebar meshes. Within the block, we embedded 18 sensors that serves both as sources and receivers. Because of coupling issues, we only use 10 of the sensors (Figure 5). Table 1 gives the locations of the used sensors. These ten sensors are connected to an ultrasound transmitting and receiving equipment via a multiplexer. The multiplexer connects an amplified signal generated by a function generator to a given sensor selected at a given time (Figure 6). This selected sensor acts as the source while the rest of the sensors serve as receivers. The recorded signals are sent to an Analog-to-Digital (A/D) device and then to data storage. A given load experiment results to a total of 90 traces. For the source signal, we use a 60 kHz Ricket wavelet.

To induce a local change on the concrete block, we apply stress loading on a 30 mm borehole drilled through the block along the z-axis. The borehole is located at \((x = 0.95 \text{ m}, \ y = 0.325 \text{ m})\). A 20 mm bolt is put through the borehole and fastened on both sides with a load distribution steel plates and nuts. The load steel plates are 0.1 x 0.1 m² wide and 10 mm thick. On one side of the block, a calibrated load cell (a piezoelectric sensor) is placed between the load steel plate and an additional disc. The load cell is used for current supply and voltage measurement. This experiment which was designed and setup by the Federal Institute for Materials Research and Testing, Berlin, Germany (known as BAM) group involves monitoring series of stress loading ranging from 0 kN to 100 kN and back to 0 kN.

Figure 7 shows typical signals recorded at sensor 17 when sensor 16 acts as a source. At time zero each receiver records a event from source induced electrical response. We use this electrical event (Figure 7 (black ellipse)) to book-keep the onset time of the source signals. This event is removed prior to time-lapse analysis of the signals. The rest of the recorded scattered waves in the signals consist of ballistic and coda waves resulting from
wave scattering within the 3D concrete block. The two signals in Figure 7 are from repeated experiments on a given stress load displaying the strong repeatability of the signals.

### Table 1. Sensor coordinates.

<table>
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<th>x (mm)</th>
<th>y (mm)</th>
<th>z (mm)</th>
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</tr>
<tr>
<td>Load</td>
<td>950</td>
<td>325</td>
<td>0 - 500</td>
</tr>
</tbody>
</table>

#### 4.2 Time-lapse inversion

Here, we invert for weak changes within the heterogeneous 3D concrete block. To invert for the changes induced by the stress loading, we use the decorrelation of the time-lapse coda. These stress loading induces relatively more significant and consistent perturbation on the amplitude of the coda signals than on the phase of the coda. Figure 8 shows the average decorrelations estimated from the time-lapse scattered waves arriving between 0.28 ms and 0.58 ms after the first arrival. To invert the change within the concrete block due to the stress loadings, we solve equation 7. In this inversion, we compute the sensitivity kernel using radiative transfer model of the scattered intensity in the block (Kanu and Snieder, 2014). We currently assume acoustic wave
scattering but future inversion of the weak change in the concrete block will incorporate elastic scattering. To estimate the sensitivity kernel we estimate the average mean free path by fitting the intensities of the coda waves using radiative transfer intensity which depends on the mean free path. We estimate the mean free path to be approximately 1.52 m which corresponds to a mean free time of 0.362 ms using P-wave velocity. The time window we use for the inversion extends beyond the mean free path time. Figure 9 shows the inverted solution of the weak changes within the 3D block at \( z = 0 \text{m} \) for stress loading of 5-10 kN and 5-15 kN using the time window in Figure 9 (top). The resolved change closely localizes the change to the point of the stress loading for both stress experiments. The resolution of the weak change depends on the source-receiver coverage and the scattered phases we use for the inversion due to the spatial broadening of the sensitivity kernel with traveltime (Kanu and Snieder, 2014).

5 CONCLUSIONS

In this study, we demonstrate the capability of resolving localized weak changes within heterogeneous scattering media using multiply scattered waves. Our imaging algorithms build on from the work of Pacheco and Snieder (2005) and Rossetto et al. (2011). Using estimated time-lapse traveltime change, we demonstrate how well we can localize a weak velocity change within a heterogeneous 3-layer scattering model using a numerical example. We image this localized velocity change in the 3-layer model without assuming any homogeneity of the scattering model. Our imaging recipe resolves well the location and amplitude of the velocity change especially early in the coda wave. However, the resolution of the imaged velocity change progressively decreases with increasing traveltime of the coda waves we use in the imaging. The decrease in the resolution of the velocity change results from the spatial broadening of the sensitivity kernel with traveltime and the strong scattering near the region of velocity change. Furthermore, we apply our imaging algorithm to image a weak change on a physical 3D scattering concrete block induced by localized stress loading. The stress loading is applied on the surface of the concrete block.

ACKNOWLEDGMENTS

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REFERENCES


Figure 9. Resolved change due to stress loading from 5 kN to 10 kN (left) and from 5 kN to 15 kN (right) at X. Top inset: Time-lapse coda showing the time-window used for the inversion.
A comparison of three methods for estimating velocity changes between time-lapse microseismic signals

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ABSTRACT

Time-lapse seismic signals provide information about subsurface velocity changes. The accuracy with which we measure these velocity changes is dependent on the accuracy of time shift measurements. We compare three methods of estimating time shifts using real and synthetic microseismic time-lapse signals, and use these time shifts to estimate relative velocity changes. We compare: 1) time-shift cross-correlation, 2) stretch method, and 3) smooth dynamic time warping (SDTW); SDTW provides the most detailed estimation of both time shifts and relative velocity changes among the three methods.

1 INTRODUCTION

Subsurface time-lapse velocity changes can be monitored using repeating earthquakes and microseismic events (Poupinet et al., 1984). Velocity changes might be caused by fluid injection from hydraulic activities (Davis et al., 2003), hydrocarbon migration and production (Zoback and Zinke, 2002), and stress-induced changes. Velocity changes that are weak, localized, or embedded within a noisy signal are challenging to detect. We seek a method that estimates the relative velocity changes in the presence of noise and without any prior knowledge of the subsurface velocity structure.

These velocity changes are heterogeneous and vary non-linearly with traveltime. Estimating accurate velocity changes requires that we use a high-resolution traveltime estimation method. Windowed cross-correlation (sometimes known as moving window cross-correlation) is a well known method used for estimating time shifts. Another well-known method, the stretching method, better estimates traveltimes and also directly estimates relative velocity changes. We introduce a new technique for time-lapse traveltime estimation called dynamic time warping, which is a well-known signal processing method. To compute these traveltimes, we use a modified version called smooth dynamic time warping (SDTW) (Hale and Compton, 2013). We then compute relative velocity changes from these time shifts. To compare these three methods, we use the time shifts between both synthetic and recorded microseismic time-lapse signals using each method and compute relative velocity changes.

2 METHODS

Time-lapse velocity changes within the subsurface, like in a stimulated reservoir, induce travel time delays on seismic waves. We estimate these delays and use them to compute relative velocity changes.

2.1 Windowed cross-correlation

Windowed cross-correlation of paired signals is a common technique used for time-shift estimation. This technique has specifically been used for time shift estimation in time-lapse studies (Sens-Schönfelder and Wegler, 2006) and for computing traveltime misfits between the real data and model predicted data in geophysical inverse problems (Luo and Schuster, 1991; Van Leeuwen and Mulder, 2010). Its extensive use stems from the efficient computation cost of cross-correlation.

For a constant relative velocity change $\delta v/v$, we can...
estimate the relative velocity change (Snieder, 2006)

$$\frac{\delta v}{v} = -\frac{\delta t}{t}$$  \hfill (1)

where $\delta t$ is the traveltime change between the time-lapse seismic signals and $t$ is the traveltime of the seismic wave.

The time shifts between the seismic codas can be estimated using the maximum cross-correlation given by

$$R(t_s) = \frac{\int_{t-w}^{t+w} d(t') d(t' + t_s) dt'}{\sqrt{\int_{t-w}^{t+w} d^2(t') dt' \int_{t-w}^{t+w} d^2(t') dt' }}$$  \hfill (2)

where $d(t)$ is the baseline coda and $\hat{d}(t)$ is the time-lapse coda. The time shift $\delta t$ between the two cross-correlated codas corresponds to the timeshift $t_s$ that maximizes $R(t_s)$ (Snieder et al., 2002). The maximum cross-correlation is computed over a time window $2w$, centered at traveltime $t$. The accuracy of the relative velocity change depends on the accuracy and stability of the estimated time shifts. The estimation of time shifts is influenced by cycle-skipping, especially for large traveltime changes, while the stability of the estimated time shifts depends on the size of the cross-correlation window.

Figure 2 shows the time shifts estimated from noise-free synthetic time-lapse codas using three different window sizes. By increasing the window size, the stability of the estimated time shifts increases. However, each time shift estimation is influenced by cycle-skipping, regardless of the window size. For instance, at $t = 8.2$ s in Figure 2, there is a discontinuity in the estimated time shifts. This discontinuity corresponds to a cycle-skip. The cycle-skip is solely dependent on the time shifts between the signals and the dominant period of the signals.

Time shifts between time-lapse codas can also be estimated in the frequency domain using the moving-window cross-spectrum technique (Poupinet et al., 1984). Using this technique, the time shifts $\delta t$ are estimated from the phase spectrum $\phi(f)$:

$$\delta t = \frac{\phi(f)}{2\pi f}.$$  \hfill (3)

### 2.2 Stretching method

The stretching method proposed by Hadziioannou et al. (2009) provides a direct estimate of the relative velocity change $\delta v/v$ by shrinking or stretching the time-lapse coda signal relative to the baseline coda signal using an estimated stretch factor $\epsilon$. We stretch the signals by

$$\hat{d}(t) \rightarrow \hat{d}(t(1 + \epsilon)),$$  \hfill (4)

where $\rightarrow$ is the stretching operation.

The estimation of the optimal stretch factor $\epsilon$ can be obtained using either the cross-correlation

$$R(\epsilon) = \frac{\int_{t_{min}}^{t_{max}} d(t') \hat{d}(t'(1 + \epsilon)) dt'}{\sqrt{\int_{t_{min}}^{t_{max}} d^2(t') dt' \int_{t_{min}}^{t_{max}} \hat{d}^2(t'(1 + \epsilon)) dt'}}.$$  \hfill (5)

where $t_{min}$ and $t_{max}$ are the minimum and the maximum traveltimes of the stretched time-lapse signals, or the $L_2$ norm of the difference between the time-lapse and baseline coda signals:

$$R(\epsilon) = ||\hat{d}(t(1 + \epsilon)) - d(t)||_2.$$  \hfill (6)

The optimal $\epsilon$ is determined by the $\epsilon$ that maximizes the cross-correlation or minimizes the $L_2$ norm between the baseline coda and the time-lapse coda. In this study, we use the $L_2$ norm to pick the optimal $\epsilon$ value. This optimal stretching factor $\epsilon$ is equal to $-\delta t/t$, which, according to equation 1, is also equal to the estimated relative velocity change $\delta v/v$.

Hadziioannou et al. (2009) show, using laboratory data, that the stretching method is a more accurate and robust estimate of $\delta v/v$ compared to the window cross-correlation method. The stretching method is limited to a constant $\delta v/v$ over the window that the stretch factor is computed. To compute a time-varying $\delta v/v$, we compute the stretch factor over smaller time windows, or we use a predefined functional relationship between the stretch factor and traveltime $\epsilon = f(t)$. To use this predefined function, we must know how the relative velocity change varies along the seismic coda. By computing the stretch factors with small coda time windows (windowed stretching method), we only need to assume the stretching window length. In the windowed stretching method, the stability of the extracted stretch factors depends on the time window length relative to the dominant period of the time-lapse signals; this dependence is similar to that found with the windowed cross-correlation method. Figure 3 shows the impact of the window length on the stability of the estimated velocity changes. The stability of the estimated relative velocity changes increases
Estimation of relative velocity changes from time-lapse events

2.3 Smooth dynamic time warping

Dynamic time warping (DTW) is a widely used algorithm in speech-processing and has recently been modified and applied to geophysics (Hale, 2013; Hale and Compton, 2013; Compton and Hale, 2013). We use Hale and Compton’s (2013) modified form of DTW called smooth dynamic time warping (SDTW) to compute time shifts between time-lapse signals; this modification of DTW samples time shifts at coarse intervals, which decreases the resolution of (or smooths) the time shifts. SDTW computes a globally optimal solution to a non-linear minimization problem (the alignment between two time-lapse signals), subject to constraints on time shifts.

Specifically, we align the baseline signal $d(t)$ to the time-lapse signal $\hat{d}(t)$ such that

$$\hat{d}(t) \approx d(t - u(t)),$$

where $u(t)$ are the time shifts that optimally align the time lapse signals. These shifts are subject to constraints on $du/dt$. We follow the same analysis as Muñoz and Hale (2014) to relate $du/dt$ to relative velocity changes:

$$\frac{du}{dt} = \frac{\dot{v}(t) - v(t)}{v(t)} = \frac{\delta v}{v},$$

where $\dot{v}$ is the velocity of the time-lapse signal, $v$ is the velocity of the baseline signal, and $\delta v/v$ is the relative velocity change. We limit these velocity changes to constrain $du/dt$:

$$\min \left(\frac{\delta v}{v}\right) \leq \frac{du}{dt} \leq \max \left(\frac{\delta v}{v}\right).$$

Following Compton and Hale (2013), we use an amplitude-aligned, non-uniform grid (computed from the baseline signal) to estimate time shifts. These grid points align with the largest amplitudes within a time window of at least 0.25 s. We show an example of these grid points in section 3. The grid spacing prevents us from sampling time shifts too finely. At the grid points, we are more confident with our time shift estimates and therefore our relative velocity change estimates compared with time shifts estimated at blindly sampled coarse grid points.

Figure 4a shows time shifts, computed using SDTW, that align the synthetic microseismic baseline and time-lapse signals shown in Figure 1, and Figure 4b shows the associated relative velocity changes in blue compared to the exact velocity changes in red. These computed relative velocity changes align well with the exact values.

A geophysically reasonable assumption is that relative velocity changes are smooth and continuous. Therefore, we interpolate time shifts between grid points using cubic splines, and because cubic splines have continuous first derivatives, our resulting relative velocity changes are smooth. We illustrate the sensitivity of constraints computed using equation 9 in Figure 8b, where the variation in relative velocity change is larger in the late times of the time-lapse signal than at the earlier times; this instability may be attributed to the decrease in signal to noise ratio with time.
3 REAL MICROSEISMIC SIGNALS

We use recorded time-lapse microseismic signals from a geothermal field to compare the three time shift estimation methods. These microseismic signals are shown in Figure 1; their magnitude is approximately 1.5 $M_w$. We anticipate a non-zero relative velocity change between the microseismic signals due to the variation in the subsurface properties over time. These recorded microseismic signals are noisy, which will affect the accuracy of the estimated time shifts and relative velocity changes (Weaver et al., 2011).

We compute these time shifts using the three previously discussed methods: windowed cross-correlation, window stretching method, and SDTW. Figure 5 shows the estimated time shifts using the full-bandwidth microseismic signals. The time shifts estimated using windowed cross-correlation and the window stretching method are computed with overlapping window sizes of 1.0 s. Computing time shifts with the overlapping windows provides a detailed estimate of the time shifts. However, this implies that the individual measurements of the time shifts are correlated. Figure 5 shows that the estimated time shifts computed using windowed cross-correlation and stretching method have less variability compared to the SDTW estimates, especially at times with high signal-to-noise ratios. This is due to an averaging effect of the windowing. Late in the coda, for example at $t > 3.5$ s, the windowed cross-correlation and stretching method show more variability in time shifts compared to the early coda. This variability in the late coda is due to the low signal-to-noise ratio; this effect is also noted when computing time shifts using SDTW, as shown in Figure 5b. However, the estimated time shifts using SDTW are more continuous. In Figures 5d and 6d, we bound $\delta v/v$ to $\pm 10\%$.

We also compute the time shifts using the filtered time-lapse signals shown in Figure 6 to improve our time shift estimates. We filter the high frequency noise from the previously shown time-lapse signals using a 5-15 Hz bandpass filter; this range spans the dominant frequencies of the signal. Figure 6 shows estimates of time shifts from these filtered signals using all three methods. These estimates contain smaller time shift variations compared to the time shifts estimated from unfiltered recorded data shown in Figure 5. The time shifts computed using SDTW show a larger variability with time compared to the time shifts estimated from windowed cross-correlation and stretching method. The averaging effect of the time windowing of the time-lapse signals via the cross-correlation and stretching methods reduces this variability of the time-shifts across the coda. However, the high variability in SDTW time shifts leads to a good signal alignment, as illustrated by Figure 7a. The estimated time shifts from these three methods are similar from $t = 1$ s to $t = 2.7$ s (Figure 6). For traveltimes greater than 2.7 s, the SDTW estimated time shifts deviate from the estimated time shifts that are
Estimation of relative velocity changes from time-lapse events

Figure 7. The 5-15Hz bandpassed time-lapse microseismic baseline signal (a) (blue), time-lapse signal (a) (red), and aligned time-lapse signal (a) (black). The time shifts (b) used to align the time-lapse signal to the baseline signal are computed using SDTW.

computed using cross-correlation and stretching methods. The time shifts computed using SDTW are dependent on the sampling of amplitude-aligned grid points and the du/dt constraints used (equation 9).

From these time shifts, estimated using SDTW, we compute relative velocity changes using equation 8, as shown in Figure 8. We vary these relative velocity changes by varying the bounds on the maximum and minimum relative velocity change to: ±5%, ±10%, ±15%, and ±20%. As shown in Figure 8b, if we vary these constraints, we also vary the resulting relative velocity changes. When the signal to noise ratio is large, the relative velocity change is consistent, as seen in early times in the signal. When the signal to noise ratio is low, the relative velocity change varies more as we vary the constraints on δu/v. Figure 8 can be used to measure the time-varying uncertainty of our relative velocity change estimation. The red points shown in Figure 8a align with peaks and troughs of amplitudes in the signal. These points correspond to the grid points used to coarsely sample time shifts in SDTW. Notice a gap in grid points from 0 s to about 1 s; this is gap that we compute by measuring the first arrival time among the time-lapse signals.

4 MONITORING WITH DOWNHOLE ARRAYS

To compare the ability of these time shift estimation methods in identifying localized velocity changes within the subsurface, we consider a synthetic example with a time-lapse monitoring source (red) and receiver (blue) setup given in Figure 9a that shows both the velocity (scattering) model and the time-lapse velocity change for our monitoring problem. The velocity model is a 3-layer model with each layer having scattering properties of different statistical characteristics. The statistical characteristics of the top and the bottom scattering layers are homogeneous and structurally isotropic, while the middle layer is heterogeneous and characterizes a highly fractured reservoir. There is a 5% relative velocity change shown in Figure 9b in a black rectangle at around 3 km in depth. To measure this localized change with time-lapse signals, we use two vertical receiver arrays representing two boreholes lying on opposite sides of the localized change. These arrays record scattered waves generated by the three sources indicated by the red circle in Figure 9b. We assume acoustic wave propagation and do not account for the effect of source radiation and elastic wave modes.

Figures 10a and 10b show the time shifts and the corresponding relative velocity changes computed using the stretching method and SDTW. The estimated time shifts show similar time shift variation along the coda between methods. Similar to the time shifts estimated from the recorded time-lapse microseismic signals (section 3), the variability in the time shifts from SDTW is larger than the time shifts variations from the stretching method. This difference in the time shift variation is a function of how time shifts are estimated with each method.

Time shifts computed using SDTW vary more with time compared to the windowed stretching method. Because SDTW finds the optimal alignment between two signals (subject to constraints), the computed time shifts align the microseismic signals better than the stretching method. Figure 10b also shows the time-varying relative velocity changes computed using SDTW and stretching method; SDTW shows more variability in relative velocity change with time. This estimated velocity change is a fraction of the true velocity change, where the fraction depends on the relative time
the scattered wave spends in the region of change relative to the traveltime of the scattered wave (Planès et al., 2014).

Figure 11 shows source-receiver estimated time shifts relative to the location of the velocity change both in the early and late coda. Both time shift estimation methods give similar source-receiver distribution of the time shifts. Early in the coda waves (near \( t = 1.6 \) s in Figure 11a), the estimated time shifts are larger for source-receiver paths that intersect the localized velocity change. This distribution of time shifts in the early coda is also reflected in the estimated velocity changes shown in Figures 12a and 12b. Relative velocity changes computed using SDTW are a better indicator of the localized velocity change in our model than the relative velocity changes computed using the stretching method.

Later in the coda (near \( t = 3.4 \) s in Figure 11a), the time shifts average across the source-receiver pairs such that the source-receiver pairs below the region of change are affected by the localized change. This behavior of the time shifts in the late coda results from scattered waves that travel along paths other than the direct source-to-receiver path. We need the statistical properties of the scattering model to understand the behavior of the time shifts and the scattered waves late in the coda. For more details on the behavior of the scattered waves in a random scattering medium, we refer the reader to Kanu and Snieder (2014). However, the time shifts in the early coda give a preliminary indication of the location of the velocity change.

5 CONCLUSION

In this paper, we compare time shift and relative velocity change estimations using three methods: time shift cross-correlation, stretching method, and smooth dynamic time warping (SDTW). We compare these methods using both synthetic time-lapse signals and recorded time-lapse microseismic signals. The synthetic signals allow for a comparison of the results of the three methods with known exact values, while the recorded microseismic signals allow us to compare the methods in the presence of noise.

Compared to the time shift cross-correlation and the stretching method, time shifts computed using SDTW are more detailed. This is because the SDTW time shifts are computed by optimally aligning two signals. In the time shift cross-correlation and stretching methods, the time shifts and relative velocity changes are estimated within windows. This leads to an averaging of the estimated time shift or velocity change values within the time window, which smooths the variations.
Figure 11. The early (near $t = 1.6$ s) and late (near $t = 3.4$ s) coda of the time-lapse signals (a) are highlighted. Time shifts from the early coda are computed using the stretching method (b) and SDTW (c), and time shifts in the late coda are also computed using the stretching method (d) and SDTW (e). The colored lines indicate the magnitude of the estimated traveltime changes.

in the time shifts. However, the averaging by the cross-correlation and stretching methods reduces the impact of signal noise on the time-shift estimations. SDTW may be inaccurate in the presence of noise if time shifts are computed on a finely sampled grid, but if we use an amplitude-aligned coarse grid, the effect of noise on the resulting time shifts is reduced. We are also able to constrain relative velocity changes by imposing constraints on $\delta v/v$, thus further limiting potential errors in our time shifts due to noise. We use the amplitude-aligned grid points and constraints on relative velocity changes that minimize the effect of noise and provide an optimal tie of the time-lapse signals. SDTW is also unaffected by cycle-skipping, which causes errors in estimating time shifts using the cross-correlation and the stretching methods.

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