Asymptotically True-amplitude One-way Wave Equations in t: modeling, migration and inversion

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SUMMARY:

Currently used one-way wave equations in depth fail at horizontal propagation. One-way wave equations in time do not have that shortcoming; they are omni-directional in space. In these equations, spatial derivatives appear in a pseudo-differential operator—the square root of the Laplacian. With an appropriate definition of this operator, we have proved via ray theory that the solutions of one-way wave equations in time asymptotically approximate the solutions to the two-way wave equation to leading order for forward or reverse time propagation. For us "true-amplitude" is meant in this ray-theoretic (asymptotic) sense. The inverse series in powers of $i\omega$ in the frequency domain becomes a series in progressing waves in the time domain. The propagation of the leading order progressing wave is governed by the eikonal equation for the two-way wave equation and the slowly varying amplitude of this leading order progressing wave satisfies the same transport equation as for the two-way wave equation. This theory provides a solid theoretical base for the Explicit Marching algorithm for solving reverse time migration and anticipates an inversion—a true-amplitude reverse time migration. We present the correct initial value problems for forward and reverse time Green's functions. These Green's functions are the *analytic* extensions of the Green's functions for the two-way wave equation with their imaginary parts being the Hilbert transforms of those real (two-way) Green's functions. The Kirchhoff approximation of asymptotic ray theory in frequency domain applies to progressing waves in time domain, except that the incident wave must also be the analytic extension of the data for the two-way wave equation. A Green's identity relating solutions of one-way wave equations and their adjoint is derived. This allows us to develop Kirchhoff integral representations from propagation of surface data into the Earth and the propagation of reflection data to the upper surface. Those identities plus identification of the adjoint operator for the forward modeling operator lead to migration and inversion formulas using our analytic Green's functions. Observed data at the upper surface must be extended to analytic data in order to apply the inversion theory.

INTRODUCTION:

True-amplitude one-way wave equations in depth (Zhang, 1993; Zhang et al. 2003) yield high quality solutions for forward modeling problems and inversion. A shortcoming of these equations is that they yield poor results *near* and fail *at* horizontal propagation. In theory, the separation into equations for up-going and down-going wave equations has a pathology in the neighborhood of horizontal propagation where the terminology, upward and downward, loses meaning.

Zhang et al. (2007) derived a new equations from the two-way wave equation. They involve the first derivative term in time and so-called *analytic Green's functions*. These analytic Green's functions are complex-valued with the imaginary part being the Hilbert transform in time of the real-valued Green's function of the two-way wave equation. These equations can handle horizontal propagation and turning waves. Further, they provide a new way of doing reverse-time migration, called "Explicit Marching" (EM) method. Unlike the conventional finite-difference methods, EM does not suffer from stability and numerical dispersion problems. Unlike one-way wave equations in depth, the pseudo-differential operators involved in EM are non-singular and the new equations can be efficiently solved numerically.

Here, we analyze these temporal one-way equations in *time* from a theoretical point of view. The methodology we use is similar to what we have been using to analyze the one-way wave equation in depth. We apply asymptotic ray theory in time using smoothness of the functions as a basis for ordering one-way wave equations in *time*. This is known as the progressing wave formalism; see Lewis [1964]. We show that the eikonal equation for traveltime and the transport equation for leading order amplitude in the asymptotic solution of our new wave equations are equivalent to the eikonal and transport equations for the two-way wave equation. We further demonstrate through analysis of the Green's function that source problems for the two-way wave equations become initial or final value problems for the one-way wave equations. We further demonstrate that boundary data for the two-way wave equations become source problems for the one-way wave equation. We provide formulas for forward modeling of Kirchhoff-approximate data on a surface at depth and for back-projection of observed data at receivers. Viewing the forward modeling of reflection data as a linear integral operator, we are able to identify the adjoint of this operator. That provides a Kirchhoff migration formula for these data. We then derive an pseudo-inverse operator by normalizing this adjoint operator. This method is demonstrated for common-shot migration and inversion.

We can only report on this list of results here. Details of the theory are available in the preprint, Bleistein et al (2008).

ONE-WAY WAVE EQUATIONS IN TIME AND PROGRESSING WAVES:

The one-way wave equations that we will discuss below are

$$\mathscr{L}_{\pm}W = \frac{1}{\mathrm{v}}\frac{\partial W}{\partial t} \pm \sqrt{\nabla^2}W = 0. \tag{1}$$

We introduce a symbolic correspondence for the square root of the Laplacian appearing here:

$$\nabla \;\leftrightarrow\; i\mathbf{k}, \qquad \mathbf{k} = (k_x, k_y, k_z), \tag{2}$$

and

$$\sqrt{(\nabla)^2} \;\leftrightarrow\; \sqrt{(i\mathbf{k})^2} = ik. \tag{3}$$

If the wave speed v were constant, we could think of $\mathbf{k}$ as just the components of the spatial Fourier transform. In fact, this is still reasonable when v is not constant, but only in a leading order asymptotic sense for "large" values of $|\mathbf{k}| = k$. By imitating the methodology of Zhang et al (2003), we can show that

$$ik = i\,|k_z|\,[I(\mathbf{k})+1],\; I(\mathbf{k}) = \frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-s^2\mathbf{k}_T^2}}{k_z^2+s^2\mathbf{k}_T^2}ds,\; \mathbf{k}_T = (k_x,k_y). \tag{4}$$

Both the factor $i\,|k_z|$ and the integral $I(\mathbf{k})$ require interpretation as pseudo-differential operators. We interpret $i\,|k_z|$ as follows.

$$k_z > 0,\;\; i\,|k_z| = ik_z \leftrightarrow \frac{\partial}{\partial z}; k_z < 0,\;\; i\,|k_z| = -ik_z \leftrightarrow -\frac{\partial}{\partial z}. \tag{5}$$

We remark that this preliminary distinction of the vertical direction is only a *device* that allows us to develop a ray theory for our progessing waves and leads to an eikonal and transport equation with no distinguishing directionality. We could have as easily distinguished x or y and end up in the same place.

As for the integral $I(\mathbf{k})$ in Equation 4, the multiplication in the numerator of the integrand merely means differentiation. The symbolic operator in the denominator is interpreted as the *inverse* of a differential operator; that is, convolution with a Green's function or solution to an appropriate differential equation. Thus, we introduce an auxiliary function $W_T(\mathbf{x},t,s)$ that satisfies the differential equation,

$$\frac{\partial^2 W_T}{\partial z^2} + s^2\nabla_T^2 W_T = \nabla_T^2 W,\;\; \nabla_T = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right). \tag{6}$$

This is exactly what we need to interpret $I(\mathbf{k})W$ with I and ik defined below equation 4. We now have a compound of $\pm$ signs due to the signs in the differential Equation 1 and the signs of 5. Therefore, below we proceed only with $\mathscr{L}_+$ of Equation 1 and then state results for $\mathscr{L}_-$, allowing $\pm$ to correspond to the signs of $i\,|k_z|$ in Equation 5. Then, for the forward going wave equation 1, we write

$$\mathscr{L}_+ W = \frac{1}{\mathrm{v}}\frac{\partial W}{\partial t} \pm \frac{\partial W}{\partial z} \pm \frac{\partial}{\partial z}\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-s^2}\,W_T\,ds = 0, \tag{7}$$

Here, the upper sign corresponds to downgoing waves and the lower sign corresponds to upgoing waves. Those signs are opposite ($\mp$) for $\mathscr{L}_-$).

We introduce a sequence of progressing wave functions $F_0[\tau(\mathbf{x})-t], F_1[\tau(\mathbf{x})-t],\ldots$, with the property that

$$F'_{n+1} = F_n,\;\; n = 0,1,2,\ldots\;. \tag{8}$$

Here the prime $\{'\}$ means derivative with respect to total argument of the function. With this definition, each wave function is smoother than its predecessor. This is the analog of the sequence of inverse powers of $i\omega$ in the asymptotic series in the frequency domain. We then assume that the solution W can be written as a series in these functions as follows.

$$W(\mathbf{x},t) = B_0(\mathbf{x})F_0[\tau(\mathbf{x})\mp t] + B_1(\mathbf{x})F_1[\tau(\mathbf{x})\mp t] + \ldots\;. \tag{9}$$

For leading order asymptotics, that is, to determine the governing equations for τ and B_0 we will not need any further terms in the series.

The procedure for asymptotic analysis of the one-way wave equations 1 is as follows. Write down progressing wave series for both W and W_T. Use the spatial differential equation 6 to determine the coefficients of the series for W_T in terms of the coefficients of the series for W. Those coefficients are also functions of s Substitute the series for W_T into the integral in the one-way wave equations 7 just above, and carry out the integrals with respect to s. What results is a progressing wave series totally in terms of the traveltime and amplitudes of the progressing wave representation for W. The most singular part of this equation will be the coefficient of F'_0 and the next order will be the coefficient of F_0, itself. Setting those two coefficients equal to zero yields the eikonal equation and transport equation that we seek, namely,

$$\sqrt{\mathbf{p}^2} = 1/\mathrm{v},\;\text{ and }\;\nabla\cdot\left[\mathrm{v}B_0^2\mathbf{p}\right] = 0,\;\; \mathbf{p} = \nabla\tau,\;\; p^2 \equiv \mathbf{p}\cdot\mathbf{p}. \tag{10}$$

Our care with signs has led to the same eikonal equation for upward and downward propagating waves, thereby not distinguishing them by direction. The transport equation conserves energy in ray tubes as it should. If we denote by A_0 the same leading order coefficient for the progressing wave solution of the two-way wave equation, then A_0 satisfies the transport equation

$$\nabla\cdot\left[A_0^2\mathbf{p}\right] = 0. \tag{11}$$

This is the same equation that we would obtain in frequency domain. Thus, there is a minor scale difference between the conserved quantity in ray tubes for the solution of the one-way and two-way wave equations.

In contrast to the one-way equations in depth, there is no need for an additional zeroeth order pseudo-differential operator in this wave equation to achieve this conservation. Exactly the same eikonal and transport equation are obtained for the operator $\mathscr{L}_-$.

ANALYTICAL/NUMERICAL SOLUTION TECHNIQUE:

The analytical scheme used in the previous section to derive the asymptotic solutions of the one-way wave Equation 1 is not a viable solution technique. Instead, we use a spectral method. To this end, we introduce the spatial Fourier transform,

$$\mathscr{F}[W(\mathbf{x},t)] = \tilde{W}(\mathbf{k},t) = \int_{-\infty}^{\infty} W(\mathbf{x},t)\exp\{-i\mathbf{k}\cdot\mathbf{x}\}d^3x, \tag{12}$$

and rewrite the one-way wave equations 1 as

$$\frac{1}{\mathrm{v}}\frac{\partial W}{\partial t} \pm \mathscr{F}^{-1}[ik\mathscr{F}[W(\mathbf{x},t)] = 0. \tag{13}$$

That is, we effect the square root operator by computing the forward Fourier transform of W, multiplying by ik, and computing the inverse transform.

The Green's functions are solutions of the initial value problems

$$\mathscr{L}_\pm G_\pm = \frac{1}{\mathrm{v}}\frac{\partial G_\pm}{\partial t} \pm \sqrt{\nabla^2}G_\pm = 0,\;\; G_\pm(\mathbf{x},0) = \mp\frac{\mathrm{v}(\mathbf{0})}{ik}\delta(\mathbf{x}). \tag{14}$$

For homogeneous media, the solutions of these equations are

$$G_\pm(\mathbf{x},t) \quad = \quad \pm H(\pm t)\frac{1}{4\pi r}\Big\{\delta(r/\mathrm{v}-t) - \delta(r/\mathrm{v}+t)$$

$$+\frac{i}{\pi}\left[-\frac{1}{r/\mathrm{v}-t}+\frac{1}{r/\mathrm{v}+t}\right]\Bigg\}. \quad (15)$$

The real parts of $G_\pm$ are the causal/anti-causal Green's functions of the two-way wave equation. The imaginary parts here yield the analytic solutions. The complex values arise from the implementation of the operator ik. This operator yields analytic solutions whenever it is applied. Thus, the solution obtained from a problem with real data will be the analytic solution with its real part being the solution that we really want.

Note that these Green's functions have the form of our progressing waves. That leads us directly to the structure of the Green's functions for heterogeneous media; namely,

$$\begin{aligned} G_\pm(\mathbf{x},\mathbf{x}_0,t-t_0) \;\sim\; &\pm B(\mathbf{x},\mathbf{x}_0)\Bigg\{\delta\Bigg(\tau(\mathbf{x},\mathbf{x}_0)\mp(t-t_0)\Bigg) \\ &\pm\frac{i}{\pi}\cdot\frac{1}{\tau(\mathbf{x},\mathbf{x}_0)\mp(t-t_0)}\Bigg\}, \quad (16) \\ \pm(t-t_0)>0, \quad &\lim_{\mathbf{x}\to\mathbf{x}_0}\{|\mathbf{x}-\mathbf{x}_0|B(\mathbf{x},\mathbf{x}_0)\}=\frac{1}{4\pi}. \end{aligned}$$

We can show that the adjoints (†) of the differential operators are

$$\mathscr{L}_\pm^\dagger W=-\mathscr{L}_\mp W=-\left\{\frac{\partial W}{\partial t}\mp[ikW]\right\}. \quad (17)$$

Using these adjoints, we can obtain a Green's identity,

$$\begin{aligned} &\int dV'\int_{t_-}^{t_+}dt'\left\{U\mathscr{L}_\pm W-W\mathscr{L}_\pm^\dagger U\right\}(\mathbf{x}',t') \\ &\qquad=\int\frac{dV'}{\mathrm{v}(\mathbf{x}')}U(\mathbf{x}',t')W(\mathbf{x}',t')\Bigg|_{t_-}^{t_+}. \quad (18) \end{aligned}$$

This identity allows us to model downward continuation of observed surface data, $D(x',y,t')$ at $z'=0$ as

$$\begin{aligned} W(\mathbf{x},t)=&\int_{z'=0} dx'dy'\int_t^\infty dt' D(x',y',t')\frac{1}{\mathrm{v}(\mathbf{x}')}\frac{\partial G_+(\mathbf{x}',\mathbf{x},t'-t)}{\partial t'} \quad (19)\\ =&-\int_{z'=0} dx'dy'\int_t^\infty dt' D(x',y',t')\frac{1}{\mathrm{v}(\mathbf{x}')}\frac{\partial G_-(\mathbf{x}',\mathbf{x},t-t')}{\partial t'}, \end{aligned}$$

with the source of the delta function here being at $(\mathbf{x},t)$ rather than at $(\mathbf{0},0)$ as in differential equations 14 for the Green's functions.

The Green's identity 18 also allows us to model upward propagation from a reflector S when we use Kirchhoff-approximate boundary data:

$$\begin{aligned} W(\mathbf{x},t) \;=\; &-\int_S dS'\int_0^t dt' D(x',y',t')\frac{1}{\mathrm{v}(\mathbf{x}')}\frac{\partial G_-(\mathbf{x}',\mathbf{x},t'-t)}{\partial t'} \quad (20)\\ =\;&\int_S dS'\int_0^t dt' D(x',y',t')\frac{1}{\mathrm{v}(\mathbf{x}')}\frac{\partial G_+(\mathbf{x}',\mathbf{x},t-t')}{\partial t'}. \end{aligned}$$

Detailed derivations of all results presented here can be found in Bleistein et al (2008).

COMMON-SHOT MIGRATION/INVERSION:

We use the Kirchhoff approximation for reflected data from a point source $\mathbf{x}_s$. We are then able to write down the upward propagation of the reflected data as a volume integral as follows.

$$\begin{aligned} W(\mathbf{x}_r,t) \;&=\; \int d^3x'\int_0^t dt'\frac{\mathscr{R}_0(\mathbf{x}',\mathbf{x}_s)}{\mathrm{v}(\mathbf{x}')} \\ &\qquad\cdot G_+(\mathbf{x}',\mathbf{x}_s,t')\frac{\partial G_+(\mathbf{x}',\mathbf{x}_r,t-t')}{\partial t'}, \\ \mathscr{R}_0(\mathbf{x}',\mathbf{x}_s) \;&=\; R(\mathbf{x}',\mathbf{x}_s)\gamma(\mathbf{x}'). \end{aligned} \quad (21)$$

Here, $\mathscr{R}_0$ is the reflectivity function that we seek in the inverse problem. The function $\gamma(\mathbf{x}')$ is the *singular function* of the reflecting surface. It is a delta function, $\delta(n)$ of signed normal distance to the reflector. This function allows us to recast the surface integral of model data in Equation 20 into a volume integral. The function $R(\mathbf{x}',\mathbf{x}_s)$ is the ray-theoretic reflection coefficient at $\mathbf{x}'$ for the ray incident from the source $\mathbf{x}_s$.

The function $\mathscr{R}_0$ corresponds to the reflectivity β of Bleistein et al [2001]. This notation is consistent with the notation in Bleistein et al (2005). When we divide the integrand by $|\nabla[\tau(\mathbf{x},\mathbf{x}_s)+\tau(\mathbf{x},\mathbf{x}_r)]|$ we obtain the reflectivity β_1 of Bleistein et al (2001), denoted by $\mathscr{R}$ in Bleistein et al (2005).

We remark that our modeled Kirchhoff-approximate data uses the analytic Green's function as the incident wave. Furthermore, the upward propagation of those data yields the analytic extension of the standardly observed data of the two-way wave equation. Thus, in using these formulas to migrate or invert observed data for the two-way wave equation, we must first generate its extension to analytic data.

If we think of the modeling of surface data in equation 21 as the operator $W(\mathbf{x}_r,t)=K[\mathscr{R}_0(\mathbf{x}')]$, then the adjoint operator acting on the data—$K^\dagger[W(\mathbf{x}_r,t)]$ migrates the data to produce an image of the reflector. That is,

$$\begin{aligned} K^\dagger[W](\mathbf{x},t) \;&=\; 2\pi\int_0^\infty dt G_-(\mathbf{x},\mathbf{x}_s,-t)W(\mathbf{x},t) \\ &=\; -2\pi\int_0^\infty dt G_+(\mathbf{x},\mathbf{x}_s,t)W(\mathbf{x},t) \quad (22) \end{aligned}$$

migrates the data. Here, $W(\mathbf{x},t)$ is produced by using the downward continuation of data of Equation 19 applied to the data W rather than the data D.

We are also able to derive an asymptotic approximation of the compound operator $||K^\dagger K||$ to obtain the necessary normalization for inversion. We are then able to claim that

$$\mathscr{R}_0(\mathbf{x},\mathbf{x}_s) \;=\; K^{-1}[W(\mathbf{x}_r,t]=\frac{K^\dagger}{||K^\dagger K||}[W(\mathbf{x}_r,t] \quad (23)$$

$$= \quad \frac{1}{\mathrm{v}^2(\mathbf{x})|B_0(\mathbf{x},\mathbf{x}_s)|^2} \int_0^\infty dt G_-(\mathbf{x},\mathbf{x}_s,-t) W(\mathbf{x},t).$$

As above, $W(\mathbf{x},t)$ is the downward continued data as defined by Equation 19. The scaling factor $\mathrm{v}^2(\mathbf{x})|B_0(\mathbf{x},\mathbf{x}_s)|^2$ is an asymptotic expansion of $||K^\dagger K||$ under the assumption that the incident wave does not have a caustic near the image point $\mathbf{x}$. The interpretation of the reflectivity $\mathscr{R}_0$ in terms of a reflection coefficient as in equation 21, has not been shown to be valid near a caustic. Thus, we can content ourselves with simply regularizing this denominator so that it does not become infinite at the caustic.

Here is one recipe for such regularization. Using ray theory in ray-centered coordinates, one can show that

$$\frac{1}{|B_0(\mathbf{x},\mathbf{x}_s)|^2} = (4\pi)^2 \mathrm{v}(\mathbf{x}_s)|\det[\mathbf{Q}]|, \tag{24}$$

with $\mathbf{Q}$ the standard notation of dynamic ray theory in ray-centered coordinates. It is this determinant that goes to zero at caustics. On the other hand, in homogeneous media, $|\det[\mathbf{Q}]| = |\mathbf{x}-\mathbf{x}_s|^2$. Thus we propose the replacement

$$\frac{1}{|B_0(\mathbf{x},\mathbf{x}_s)|^2} \Longrightarrow (4\pi)^2 \mathrm{v}(\mathbf{x}_s)\left[\det[\mathbf{Q}]| - i\varepsilon|\mathbf{x}-\mathbf{x}_s|^2\right], \tag{25}$$

for some appropriately chosen "small" ε. Now, away from caustics, this factor has phase near zero and we simply ignore the imaginary part to estimate the reflection coefficient. Near a caustic, the phase of this new term moves away from zero and the estimate of a reflection coefficient is unreliable. However, the output does not have a zero at the caustic and we are not evaluating $1/|B_0(\mathbf{x},\mathbf{x}_s)|^2$ which becomes infinite at the caustic.

APPLICATIONS TO REVERSE TIME MIGRATION:

Equation (1) has been applied to reverse time migration and leads to a new algorithm called Explicit Marching (EM) method (Zhang et al., 2007). Unlike the conventional finite-difference algorithm, it is guaranteed numerically stable and dose not suffer from numerical dispersion problems. To show how the EM works, we apply it to the 2004 BP 2D data set (Billette and Brandsberg-Dahl, 2005). This is a high quality dataset generated by finite-difference modeling with shot spacing of 50m, receiver spacing of 12.5m and 15000m maximum offset. For such a data set, the EM method handles complex velocity fairly well and gives good delineation of the salt boundaries (Figure 1) especially the steeply dipping salt flanks and the overturned salt edges, which require high angle propagation or turning waves to image clearly.

CONCLUSIONS:

We have proved that the solutions of one-way wave equations in time asymptotically approximate the solutions to the two-way wave equation to leading order for forward or reverse time propagation. This provides a solid theoretical base for Explicit Marching algorithm. We present initial value problems for Green's functions. The Kirchhoff approximation of asymptotic ray theory in frequency domain applies to progressing waves in time domain. A Green's identity relating solutions of the one-way wave equations and their adjoint is derived. This allows us to develop Kirchhoff integral representations from propagation of surface data into the Earth and the propagation of reflection data to the upper surface. We were then able to derive Kirchhoff migration and inversion formulas.

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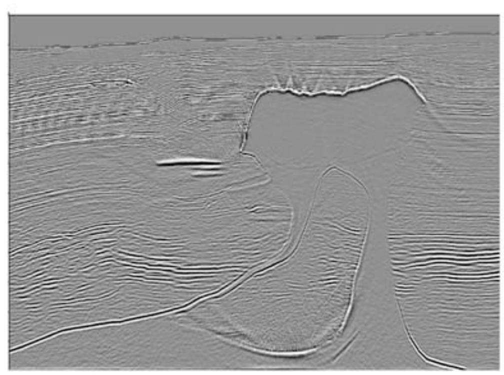
(a)



(b)

Figure 1: BP 2004 velocity model (left) and its reverse-time migration images with two-step explicit marching method (right).

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