True Amplitude One-Way Wave Equations near Smooth Caustics: An Example

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Abstract

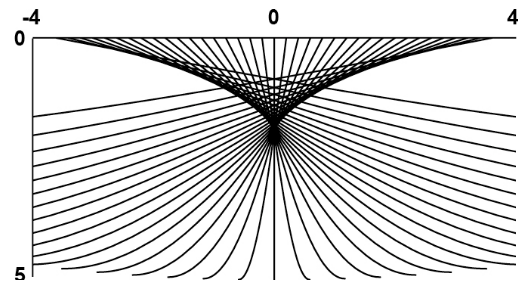
True amplitude one-way wave equations provide full waveform solutions that agree with leading order WKBJ solutions of the full, two-way wave equation away from caustics and other anomalies of the ray field. Both a proof of this fact and numerical examples are available in the literature. Near smooth caustics, ray-theoretic solutions fail for both the one-way and two-way wave equations; the WKBJ amplitude becomes infinite on the caustic although this is not true for the exact solution. For the full wave equation, the asymptotic solution in the neighborhood of smooth caustics is well understood A theory exists for generating asymptotic expansions in terms of Airy functions. These asymptotic solutions having the property that they remain finite on the caustic and accurately represent exact solutions in the neighborhood of the caustic. A corresponding theory for the one-way wave equation has proven to be elusive. Here, however, we present an example in which the one-way wave equation admits the same leading order asymptotic solution in the neighborhood of a smooth caustic as the two-way wave equation. This suggests that the derivation of a ray-theoretic asymptotic solution in the neighborhood of a smooth caustic should ultimately be achieved for the one-way wave equation, as well.

Introduction

In a series of papers [G. Zhang, 1993; Y. Zhang et al, 2001, 2002, 2003] we presented a technique for modeling wave propagation with one-way wave equations. These equations provide a correct dynamic description as well as correct kinematic description of the wave propagation when compared to the WKBJ approximation for the full wave equation. When used in wave equation migration (WEM) the imaging condition provides correct amplitude as well as location. Earlier versions of WEM produced correct location, but did not recover the WKBJ reflection amplitude. "Correct amplitude" refers to the relationship between this imaging condition and the Kirchhoff inversion formula [Keho and Beydoun, 1988; Bleistein, et al, 2001] for common shot data. Specifically, the imaging condition for these true amplitude one-way wave equations reduces to the Kirchhoff inversion formula when the full waveform Green's functions of the one-way equations can be replaced by their ray-theoretic approximations. The cited Kirchhoff inversion formula has been proven to asymptotically yield an output in known proportion to the ray-theoretic or WKBJ reflection coefficient at specular incidence angle.

The WKBJ method is known to fail in the neighborhood of caustics, yielding infinite amplitudes. The exact solution is bounded on a smooth caustic, so the infinite amplitude is a property of the WKBJ approximation and not of the exact solution. What is needed to correct this defect of the asymptotic solution is an expansion in terms of a higher function with a second parameter in addition to (dimensionless) frequency. The second parameter characterizes the distance from the caustic in such a manner as to effect the transition of the solution from WKBJ form to something else when that distance between observation point and a caustic is "nearly" zero in a sense defined by the length scales of the high frequency approximations under which these methods operate.

Caustics can be smooth, such as in the wake of a boat or the bright spots on the floor of a pool in sunlight, or they can be cusped, such as at the focal point of a less-then-prefect (real-world) lens. The ridge of the wake, the bright spot on the pool floor or the ability of a lens to burn paper, are all indicators of higher amplitude of the solution. In fact, the asymptotic order in "high frequency" changes in the neighborhood of a smooth caustic or a cusp.



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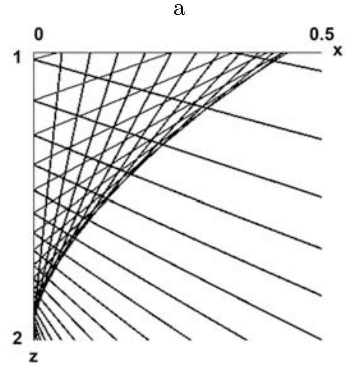

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Fig. 1:
a: A family of rays in a v(z) medium forming a caustic in the subsurface. b: Blow-up of the right side of Part a.

Figure 1a is an example of a caustic in a v(z) medium. The data were created to assure that the upward propagating rays at $z = 0$ are directed away from the z-axis in a medium of increasing velocity. For some of the rays, the plot extends downwards to their turning points; for others, the turning points are beyond the transverse range of the plot. The theory guarantees that the caustic will lie above the turning point. In the plot, two caustics converge at a cusp. We are interested here in the smooth sections. At any point near and above either of them, we see two rays with nearly equal dip angles and a third ray that travels through the particular branch of the caustic and is later tangent to the opposite smooth branch. The enlarged image of one side of the cusp in Figure 1b makes this easier to see. We focus on the first two rays with nearby dip angles. As the observation point moves towards the caustic, these two rays come progressively closer in dip, finally coalescing into a single ray on the caustic. Associated with these two rays are two wave field contributions that coalesce into a single more exotic waveform on the caustic but provide only an attenuated contribution to the wave field on the other side of the caustic. When we think of the upward trajectory of the rays, we see that one of these two rays has not yet arrived at the caustic, while the other has already passed the caustic tangentially and moved on.

Kravstov [1964a, 1964b] and Ludwig [1966] have extended WKBJ ray theory for the case of a smooth convex caustic for the acoustic wave equation. There is also a parallel theory in the uniform asymptotic expansion of integrals (also cited by Ludwig). See, for example, Chester, et al, [1957]. In Fourier representations of wave fields, the application of stationary phase to the integral will typically pick out the rays of the WKBJ method as the condition that the phase be stationary. In the simplest case, the stationary points will be well separated; the second derivative of the phase at the stationary point(s) will be nonzero. However, for points in the physical space on a smooth caustic, two nearby stationary points coalesce to yield a higher order stationary point; the second derivative will be zero there, but the third derivative is not.

The transition in asymptotic order is easy to see in these asymptotic expansions. For simple stationary points, where the second derivative is nonzero, it is well known that the method of stationary phase introduces a factor of $\omega^{-1/2}$ into the asymptotic approximation of the wave field. When the second derivative is also zero, but the third derivative is not zero, the stationary phase formula introduces a factor of $\omega^{-1/3}$ into the approximation of the wave field. This second amplitude factor for the second order stationary point is larger than the previous amplitude for the simple stationary point by a multiplier of $\omega^{1/6}$. This is not very strong; for $\omega = 64$, $\omega^{1/6} = 2$. However, the simple WKBJ expansion has no mechanism to effect this change in order in frequency and, hence, it fails at the caustic.

The simplest phase to have two stationary points must of necessity be a cubic in the integration variable, so that its first derivative is quadratic. Then, by making the stationary point locations a function of a parameter that characterizes their separation, we can effect the transition from two separated stationary points to one higher order stationary point. The special function having this integral representation and properly describing the physics of the wave field is the Airy function of the first kind, traditionally denoted by Ai. Further, we choose this Airy function with negative argument and call it $\mathrm{V}(\sigma) = \mathrm{Ai}(-\sigma)$. When σ is negative, this function decays exponentially towards zero as $\sigma \to -\infty$; when σ is large and positive, the asymptotic expansion contains two wave forms, one with negative travel time and one with positive travel time.

This description of the Airy function mirrors the description above of the nature of the wave field near the smooth part of the caustic. The two wave forms of the Airy function describe the two phases on the waves with nearby dip angles; the negative phase corresponds to the wave that has not yet arrived at the caustic while the positive phase corresponds to the wave that has already passed through the caustic. While this understanding and description of uniform asymptotic expansions near smooth caustics is complete for solutions of the two-way wave equation, the situation is not as clear for the true-amplitude one-way wave equation. Recall that the latter has only been proven to be "true-amplitude" in the sense of non-uniform ray theory. The extension to uniform ray theory following Kravstov and Ludwig to the true-amplitude one-way wave equation is much more difficult and has not as yet been completed. Here, we report on a simpler test of the uniform theory near a smooth caustic for the one-way wave equation.

Analysis

We start with the true amplitude upgoing one-way wave equation in the (k_x, z, ω)-domain,

$$\frac{dW}{dz} - ik_z W + \frac{1}{2k_z}\frac{dk_z}{dz} = 0, \quad k_z = \frac{\omega}{\mathrm{v}(z)}\sqrt{1 - \frac{(\mathrm{v}k_x)^2}{\omega^2}}. \tag{1}$$

The equation can be found in Zhang, et al [2003]. The final data we choose is best described initially in the (x, z, ω)-domain:

$$W(x, 0, \omega) = \exp\{-i\omega a x^2/2\}, \quad a > 0. \tag{2}$$

Note the quadratic phase here. For this case, the horizontal slowness will be linear, changing sign across $x = 0$. This is exactly the character of the horizontal slowness at the upper surface in Figure 1. Indeed, that plot was generated with in units of km and s and

$$a = .25, \quad \text{and} \quad 1/\mathrm{v}^2(z) = p^2(z) = .25 - .01z^2$$

The solution to the problem defined by Equations (1) and (2) is

$$W(x, z, \omega) = \sqrt{\frac{2\pi}{|\omega| a}}\exp\{-i\pi/4\cdot\mathrm{sgn}(\omega)\} \cdot \int_{-\infty}^{\infty}\frac{dp_1}{p_3(p_1, z)}\exp\{i\omega\Phi(x, z, p_1)\},$$

$$\Phi(x, z, p_1) = \int_0^z p_3(p_1, z')dz' + p_1^2/2a - p_1 x. \tag{3}$$

This is an exact solution to the true-amplitude one-way wave equation (1) and it is also the leading order asymptotic solution to the full two-way wave equation from which Equation (1) is derived. Asymptotic analysis of this solution is carried out by the method of stationary phase. We list the main features of this analysis.

1. In the V-shaped region of Figure 1a, there are three stationary points in the variable p_1 at each point (x, z) with stationarity providing the equations of the three rays intersecting as each (x, z) as seen in the figure. Each of these contributions is $O(1)$ in ω.

2. For the point (x, z) near the smooth part of the caustic, two of the stationary p_1-values are near one another and they coalesce when (x, z) moves to the caustic. There, the second derivative of the phase is zero and the third derivative is not. On the caustic, this contribution, denoted by W_C, is larger than the off-caustic contributions, namely, $W_C = O(\omega^{1/6})$.

3. At the cusp, all three stationary points coalesce, with the first, second and third derivative of the phase being zero, but the fourth derivative being nonzero. Here, with all three stationary points coalescing into one of higher order, $W = O(\omega^{1/4})$. We will not discuss the neighborhood of the cusp at all.

If the observation point is away from the cusp, the structure of this phase conforms to the requirements for a uniform asymptotic expansion near the smooth caustic in terms of the Airy function,

$$V(s) = \mathrm{Ai}(-s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{i[\sigma^3/3 - s\sigma]\}d\sigma. \quad (4)$$

and its derivative $\mathrm{V}'(s)$. Figure 2 is a graph of the Airy

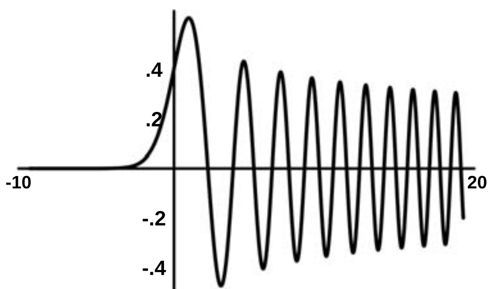


Fig. 2:

The Airy function, $V(s)$.

function $V(s)$ for the range $-10 < s < 20$. This function is sinusoidal–wave-like–for s positive and decays exponentionally for s negative. The caustic will be located at $s = 0$ in the application. In the frequency domain, the sinusoid consists of two imaginary exponentials in the of opposite sign that lead to two waves back in the time domain, one wave approaching the caustic, the other leaving the caustic as time progressed.

The uniform asymptotic expansion in terms of this Airy function will accurately describe the contributions from the two nearby stationary points. The full asymptotic solution will still include a contribution from the third stationary point in p_1. This one is bounded away from the previous two, even though in the physical space, its ray crosses the two rays with nearby stationary points that produce the near-caustic contribution.

We proceed, then, to obtain the following uniform asymptotic expansion of the contribution from the two nearby stationary points. The structure of this phase in the solution representation for W in Equation (3) conforms to the requirements for a uniform asymptotic expansion in terms of the Airy function that we called V and its derivative with respect to argument. Thus, we obtain the following uniform asymptotic expansion of the contribution from the two nearby stationary points.

$$W_C(x,z,\omega) \sim \omega^{1/6}\sqrt{\frac{2\pi}{a}}\exp\{i\omega\Psi - i\pi\mathrm{sgn}(\omega)/4\} \quad (5)$$

$$\cdot\left[A_+\mathrm{V}(\omega^{2/3}\nu) + \frac{A_-}{i\sqrt{\nu}\omega^{1/3}}\mathrm{V}'(\omega^{2/3}\nu)\right].$$

In this equation

$$\Psi = \frac{\Phi(x,z,p_-) + \Phi(x,z,p_+)}{2},$$

$$\nu = \left[\frac{3(\Phi(x,z,p_-) - \Phi(x,z,p_+))}{4}\right]^{2/3}, \quad (6)$$

$$A_\pm = \frac{1}{2}\left[\frac{1}{\sqrt{p_3(x,z,p_-)}}\frac{\sqrt{2}\nu^{1/4}}{\sqrt{|\Phi''(x,z,p_-)|}} \pm \frac{1}{\sqrt{p_3(x,z,p_+)}}\frac{\sqrt{2}\nu^{1/4}}{\sqrt{|\Phi''(x,z,p_+)|}}\right]$$

and we have denoted the two nearby stationary points by $p_\pm(x,z)$. The difference of the travel times at the two stationary points appears in the argument of the Airy function, while the sum of the travel times appears in the overall phase multipler, Ψ. Similarly, $A_\pm$ are the sum and difference, respectively, of the amplitudes of the integral in Equation (3) evaluated at the two stationary points. One can show that $A_\pm$ remain finite in all of the limits of interest, even though various elements of these expressions are zero on the caustic or at the turning point. This is in contrast to the non-uniform expansion whose amplitudes become infinite on the caustic through a factor of $1/\sqrt{|\Phi''(x,z,p_\pm)|}$ in the separate contributions to the nonuniform expansion. Further, this uniform expansion effects the transition from $W = O(1)$ in ω away from the caustic to $W = O(\omega^{1/6})$ on the caustic. Most importantly, for the objective of this exercise, this uniform asymptotic expansion is exactly the same as we obtain for the same problem for the full wave equation.

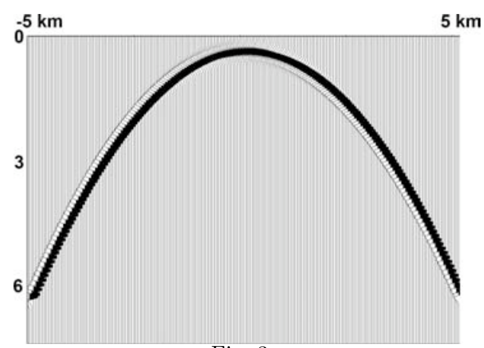


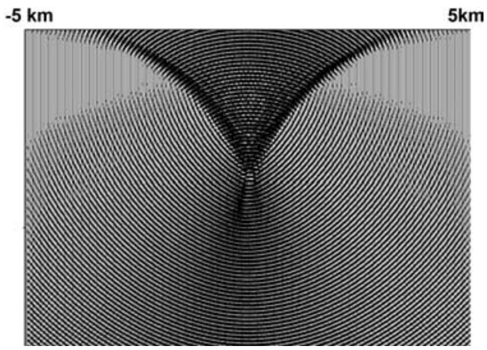
Fig. 3:

400 traces of input data for which the rays are as depicted in Figure 1. Arrival time of the pulse at $z = 0$ is $t = 0.1 + 0.06255X10^{-6}x^2$.

Finally, as regards this analysis, we note that the denominators in the expressions for $A_\pm$, appearing in Equation (6) both contain the product, $\sqrt{p_3}\sqrt{|\Phi''|}$. At the turning point, the first of these factors is zero, while the second factor is infinite. In fact, the product is finite and nonzero. This product also appears in the nonuniform asymptotic expansion of the wave forms. This anomalous behavior is a consequence of using z as the independent variable along the rays. The solution, itself is well-behaved and the finite limit does reflect its finite behavior at the turning point of each ray.

Numerical test

In addition to this analytical solution we solved the same problem numerically to confirm the asymptotic analysis. Figure 3 shows the observed data for the numerical test. These data represent a band limited delta function whose arrival at the upper surface is delayed quadratically with



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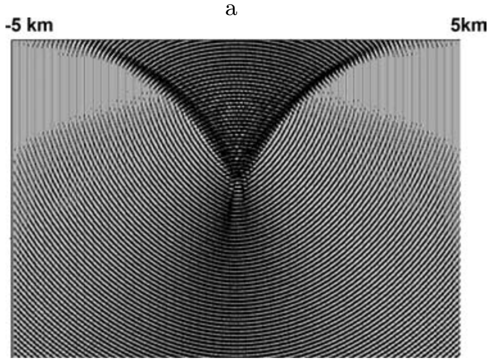

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Fig. 4:

a: Real part of the back-projected wave at 20hz. Note the match of the caustic with the rays of Figure 1. b: Imaginary part of the back-projected wave.

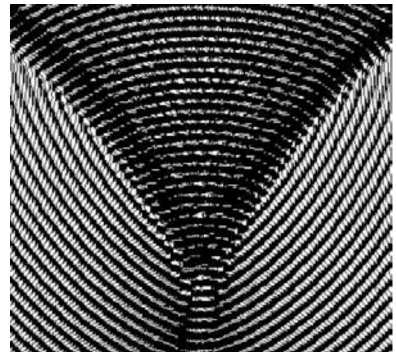

Fig. 5:

Enlargement of the center of Figure /ref-fig:outputdatab, the imaginary part of the wave field. The interference pattern just above the cusp is caused by the convex-up wave propagating from outside the caustic on its traverse to the branch of the caustic across the V-shaped region. The convex-down wave fronts arise after the rays of the previous wave front pass tangentially through the caustic.

offset from the center of the acquisition range. The output is shown as wave fronts at 20hz in Figures 4a and Figures 4b. They clearly show the same caustic as the rays of Figure 1 show, now through the high intensity amplitude on the caustic. The wave fronts that have passed through the caustic are convex-upward; the wave field from outside the caustic region is convex-downward. The transition occurs at the caustic. The continuation of the convex-downward propagating wave in the V-shaped region formed by the caustic is best seen in the interference patter near the cusp, displayed in an enlargement in Figure 5. Note that the wave field remains finite, although certainly stronger near the smooth part of the caustic, where we generated an analytical uniform expansion, and near the cusp, where we did not generate a uniform analytical result.

Summary and Conclusions

We have analyzed the behavior of the solution to a true-amplitude one-way wave equation in the neighborhood of a smooth caustic. Both the analytical solution and the numerical solution demonstrate that the one-way equation provides solutions that remain bounded near the caustic. Asymptotic analysis of the analytical solution exhibits the same behavior as the analysis for solutions of the (full) two-way wave equation. That is, the integral representation of the solution of the true-amplitude one-way admits an asymptotic expansion in terms of an appropriate Airy function and its derivativee, with the argument of the Airy function, additional phase shift and amplitudes being exactly as they are for the asymptotic solution of the full wave equation.

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