# Theory of true amplitude one-way wave equations and true amplitude common-shot migration

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## Abstract

One-way wave operators are powerful tools for forward modeling and inversion. Their implementation, however, involves introducing the square-root of an operator as a pseudo-differential operator. A simple factoring of the wave operator produces one-way wave equations that yield the same traveltimes as the full wave equation, but do not yield accurate amplitudes except for homogeneous media. Here, we present augmented one-way wave equations that yield solutions for which the leading order asymptotic amplitude as well as the traveltime satisfy the same differential equations as do the corresponding functions for the full wave equation. Exact representations of the square-root operator appearing in these differential equations are elusive, except in cases in which the heterogeneity of the medium is independent of the transverse-spatial variables. Here, we introduce a representation of the square-root operator as an integral in which a rational function of the transverse Laplacian appears in the integrand. This allows for an explicit asymptotic analysis of the resulting one-way wave equations. We have proven that ray theory for these one-way wave equations leads to one-way eikonal equations and the correct leading order transport equation for the full wave equation. By introducing appropriate boundary conditions at $z = 0$ , we generate waves at depth whose quotient leads to a reflector map and estimate of the ray-theoretical reflection coefficient on the reflector. Thus, these true amplitude one-way wave equations lead to a "true amplitude wave equation migration (WEM)" method when we use the same imaging condition as is standardly used in WEM. We have proven that applying the WEM imaging condition to these newly defined wavefields in heterogeneous media leads to the Kirchhoff inversion formula for common-shot data. Computer output using numerically generated data confirms the accuracy of this inversion method. However, there are practical limitations. The observed data must be a solution of the wave equation. Therefore, the data over the entire survey area must be collected from a single common-shot experiment.

## Introduction

One-way wave equations provide fast tools for modeling and migration. These one-way equations allow us to separate solutions of the wave equation into downgoing and upgoing waves except in the limit of near-horizontal propagation. The original one-way wave equations used for wave equation migration (WEM) [Claerbout, 1971, 1985] were designed to produce accurate traveltimes, but were never intended to produce accurate amplitudes, even at the level of leading order asymptotic WKBJ or ray-theoretic amplitudes. As such, classic WEM provides a reflector map consistent with the background propagation model, but with unreliable amplitude information.

Here we describe a modification of those one-way wave equations to produce equations that provide accurate leading order WKBJ or ray-theoretic amplitude as well as accurate traveltime. The necessary modification of the basic one-way wave equations is motivated by considering depth-dependent ($v(z)$) medium. In that case, through the use of Fourier transform in time and transverse spatial coordinates $(x, y)$, we reduce the problem to the study of ordinary differential equations. There, it is relatively simple to see how to modify Claerbout's equations used in order to obtain equations that provide leading order WKBJ amplitudes, as well. This leading order amplitude is what we mean by "true amplitude" for forward modeling.

For heterogeneous media, $v = v(x, y, z)$, the same one-way wave equations still provide true amplitudes. However, now the transverse wave vector $(k_x, k_y)$ must be interpreted as differentiations in the corresponding dual spatial variables. Further, our modified one-way wave equations involve square-roots and divisions by functions of this transverse wave vector. We provide an interpretation of these operators through some basic ideas from the theory of pseudo-differential operators.

We provide a relatively simple representation of the one-way differential operators. This, in turn leads to a proof [Zhang, 1993] that the ray-theoretic solutions of these equations satisfy the separate eikonal equations for downgoing and upgoing waves, but the leading order amplitudes also satisfy the same equation–the transport equation–as does the leading order amplitude for the full wave equation.

Having these true amplitude one-way equations allows us to develop a "true amplitude" WEM for heterogeneous media. To date, we only have numerical checks on this method for $v(z)$ media, where the pseudo-differential operators revert to simple multiplications in the temporal/-transverse-spatial Fourier domain. However, we are able to prove that the reflection amplitude agrees with the amplitudes generated by Kirchhoff inversion (true amplitude Kirchhoff migration) as developed by one of the authors [Bleistein, 1987, Bleistein et al, 2001] and colleagues. This proof is valid in heterogeneous media.

This is a common-shot inversion, requiring data with the receiver array covering the entire domain of the survey.

## Dynamically correct one-way wave equations

We begin by considering the wave equation in three spatial dimensions and time:

$$\frac{1}{v^2}\frac{\partial^2 W}{\partial t^2} - \nabla^2 W = 0. \qquad (1)$$

Let us first consider this equation in a homogeneous medium and apply Fourier transform in time and the transverse spatial variables:

$$\frac{\partial^2 W}{\partial z^2} + k_z^2 W = \left[\frac{\partial}{\partial z} \mp ik_z\right]\left[\frac{\partial}{\partial z} \pm ik_z\right] W = 0, \qquad (2)$$

where

$$k_z = \mathrm{sign}(\omega)\sqrt{\frac{\omega^2}{v^2} - \bar{k}^2} = \frac{\omega}{v}\sqrt{1 - \frac{\left(v\bar{k}\right)^2}{\omega^2}}, \qquad (3)$$

and $\bar{k}$ is the transverse wave vector, $\bar{k} = (k_x, k_y)$, $\bar{k}^2 = k_x^2 + k_y^2$. For constant wavespeed, the separate one-way equations implied in (2) have exact solutions that are solutions of the two-way wave equation:

$$\left\{\frac{\partial}{\partial z} \pm ik_z\right\} A_{\pm} \exp\{\mp ik_z z\} = 0, \qquad (4)$$

Here, the $A$'s are constants.

For wavespeed $v(z)$, these solutions are no longer valid. For (2) we would then content ourselves with asymptotic WKBJ solutions. Then, we would want the solutions of the one-way equations to agree, at least asymptotically, with those solutions of the two-way equation (1). For the one-way operators in (4), the exact solutions have the same traveltimes as the WKBJ solutions to the leftmost operator equation in (2), but do not have the same amplitude. This leads us to look for ways to modify the two one-way equations in (4), so that the new equations provide a transport equation that yields the same amplitude as in (2). We are able to show that the correct modification of the equations in (4) are

$$\left\{\frac{\partial}{\partial z} \pm ik_z - \frac{\omega^2}{2v^3(z)k_z^2}\frac{dv(z)}{dz}\right\} W = 0. \qquad (5)$$

For heterogenous media in which $v = v(x, y, z)$, the traditional WEM continues to use the one-way wave operators in (4). The transverse wave vectors in those equations are interpreted as derivatives in the transverse directions and various rational approximations are made to avoid the square-root of differential operators implicit in the representation. In fact, interpretation and manipulation of such operators is a major component of the theory of pseudo-differential operators. Through proper interpretation, these operators can be analyzed rigorously. In G. Q. Zhang [1993], the second author has done just this. To explain, we think of the transverse wave vector and frequency in (5) as being symbolic place-holders for differentiations:

$$i\omega \;\Leftrightarrow\; \partial/\partial t; \quad i(k_x, k_y) \;\Leftrightarrow\; -\left(\partial/\partial x, \partial/\partial y\right) = -\nabla_T.$$

We think of $ik_z$ as a *symbol* for a differential operator. More precisely, we rewrite (5) as

$$\mathcal{L}_{\pm} W = \left[\frac{\partial}{\partial z} \pm \Lambda\right] W - \Gamma W = 0, \qquad (6)$$

with $\Lambda$ and $\Gamma$ being pseudo-differential operators with symbols $\lambda$ and $\gamma$ given by

$$\begin{aligned}\lambda &= ik_z = \frac{i\omega}{v(\vec{x})}\sqrt{1 - \frac{\left(v(\vec{x})\bar{k}\right)^2}{\omega^2}}, \quad \vec{x} = (x, y, z), \\ \gamma &= \frac{v_z(\vec{x})}{2v(\vec{x})}\left(1 + \frac{\left(v(\vec{x})\bar{k}\right)^2}{\omega^2 - \left(v(\vec{x})\bar{k}\right)^2}\right), \quad \bar{k} \equiv (k_x, k_y).\end{aligned} \qquad (7)$$

Here, it is easy to check that the expression for $\gamma$ is exactly the same as the last factor in (5).

G. Q. Zhang's [1993] analysis of (6) was facilitated by his identity

$$\lambda = ik_z = \frac{i\omega}{v}\left\{1 - \frac{1}{\pi}\int_{-1}^{1}\sqrt{1 - s^2}\,\frac{\left(v\bar{k}\right)^2}{\omega^2 - s^2\left(v\bar{k}\right)^2}\,ds\right\}. \qquad (8)$$

With this identity, we see that essentially the same denominator appears in the symbols $\lambda$ and $\gamma$, namely We view

$$-\omega^2 + s^2\left(v\bar{k}\right)^2 \quad \Leftrightarrow \quad \frac{\partial^2}{\partial t^2} - s^2(v\nabla_T)^2 = L_T(s),$$

Then, we can view the inverse $\left(-\omega^2 + s^2\left(v\bar{k}\right)^2\right)^{-1}$ as corresponding to the inverse of the differential operator $L_T(s)$. Such an inverse suggests solving the differential equation or convolving with an appropriate Green's function. This interpretation allows us to give meaning to the pseudo-differential operators in (6), as follows. We introduce an auxiliary function $q(s; x, y, z, t)$ through the equation

$$L_T(s)q(s; \cdots) = (v\nabla_T)^2 W(x, y, z; t) \quad z > 0,\ t > 0. \qquad (9)$$

We then rewrite (6) as

$$\begin{aligned}&\frac{\partial W}{\partial z} \pm \frac{1}{v}\frac{\partial W}{\partial t} \mp \frac{1}{\pi v}\frac{\partial}{\partial t}\int_{-1}^{1}\sqrt{1 - s^2}\,q(s; \cdots)\,ds \\ &\qquad + \frac{v_z}{2v}\left[W + q(1; \cdots)\right] = 0.\end{aligned} \qquad (10)$$

Through this device of introducing the identity in (8) and the auxiliary function $q$ in (9), we are able to interpret the pseudo-differential operators in (6) in terms of traditional differential operators and solutions of traditional differential equations. The proof by G. Q. Zhang [1993] that the correct eikonal equations and transport equation arise from (6) is based on this interpretation of that equation.

## True amplitude wave equation migration

Here we describe our proposed true amplitude WEM motivated by the dynamically correct one-way wave equations of the previous section. We introduce $p_D$ and $p_U$ as solutions of the following problems.

$$\left\{\begin{aligned}&\left(\frac{\partial}{\partial z} + \Lambda - \Gamma\right) p_D(\vec{x}; \omega) = 0, \\ &p_D(x, y, 0; \omega) = -\frac{1}{2}\Lambda^{-1}\delta(\vec{x} - \vec{x}_s),\end{aligned}\right. \qquad (11)$$

and

$$\begin{cases} \left(\frac{\partial}{\partial z} - \Lambda - \Gamma\right) p_U(\vec{x};\omega) = 0, \\ p_U(x,y,0;\omega) = Q(x,y;\omega). \end{cases} \tag{12}$$

In these equations, $\vec{x}_s = (x_s, y_s, 0)$, $p_D$ is the downgoing response to the impulsive boundary condition at $z = 0$ and $p_U$ is the upgoing wave that must equal the observed data, $Q$ at the upper surface. In comparison with standard WEM, we have introduced the additional operator $\Gamma$ into the differential equations and we have also modified the boundary data for $p_D$. (Standard WEM would use only a delta function in the boundary data.) The reason is that this is the proper data to model a point source for the original wave equation. Note that this modification involves both a scaling and phase shift because of the $i$ in the definition of $\lambda$ in (8). We use the solutions $p_U$ and $p_D$ in the standard imaging condition as follows.

$$R(\vec{x}) = \int \frac{p_U(\vec{x};\omega)}{p_D(\vec{x};\omega)} d\omega. \tag{13}$$

See Zhang et al. [2001, 2002].

## Comparison of true amplitude WEM and Kirchhoff inversion

Relying on G. Q. Zhang's [1993] proof of the equivalence of the solutions of the one-way wave equations with the solutions of the full wave equation, we derive the asymptotic form of (13) in terms of the traveltimes and amplitudes of the full wave equation. That result is

$$R(\vec{x}) = 2\int i\omega \frac{\cos\alpha_r}{v_r} \frac{A(\vec{x}_r,\vec{x})}{A(\vec{x},\vec{x}_s)} \cdot e^{i\omega\{\varphi(\vec{x}_r,\vec{x})+\varphi(\vec{x},\vec{x}_s)\}} dx_r dy_r d\omega. \tag{14}$$

Here, $v_r$ is the wave speed at the receiver point and $\alpha_r$ is the emergence angle of the ray from the image point to the receiver point. Furthermore, the amplitudes and phases are solutions of the eikonal and transport equations for the full wave equation. Their equivalence with the solutions of the one-way wave equations is a consequence of G. Q. Zhang's proof. This is the formula for common-shot Kirchhoff inversion in Bleistein [1987] and Bleistein et al. [2001] as expressed by Hanitzsch [1997].

## Numerical test

To show how true amplitude common-shot migration works, we apply it to a 2-D horizontal reflector model in a medium with velocity $v = 2000 + 0.3z$. Recall from the theory that in this case, the modeling and migration can be carried out in the transverse spatial and temporal Fourier domains, with $(k_x, k_y)$ being the simple transverse part of the wave vector.

The input data (Figure 1) is a single shot record over four horizontal reflectors from density contrast. Figure 2 left shows the migrated shot record using the conventional common-shot migration algorithm with the label $U/D$ suggesting the standard upgoing and downgoing wavefields. The peak amplitudes along the four migrated reflectors are shown in Figure 2 right, normalized to the geometrical optics reflection coefficient along the reflector. This method has a phase error because the standard impulsive boundary data lacks the multiplication by $i$ in $\Lambda$ on the right side in (11). The consequent phase error has been corrected during the migration. However, the migrated amplitudes are poor, especially on the reflector at depth $z = 1000m$ along which the reflection angles vary over a wide range. (This method has incorrect angular dependence when compared to true amplitude reflectivity or the geometrical optics reflection coefficient at each point.) The wide angle peak amplitudes decrease monotonically with increasing depth. The greatest error occurs at wide angle, with the result along the shallowest reflector being the worst.

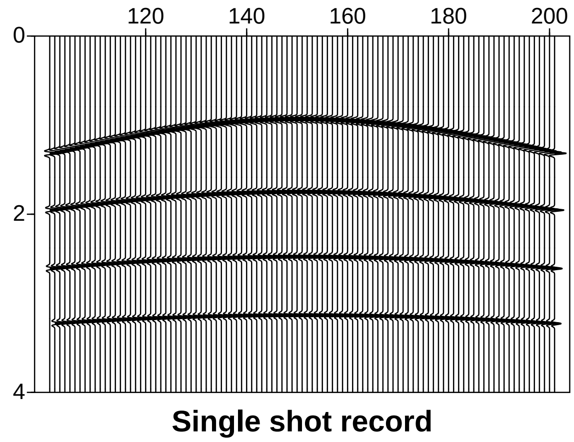


Fig. 1: Input model data.

Figure 3 left shows results of true amplitude common-shot migration (13). The peak amplitudes along the reflectors are shown in Figure 3 right. From this plot, we clearly see that the true amplitude algorithm recovers the reflectivity accurately, aside from the edge effects and small jitters caused by interference with wraparound artifacts.

## Conclusions

Common-shot migrations offer good potential of imaging complex structures, but the conventional formulations of such migrations produce incorrect migrated amplitudes. Here, we have described true-amplitude one-way wave equations that allow us to extend the standard method both for forward modeling and for wave equation migration. These modified one-way wave operators are developed with the aid of pseudo-differential operator theory. We have proven that these new one-way wave equations provide solutions that agree dynamically, as well as kinematically, with the solutions of the full wave equation. Further, we have proposed a new approach to WEM, transforming it into a true amplitude process, meaning that it produces an inversion output that agrees asymptotically with Kirchhoff inversion: it produces a reflector map with peak amplitudes on the reflector in known proportion to the geometrical optics reflection coefficient. We have proven this claim, as well. With the aid of a simple numerical example, we demonstrated that the migration method we proposed does calibrate common-shot migrations by correcting both their amplitude and phase behavior. We did this for an example in which the wave speed

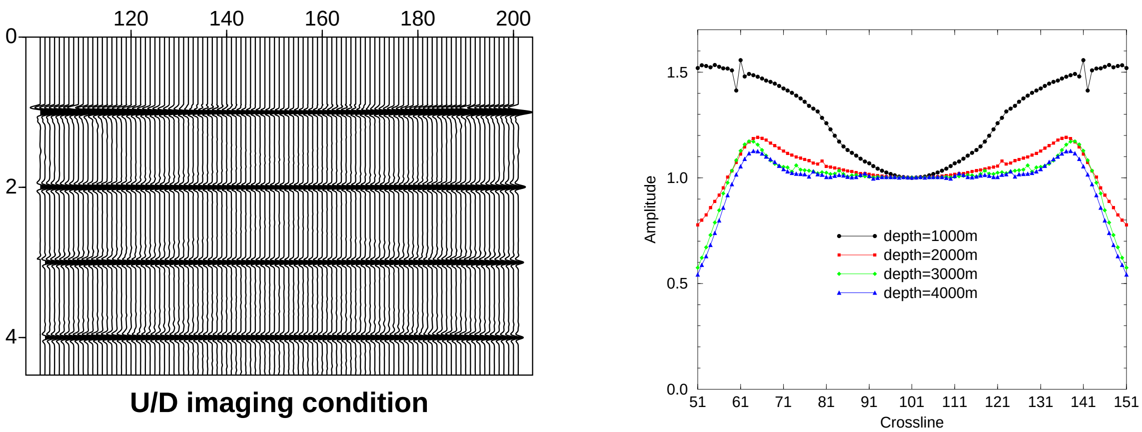


Fig. 2: Left: finite difference migration using classic WEM for imaging. Right: peak amplitudes along the four reflectors. The wide angle error decreases with depth of the reflector.

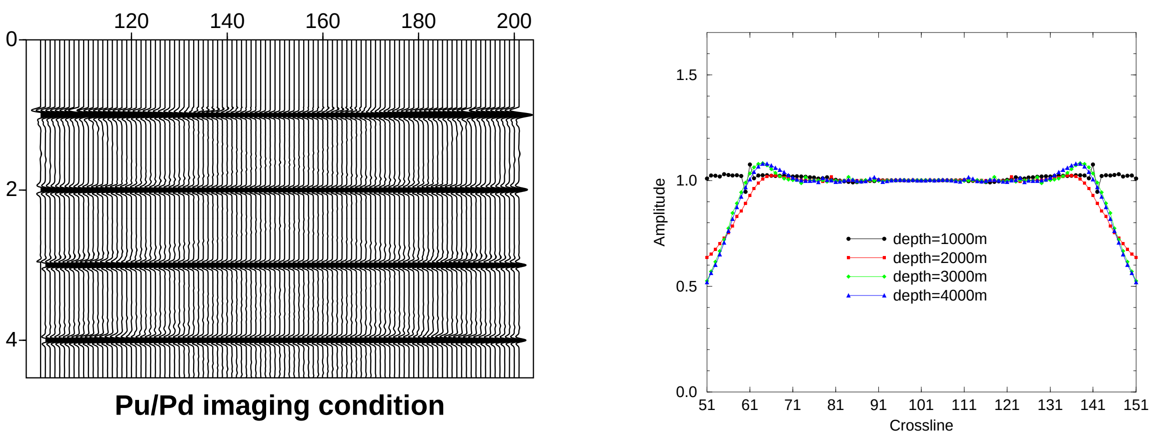


Fig. 3: Left: finite difference migration using (13) for imaging. Right: peak amplitudes along the four reflectors. The wide angle error decreases with depth of the reflector.

is depth-dependent — $v = v(z)$. The new method actually builds a bridge between true amplitude common-shot Kirchhoff migration and the migrations based on one-way wavefield extrapolation.

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