Radiative transfer in 1D, and the connection to the O'Doherty-Anstey formula

Kasper van Wijk[1,2]*, Matthew Haney*[1]*, Roel Snieder*[1] *and John A. Scales*[2]
1 *Center for Wave Phenomena, Colorado School of Mines, Golden, CO 80401*
2 *Physical Acoustics Laboratory, Colorado School of Mines, Golden, CO 80401*

Summary

There is a growing interest in incorporating multiply scattered waves into modeling the Earth's interior using radiative transfer theory. For thin beds, we derive the equivalence of the exponential decay of the transmitted wave predicted by the O'Doherty-Anstey formula with the coherent wave obtained from radiative transfer. This shows an underlying relationship between mean field theory and radiative transfer. Finally, we show how the incoherent field in radiative transfer opens the door to separate scatttering from intrinsic Q.

Introduction

Geophysicists have begun to investigate information about the subsurface hidden in multiply-scattered seismic waves in general (Hennino et al., 2001; Campillo and Paul, 2003; Wegler and Lühr, 2001), and via radiative transfer models, in particular (Wu and Aki, 1988; Margerin et al., 1999). For instance, we show that by including the late time arrivals of multiply scattered energy, scattering and intrinsic Q can be estimated, individually. Radiative transfer is a phenomenological theory for the spatial and temporal evolution of a wavefield's average intensity. We use the solution to the 1D radiative transfer equation to describe laboratory experiments of surface wave propagation through disordered grooves and compare this to the O'Doherty-Anstey Formula. The equivalence suggests that radiative transfer is a proper extension of mean field theory (a *variance field* theory) for the fluctuating, multiply-scattered waves.

The radiative transfer equation

A scalar radiative transfer equation valid for any dimension is

$$\frac{\partial I(\vec{r},\Omega,t)}{\partial t} + v\hat{n}(\Omega)\cdot\nabla I(\vec{r},\Omega,t) = -\frac{I(\vec{r},\Omega,t)}{\tau_s} - \frac{I(\vec{r},\Omega,t)}{\tau_a} + \frac{1}{\tau_s}\int \frac{1}{\sigma_s}\frac{\partial\sigma_s}{\partial\Omega'} I(\vec{r},\Omega',t)\,d\Omega' + S(\vec{r},\Omega,t), \quad (1)$$

where $I(\vec{r},\Omega,t)$ is the intensity, or average squared wavefield, at position $\vec{r}$ propagating in direction Ω, v is the group velocity of the average (coherent) wavefield, $\hat{n}$ is the unit vector in the direction of propagation, $S(\vec{r},\Omega,t)$ is the angle-resolved source function, and $\tau_{a,s}$ are the absorption and scattering mean free times, related to the mean free paths via $v\tau_{a,s} = \ell_{a,s}$. Scattering and absorption show up as loss mechanisms since both remove energy from the forward direction. Only scattering can transfer energy back into the original direction of propagation. Hence, scattering and absorption enter in fundamentally different ways. This provides the ability to separate these phenomena within radiative transfer. The Green's function for radiative transfer in 1D is

$$\begin{aligned} G(x,t) \quad &\propto \quad \exp\left(-vt(1/\ell_{ext} + 1/\ell_a)\right)\left[2\delta\left(\ell_{ext}(x-vt)\right)\right. \\ &+ \quad \left. u(vt-|x|)\left(I_0(\eta) + \sqrt{\frac{vt+x}{vt-x}}I_1(\eta)\right)\right]. \quad (2) \end{aligned}$$

The extinction mean free path ℓ_{ext} is defined as the distance over which the coherent intensity decays e^{-1}. Haney *et al.* (2003) show that in 1D $\ell_{ext} = 2\ell_s$, twice the scattering mean free path. The unit step function u assures causality and the argument of the modified Bessel functions of order zero (I_0) and one (I_1) is

$$\eta = \sqrt{(vt)^2 - x^2}/2\ell_s.$$

The Green's function for the total intensity consists of two parts. The term with the δ-function propagates like a ballistic wave and is called the *coherent intensity.* The Bessel functions describe the *incoherent intensity.*

O'Doherty-Anstey and the coherent intensity

A well known result for waves multiply scattered by a 1D layering is that obtained by O'Doherty and Anstey (1971). Their formula has subsequently been derived from mean field theory of Banik *et al.* (1985), showing that the amplitude of a wave transmitted through a stack of layers decays exponentially with distance as

$$|T| \sim \exp(-\tilde{R}(k)x), \quad (3)$$

where $\tilde{R}(k)$ represents the power spectrum of the average reflection coefficient series normalized by two-way travel distance. From the solution for the total intensity obtained in the last section, radiative transfer also predicts an exponential decay for the transmitted, or coherent, wave with distance:

$$|T| \sim \exp(-x/4\ell_s), \quad (4)$$

where the distance x has replaced vt in equation (2) since the δ-function is nonzero only at $x = vt$. The factor of 1/2 in the exponent of this equation shows up since radiative transfer predicts decay of the transmitted intensity, the

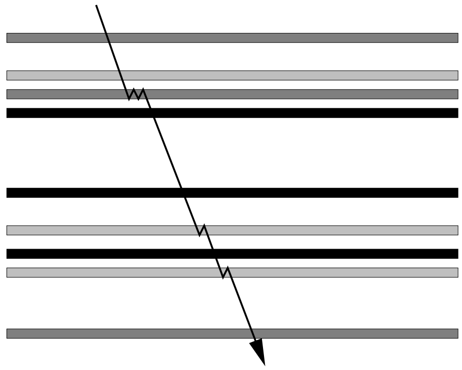

Fig. 1: A wave transmitted through a random sequence of thin beds of varying strength. The thin beds are embedded in a constant background medium.

square of the true transmission coefficient. We investigate the equivalence of these two theories for the transmission of normally incident waves through assemblages of weak 1D point scatterers (thin beds):

$$\tilde{R}(k) \stackrel{?}{=} 1/4\ell_s. \tag{5}$$

Depicted in Fig. 1 is the random medium we consider: a series of thin layers of varying strength is embedded in a constant velocity background medium. In the parlance of O'Doherty-Anstey, this would be called a *cyclic* sequence. The reflection coefficient series, $RC(x)$, for such a medium would be a series of delta functions of alternating plus and minus sign:

$$RC(x) = \sum_{j=1}^{N} R_j \left[\delta(x - d_j) - \delta(x - h - d_j)\right], \tag{6}$$

where h is the thickness of each ofx the beds, R_j and d_j represent the reflection coefficient and location of the j-th bed, respectively, and N is the number of beds.

To calculate $\tilde{R}(k)$, we take the Fourier transform of equation (5), square its magnitude to get the power spectrum, and divide by the two-way travel distance:

$$\tilde{R}(k) = \frac{1}{2L}\left|\int_{-\infty}^{\infty} RC(x)\exp(-i2kx)dx\right|^2. \tag{7}$$

Note that the Fourier transform is with respect to $2k$ and not k, similar to a Born inversion formula in 1D (Bleistein et al., 2001; Banik et al., 1985; Shapiro and Zien, 1993). Inserting equation (6) into equation (7) results in

$$\tilde{R}(k) = \frac{1}{2L}\left|\sum_{j=1}^{N} R_j \exp(2ikd_j)(1 - \exp(2ikh))\right|^2. \tag{8}$$

For thin layers, $kh << 1$, and a first-order Taylor series expansion in h leads to $1 - \exp(2ikh) \approx -2ikh$:

$$\tilde{R}(k) = \frac{4k^2h^2}{2L}\left|\sum_{j=1}^{N} R_j \exp(2ikd_j)\right|^2. \tag{9}$$

If d_j, the spacing of the thin beds, is a random variable, the cross terms in the square of the summation in equation (9) cancel in the *average*, and the squaring can be brought inside the summation:

$$\tilde{R}(k) = \frac{2k^2h^2}{L}\sum_{j=1}^{N}\left|R_j \exp(2ikd_j)\right|^2 =$$

$$\frac{2k^2h^2}{L}\sum_{j=1}^{N}|R_j|^2 = 2k^2h^2N\langle|R_j|^2\rangle/L, \tag{10}$$

where $\langle|R_j|^2\rangle$ is the mean-square of the interface reflection coefficients.

Returning to equation (5), to prove that radiative transfer and the O'Doherty-Anstey formula predict the same exponential decay for the transmitted wave, we find that

$$\ell_s = \frac{1}{8k^2h^2\langle|R_j|^2\rangle N/L}. \tag{11}$$

The quantity N/L is simply the number density of the thin beds, ρ. In the limit of weak scatterers (such that $R_j << 1$), $8k^2h^2\langle|R_j|^2\rangle = \sigma_s$, the scattering cross section (Haney et al., 2003). The presence of weak reflection coefficients is an underlying assumption in the O'Doherty-Anstey result (Banik et al., 1985), so equation (11) can be rewritten in a familiar form,

$$\ell_s = \frac{1}{\rho\sigma_s}. \tag{12}$$

This is the independent scattering approximation, which is one of the assumptions in radiative transfer, but generally violated in geophysics where geology varies within a wavelength. Equation (12) demonstrates that, for this model, the exponential decay of the transmitted wave from O'Doherty-Anstey, or mean-field theory, is equivalent to that predicted by radiative transfer. For this relation to hold, the scatterers (thin beds) have to be separated by at least a wavelength. Hence, in this model, no reflections from below the recording depth interfere with the transmitted wave. All the interference resulting in the exponential decay of the direct wave originates from peg-leg multiples within the thin beds, not between them. Fig. 2 is a conceptual diagram of this equivalence. From mean field theory, both the phase and the amplitude of the transmitted wave can be obtained. However, the incoherent energy, for which the mean is zero, falls out. Similarly, 1D radiative transfer can address the amplitude of the transmitted wave and the behavior of the incoherent intensity, but phase information is lost. Both theories agree in their region of overlap, as demonstrated by the case of random layering we considered here.

Laboratory experiment

An angle-beam transducer launches pure, effectively plane, surface waves in an aluminum block of dimensions $28 \times 23 \times 21.5$ cm^2 (Scales and van Wijk, 1999; Scales

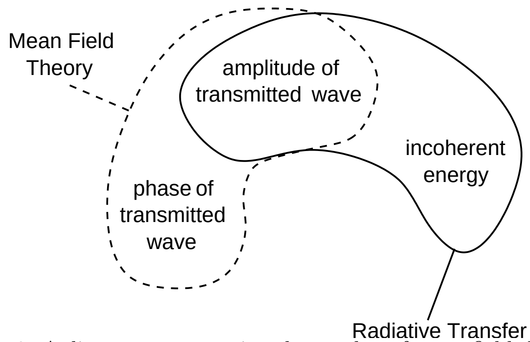

Fig. 2: A diagram representing the overlap of mean field theory and radiative transfer for the amplitude of the transmitted wave through a medium like that depicted in Fig. 1

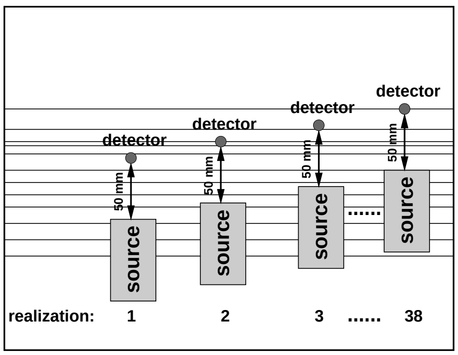

Fig. 3: Top-view of the experimental procedure to obtain an ensemble average.

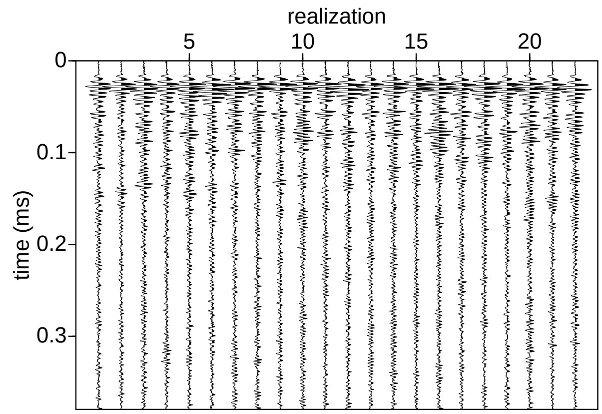

Fig. 4: Particle velocity at 50 mm source-detector offset for 22 locations in the groove sequence.

and van Wijk, 2001). The surface has a disordered pattern of aligned linear grooves that are 1 mm wide by 3 mm deep. Because the dominant wavelength of the surface waves is about 15 mm, many scatterers per wavelength are encountered as surface waves propagate perpendicular to the grooves. While plane waves are incident at right angles on grooves that extend over the entire length of the model, consistent with a 1D model, surface wave energy is lost to body wave diffractions at each groove. We treat this diffracted energy as attenuation (i.e., loss) in a 1D medium; intrinsic absorption in the aluminum is virtually zero. The wavefield is detected using a scanning laser vibrometer that measures absolute particle velocity on the surface of the sample via the Doppler shift.

To obtain ensemble measurements over the disordered medium, particle velocity measurements at fixed source-detector distances are collected for different positions in the groove sequence (Fig. 3). We used three sets of 22 realizations with the detector 25, 50 and 75 mm from the leading edge of the source. Fig. 4 shows waveforms for 50 mm source-detector spacing at 22 locations in the ensemble measurement. Around 0.025 ms all traces contain relatively large amplitudes that are in phase, comprising the coherent signal. This part of the signal is independent of local variations in the scattering distribution, whereas the later arrivals (also known as *coda*) vary in phase and amplitude for each source-detector location. This is incoherent signal due to scattering from the micro-structure in the medium.

By fitting the laboratory data, we estimate the scattering and absorption mean free paths (Q) and the energy velocity using the radiative transfer equation. Since we are able to separate both the data and the radiative transfer equation into a coherent and an incoherent part, we treat the parameter-fitting problem for each part separately. For clarity, from here on we drop scaling terms in the solution to the radiative transfer equation, since all data and simulations are normalized to amplitudes at the first source-detector offset. The Green's function in expression (2) is convolved with the band-limited input energy pulse.

The solid line in Fig. 5 is the modeled envelope of the coherent intensity for $\alpha = 1/2\ell_s + 1/\ell_a = 17.8$ m^{-1} and the group velocity $v = 1818 \pm 123$ m/s. The energy of the coherent signal propagates dispersively. As lower frequency surface waves penetrate the model deeper, they propagate relatively undisturbed by the scatterers, while higher frequencies are slowed by stronger scattering due to the grooves. We therefore model the energy velocity to be a function of frequency. The smaller, secondary peak in the coherent intensity is not modeled. This peak is most likely a part of the source wavelet, not accounted for in the model.

The incoherent intensity, which is the difference between the total and the coherent intensity, is plotted as the dashed curves in Fig. 6. The solid lines are the result of modeling the incoherent part of (2). The resulting fits in Fig. 6 are for a scattering mean free path of $\ell_s = 0.05$ m. The incoherent signal is fit only at intermediate times. At short times the incoherent data are incomplete, and at late times energy comes back into the system from reflections off the back of the model. From these computations, we obtain an absorption length of $\ell_a = 0.15$ m.

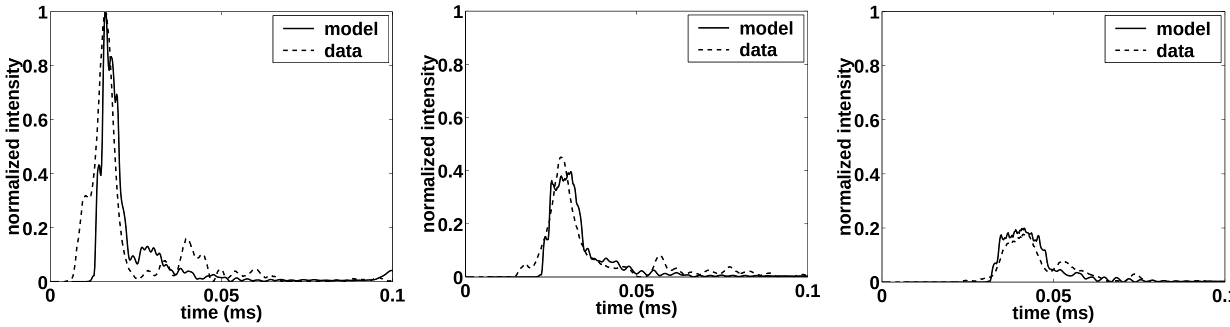

Fig. 5: Comparison between measured and modeled coherent intensities for 25 (left), 50 (middle) and 75 mm (right) offset.

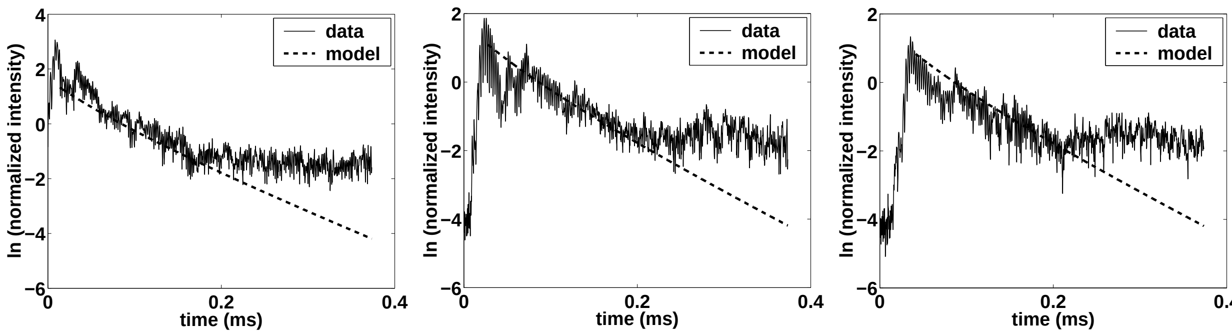

Fig. 6: Comparison between measured and modeled incoherent intensities for 25 (left), 50 (middle) and 75 mm (right) offset.

Conclusions

We have made the connection between radiative transfer theory and the O'Doherty-Anstey formula. Also, the incoherent signal in radiative transfer provides information about the scattering medium, such as independent estimates of scattering and intrinsic Q.

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