Envelopes of 2D and 3D magnetic data and their relationship to the analytic signal: Preliminary results

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Summary

Unlike the 2D case, the total gradient (magnitude of the gradient vector) in 3D is not the envelope of either the vertical or horizontal derivatives of the total-field anomaly over all possible directions of the earth's field and source magnetization. Instead, the total gradient of the reduced-to-pole data defines the envelope for the vertical derivative while the total gradient of the reduced-to-equator data defines the envelope for the horizontal derivative.

Introduction

Nabighian (1972) showed that the horizontal and vertical derivatives of the magnetic anomaly produced by a 2D source form a Hilbert transform pair and define an analytic signal. An important property of the 2D analytic signal is that its amplitude is the envelope of its underlying signal (Kanasewich, 1981) - the horizontal or vertical derivative in the 2D magnetic problem. It follows that the magnitude of the gradient of magnetic data (henceforth referred to as the total gradient) is equal to the envelope of both the horizontal and vertical derivatives over all possible inclinations. For processing magnetic data, the amplitude of the analytic signal in 2D is remarkable in that it allows one to obtain a signal that is independent of the source magnetization direction.

Attempts have been made to generalize the analytic signal to 3D (Nabighian, 1984; Craig, 1996). These studies have employed the concept of the complex field but have not defined a 3D analytic signal. In spite of this, the applied geophysical community has adopted a simple extension from 2D to 3D following Nabighian's (1984) generalized Hilbert transform and its connection to the 2D counterpart. It states that the total gradient in 3D is also the envelope of the derivatives of the magnetic anomaly over all inclinations and declinations (Roest *et al.*, 1992). This extension has been accepted without theoretical proof or numerical verification.

We show that the fact that the amplitude of the analytic signal is the envelope in 2D is a consequence of a more general theory of envelopes. This is achieved without resorting to Hilbert transforms. We extend the argument to 3D and find the envelopes of the vertical and horizontal derivatives of total-field magnetic anomaly. Numerical examples illustrate our analytic results.

The 2D Envelope

We first explore what is already known - that the amplitude of the analytic signal in 2D, or the total gradient, is indeed the envelope over all inclination directions. To begin, we examine a function of x and θ defined as:

$$f = a(x)\cos\theta + b(x)\sin\theta \tag{1}$$

which is chosen since the total-field anomaly has the form of f in 2D with θ representing the earth's inclination plus the source's inclination minus the dip (Nabighian, 1972). The parameter x is the horizontal coordinate. From Zauderer (1989), the envelope of the function f over all values of θ can be obtained by taking the derivative of f with respect to θ, setting it to zero, solving for θ as a function of $a(x)$ and $b(x)$, and substituting this expression into the definition of f:

$$\frac{\partial f}{\partial \theta} = -a(x)\sin\theta + b(x)\cos\theta = 0$$

$$\tan\theta = \frac{b(x)}{a(x)}$$

$$\sin\theta = \frac{\pm b(x)}{\sqrt{a(x)^2 + b(x)^2}} \tag{2}$$

$$\cos\theta = \frac{\pm a(x)}{\sqrt{a(x)^2 + b(x)^2}} \tag{3}$$

Substituting the expressions for $\sin\theta$ and $\cos\theta$ into equation (1), we obtain the envelope, denoted $\mathcal{E}[f]$:

$$\mathcal{E}[f] = a(x)\frac{\pm a(x)}{\sqrt{a(x)^2 + b(x)^2}} + b(x)\frac{\pm b(x)}{\sqrt{a(x)^2 + b(x)^2}}$$

$$\mathcal{E}[f] = \pm\frac{a(x)^2 + b(x)^2}{\sqrt{a(x)^2 + b(x)^2}} = \pm\sqrt{a(x)^2 + b(x)^2} \tag{4}$$

Note that the envelope is actually two functions, one positive and one negative, that bounds the function f over all possible values of the phase angle θ.

The process outlined above yields the envelope of any general function, not just that of equation (1). However, in the magnetic problem we typically deal with functions of this specific form. It also appears in the 3D problem. For functions with the form of f, an alternate, but equivalent, definition of the envelope can be formulated:

$$\mathcal{E}[f] = \pm\sqrt{f^2 + \left[\frac{\partial f}{\partial \theta}\right]^2} \tag{5}$$

Substitution of equation (1) into equation (5) will give the result of equation (4). This representation suggests a source-independent method of computing the envelope of f: all that is needed is the derivative of f with respect to θ.

Envelopes and the Analytic Signal in 2D

The directional aspects of magnetic anomalies can be most easily represented in the wavenumber domain (Blakely, 1995). For the following, we represent the total-field anomaly as ΔM and its Fourier transform as $\mathcal{F}[\Delta M]$. The wavenumber for the horizontal direction is given by k. In 2D, the Fourier transform can be written as two factors depending on the earth and source inclinations (I and I_m respectively) and another factor only dependent on the geometry of the source body, $\mathcal{F}[U]$:

$$\mathcal{F}[\Delta M] = \left[\sin I + \frac{ik}{|k|}\cos I\right]\left[\sin I_m + \frac{ik}{|k|}\cos I_m\right]\mathcal{F}[U] \tag{6}$$

Multiplying the two directional terms and simplifying yields:

$$\mathcal{F}[\Delta M] = \left[-\cos(I+I_m) + \frac{ik}{|k|}\sin(I+I_m)\right]\mathcal{F}[U] \tag{7}$$

This equation shows that, although there are two directional quantities in 2D, I and I_m, their sum acts as one effective phase angle. To keep the notation concise, ϕ will be used to represent $I + I_m$ and we refer to it generally as the phase angle. Note that equation (7) has the form of equation (1).

Now take the derivative of equation (7) with respect to ϕ:

$$\frac{\partial \mathcal{F}[\Delta M]}{\partial \phi} = \left[\sin\phi + \frac{ik}{|k|}\cos\phi\right]\mathcal{F}[U] \tag{8}$$

Using equations (7) and (8), a relationship between the Fourier transform of the total field anomaly and its phase angle derivative can be formulated:

$$\frac{\partial \mathcal{F}[\Delta M]}{\partial \phi} = \left[\frac{sin\phi + \frac{ik}{|k|}cos\phi}{-cos\phi + \frac{ik}{|k|}sin\phi}\right]\mathcal{F}[\Delta M] \tag{9}$$

A deeper relation between $\mathcal{F}[\Delta M]$ and its derivative can be obtained by rationalizing the denominator of equation (9):

$$\frac{\partial \mathcal{F}[\Delta M]}{\partial \phi} = -\frac{ik}{|k|}\mathcal{F}[\Delta M] \tag{10}$$

This equation shows that the derivative of the total field with respect to the phase angle is proportional to the total field by a factor which does not depend on the phase angle. This is a luxury only afforded in 2D.

Let us now use equation (10) to verify that the amplitude of the analytic signal is the envelope of the horizontal and vertical derivatives of the total-field anomaly, ΔM, in 2D. Both $\partial\Delta M/\partial x$ and $\partial\Delta M/\partial z$ have the form of the function f in equation (1) with ϕ in place of θ. Hence, their envelopes can be written as:

$$\mathcal{E}\left[\frac{\partial\Delta M}{\partial z}\right] = \sqrt{\left[\frac{\partial\Delta M}{\partial z}\right]^2 + \left[\frac{\partial^2\Delta M}{\partial z\partial\phi}\right]^2} \tag{11}$$

$$\mathcal{E}\left[\frac{\partial\Delta M}{\partial x}\right] = \sqrt{\left[\frac{\partial\Delta M}{\partial x}\right]^2 + \left[\frac{\partial^2\Delta M}{\partial x\partial\phi}\right]^2} \tag{12}$$

where $\mathcal{E}[\partial\Delta M/\partial z]$ and $\mathcal{E}[\partial\Delta M/\partial x]$ represent the envelopes. In equation (11), we need the quantity $\partial^2\Delta M/\partial z\partial\phi$. Using equation (10):

$$ik\mathcal{F}[\Delta M] = -|k|\frac{\partial\mathcal{F}[\Delta M]}{\partial\phi} \tag{13}$$

Taking the inverse Fourier transform of both sides of this equation yields:

$$\frac{\partial\Delta M}{\partial x} = -\frac{\partial^2\Delta M}{\partial z\partial\phi} \tag{14}$$

Substituting this back into equation (11) gives the well known result:

$$\mathcal{E}\left[\frac{\partial\Delta M}{\partial z}\right] = \sqrt{\left[\frac{\partial\Delta M}{\partial z}\right]^2 + \left[\frac{\partial\Delta M}{\partial x}\right]^2} \tag{15}$$

Applying similar logic for $\partial^2\Delta M/\partial x\partial\phi$, we obtain the envelope of the horizontal derivative:

$$\mathcal{E}\left[\frac{\partial\Delta M}{\partial x}\right] = \sqrt{\left[\frac{\partial\Delta M}{\partial x}\right]^2 + \left[\frac{\partial\Delta M}{\partial z}\right]^2} \tag{16}$$

Summarizing equations (15) and (16), the envelopes of the horizontal and vertical derivatives are the same in 2D and are equal to the total gradient:

$$\mathcal{E}\left[\frac{\partial\Delta M}{\partial z}\right] = \mathcal{E}\left[\frac{\partial\Delta M}{\partial x}\right] = |\nabla(\Delta M)| \tag{17}$$

In 2D, the amplitude of the analytic signal is the total gradient. Thus, we have verified that the amplitude of the analytic signal is the envelope of the horizontal and vertical derivatives of the total field anomaly without appealing to Hilbert transforms. The extension of the above method is needed for the more difficult problem of calculating the envelopes in 3D.

The 3D Envelopes

Above, we stated that although there are two directional quantities in 2D, I and I_m, they combine to make one phase angle, which we denoted as ϕ. As a result, in 2D there is only one envelope. In 3D, however, the four directional quantities I, I_m, D (declination of earth's field), and D_m (declination of magnetization) do not combine and reduce to a smaller set. With all four directional quantities in 3D, there are 15 different envelopes that we can speak of. Without information about the source inclination and declination, only 3 envelopes can be readily calculated from field data. Here, we focus on the envelope over all possible earth inclinations and declinations, I and D.

In analogy to equation (1), a general function with the form of a directional derivative in 3D is:

$$f = a(x, y)\sin\theta + \cos\theta[b(x, y)\sin\psi + c(x, y)\cos\psi] \quad (18)$$

Following the same steps outlined in the calculation of the 2D envelope, the envelope of equation (18) over all values of θ and ψ is given by:

$$\mathcal{E}[f] = \pm\sqrt{a(x, y)^2 + b(x, y)^2 + c(x, y)^2} \quad (19)$$

An expression for the envelope of f using only f and its angular derivatives is:

$$\mathcal{E}[f] = \pm\sqrt{f^2 + \left[\frac{\partial f}{\partial\theta}\right]^2 + \left[\frac{\partial f}{\partial\psi}\right]^2 + \left[\frac{\partial^2 f}{\partial\theta\partial\psi}\right]^2} \quad (20)$$

Substitution of equation (18) into equation (20) will yield equation (19).

As an example of this 3D generalization, we formulate the envelope of the vertical derivative of the total field anomaly over all directions of the earth's field. For this, we set f in equation (20) to $\partial\Delta M/\partial z$, θ to I, and ψ to D. The envelope is given by:

$$\mathcal{E}\left[\frac{\partial\Delta M}{\partial z}\right] = \left(\left[\frac{\partial\Delta M}{\partial z}\right]^2 + \left[\frac{\partial^2\Delta M}{\partial z\partial I}\right]^2 + \left[\frac{\partial^2\Delta M}{\partial z\partial D}\right]^2 + \left[\frac{\partial^3\Delta M}{\partial z\partial I\partial D}\right]^2\right)^{1/2} \quad (21)$$

From this expression, there are three angular derivatives needed for the calculation of the envelope: $\partial^2\Delta M/\partial z\partial D$, $\partial^2\Delta M/\partial z\partial I$, and $\partial^3\Delta M/\partial z\partial I\partial D$. All three can be calculated in the wavenumber domain with the knowledge of the earth's field direction (I and D). We denote the inverse Fourier transform by $\mathcal{F}^{-1}$ and the two horizontal wavenumbers by p and q. If ΔM has been half-reduced to the pole (the earth's field rotated to the pole, not the source magnetization), the three needed derivatives are:

$$\frac{\partial^2\Delta M}{\partial z\partial D} = 0 \quad (22)$$

$$\frac{\partial^2\Delta M}{\partial z\partial I} = \mathcal{F}^{-1}\left[(-ip\cos D - iq\sin D)\mathcal{F}[\Delta M]\right] \quad (23)$$

$$\frac{\partial^3\Delta M}{\partial z\partial I\partial D} = \mathcal{F}^{-1}\left[(ip\sin D - iq\cos D)\mathcal{F}[\Delta M]\right] \quad (24)$$

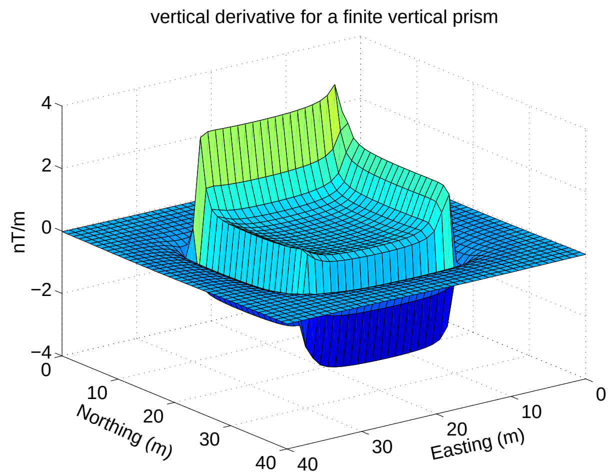

Fig. 1: The vertical derivative of the total-field anomaly for a finite vertical prism.

Equation (22) tells us that one of the four terms in equation (21) falls out if the data is half-reduced to the pole. Using equations (23) and (24), it can be shown that for the other two terms:

$$\left[\frac{\partial^2\Delta M}{\partial z\partial I}\right]^2 + \left[\frac{\partial^3\Delta M}{\partial z\partial I\partial D}\right]^2 = \left[\frac{\partial\Delta M}{\partial x}\right]^2 + \left[\frac{\partial\Delta M}{\partial y}\right]^2 \quad (25)$$

Substitution into equation (21) yields the familiar form:

$$\mathcal{E}\left[\frac{\partial\Delta M}{\partial z}\right] = \sqrt{\left[\frac{\partial\Delta M}{\partial z}\right]^2 + \left[\frac{\partial\Delta M}{\partial x}\right]^2 + \left[\frac{\partial\Delta M}{\partial y}\right]^2} \quad (26)$$

Hence, the 3D envelope of the vertical derivative over all possible inclinations and declinations of the earth's field is the total gradient of the data after half-reduction to the pole (HRTP):

$$\mathcal{E}\left[\frac{\partial\Delta M}{\partial z}\right] = |\nabla(\mathrm{HRTP}[\Delta M])| \quad (27)$$

In a similar fashion, it can be shown that the envelope of the horizontal derivative in the direction of the declination is the total gradient of the data after half-reduction to the equator (HRTE):

$$\mathcal{E}\left[\cos D\frac{\partial\Delta M}{\partial x} + \sin D\frac{\partial\Delta M}{\partial y}\right] = |\nabla(\mathrm{HRTE}[\Delta M])| \quad (28)$$

Equations (27) and (28) are the envelopes over all possible directions of the earth's field. To obtain the envelopes over both the earth's field and source magnetization, the half-reductions to the pole and equator in equations (27) and (28) should be replaced with the complete reductions

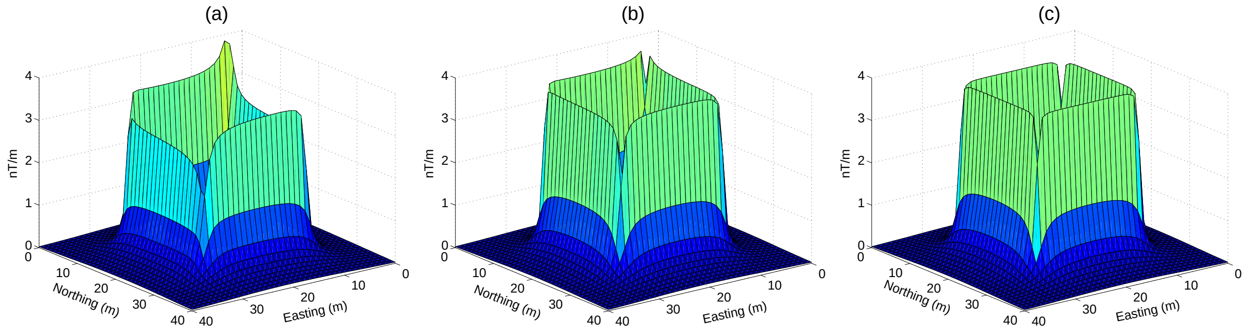

Fig. 2: (a) The total gradient for the anomaly in Fig. 1, (b) the total gradient after half-reduction to the pole (HRTP), and (c) the total gradient after reduction to the pole (RTP).

to the pole and equator (RTP and RTE). The complete reductions assume that the source magnetization direction is known.

Numerical Example

Depicted in Fig. 1 is the vertical derivative of the total field anomaly due to a finite vertical prism. The earth's field in this example has a declination of 30° to the east of north and an inclination of 45°. The source magnetization has the same declination as the earth's field, but has an inclination of 70°. Therefore, we model the case where the prism has some remanence.

For this anomaly, we computed the total gradient, the total gradient of the HRTP data, and the total gradient of the RTP data. The results are shown in Fig. 2. The envelopes we derived in the previous section demonstrated that the total gradient of the HRTP data is the envelope of the vertical derivative over all directions of the earth's field. To check this result, we obtained the envelope by first calculating the vertical derivative of the anomaly over all inclinations and declinations of the earth's field in increments of 5°. We then saved the maximum value at each observation point as the value of the envelope. The resulting envelope is identical to Fig. 2(b), verifying our arguments.

The total gradient of the RTP data, Fig. 2(c), is the envelope of the vertical derivative over all directions of the earth's field and source magnetization. In 2D, this is the same as the amplitude of the analytic signal. Unfortunately, the RTP procedure means that knowledge of the source inclination is necessary to obtain the full envelope in 3D. The total gradient, Fig. 2(a), deviates from the full envelope, Fig. 2(c), substantially. The total gradient of the half-RTP data, accessible since the earth's field direction is known, is a better approximation to the full envelope.

Conclusions

In 3D, the total gradient alone is not the envelope of the vertical or horizontal derivatives of the total-field anomaly. Using the fundamental definition of an envelope, we showed that the total gradient of the RTP data is the envelope of the vertical derivative over all directions of the earth's field and source magnetization. The envelope for the horizontal derivatives demands a reduction to the equator before taking the total gradient.

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