# Approximate reflection coefficients of $PS$-waves in anisotropic media

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## Summary

Due to the complexity of the exact reflection and transmission coefficients, their approximations are of great importance in AVO analysis, especially in anisotropic media. Whereas several approximations for anisotropic $PP$-wave reflection coefficients have been discussed previously in the literature, the influence of anisotropy is commonly ignored in the case of the converted $PS$-wave reflection coefficients. Adequate approximations for $PS$-wave reflection coefficients may provide a powerful tool for a joint $PP$ and $PS$ inversion of AVO attributes for the medium parameters.

Here, I present first-order approximations for converted-wave displacement reflection coefficients $R_{PS_1}$ and $R_{PS_2}$ at a weak horizontal interface separating two weakly anisotropic media with arbitrary symmetry. The general expressions are specified for an interface between two orthorhombic media with differently oriented vertical symmetry planes using Thomsen-type notation. I also obtained simple approximations valid for small incidence angles (approximately up to 15°). The final expressions can be applied to any combination of isotropic, VTI, HTI and orthorhombic halfspaces.


## Introduction

Recent development of acquisition techniques, such as ocean bottom cable (OBC) surveys, allows us to carry out joint analysis of both $P$- and $PS$-wave data. In terms of AVO analysis, using converted $PS$-wave reflection coefficients may provide important additional information for constraining the medium parameters.

Linearized approximations for $PP$ reflection coefficients in anisotropic media have been derived by Banik (1987), Thomsen (1993) and Rüger (1996). The most general expressions for $PP$-wave reflection coefficients in arbitrary anisotropic media have been introduced by Vavryčuk and Pšenčík (1998). Converted $PS$-wave reflection coefficients in isotropic media have been discussed, for example, in Alvarez *et al.* (1999) and Nefedkina and Buzlukov (1999). Vavryčuk (1999) gave general expressions for approximate $PS$-wave reflection coefficients in anisotropic media; however, those expressions are not well suited for AVO analysis.

In this paper, I derive azimuthally dependent weak-contrast weak-anisotropy displacement reflection coefficients $R_{PS}$ for a horizontal interface between two arbitrarily anisotropic halfspaces, following the approach by Vavryčuk and Pšenčík (1998). I specify the general expressions for some common anisotropic models using Thomsen-type notation. The accuracy of the solutions is illustrated by two numerical examples for VTI/HTI and orthorhombic/orthorhombic interfaces.

## $PS$-wave reflection coefficients for general anisotropy

In the derivation of the $PS$-wave reflection coefficients, I use the approach of Vavryčuk and Pšenčík (1998) for $PP$ reflection coefficients. Their approach is in principle similar to that of Thomsen (1993) and Rüger (1996), but with a different parameterization of the medium. It is based on an approximate solution of the algebraic system of boundary conditions. If the contrasts of the medium parameters across the interface are small, and the anisotropy in both halfspaces is weak, then the algebraic system can be linearized and the solution can be expressed in terms of perturbations of elastic parameters $\Delta A_{ij} = A_{ij}^{(2)} - A_{ij}^{(1)}$ and density $\Delta\rho = \rho^{(2)} - \rho^{(1)}$ from a certain background medium. Here, $A_{ij}^{(I)}$ are density-normalized stiffness tensor elements $a_{klmn}^{(I)}$ written using Voigt convention ($I = 1, 2$ denotes the incidence and reflecting halfspaces, respectively). The background medium is chosen to be isotropic with the $P$- and $S$-wave velocities and densities averaged as

$$\bar{\alpha} = \frac{1}{2}(A_{33}^{(1)} + A_{33}^{(2)}), \qquad \bar{\beta} = \frac{1}{2}(A_{55}^{(1)} + A_{55}^{(2)}), \qquad (1)$$
$$\bar{\rho} = \frac{1}{2}(\rho^{(1)} + \rho^{(2)}).$$

Moreover, it is necessary to specify the the polarization angle $\Phi$ between the polarization vectors of the converted $PS$-waves in the anisotropic incidence halfspace and the polarization directions $SV$- and $SH$-waves defined in the isotropic background (for more detail, see Jech and Pšenčík, 1989). Unfortunately, the angle $\Phi$ is not necessary small and thus can be neither neglected nor linearized. In principle, however, it is possible to estimate $\Phi$ from reflection data.

Using the weak-contrast, weak-anisotropy approximation, I derived the following general expressions for the reflection coefficients of two converted $PS_1$- and $PS_2$-waves generated at the horizontal interface between two anisotropic halfspaces:

$$R_{PS_1} = R_{PSV}\cos\Phi + R_{PSH}\sin\Phi, \qquad (2)$$
$$R_{PS_2} = -R_{PSV}\sin\Phi + R_{PSH}\cos\Phi.$$

The components $R_{PSV}$ and $R_{PSH}$ are relatively complicated linear functions of the medium parameters $\Delta A_{ij}$, background medium parameters (1), incidence and reflection phase angles $i$ and $j$ (respectively) and azimuth $\psi$. $R_{PSV}$ and $R_{PSH}$ will be written explicitly for particular models.

The coefficients (2) are not linear functions of the medium parameters $A_{ij}$ due to the non-linearity of the angle $\Phi$. Therefore, the term " *linearized reflection coefficient*," commonly used for $R_{PP}$ approximations, cannot be strictly applied in this case.

## $R_{PS_1}$ and $R_{PS_2}$ coefficients for orthorhombic media

For an interface separating two orthorhombic media with horizontal symmetry planes, the components $R_{PSV}$ and $R_{PSH}$ can be written as

$$\begin{aligned} R_{PSV} &= V_1 \cos i \sin i + V_2 \frac{\sin i}{\cos j} + V_3 \cos i \sin^3 i + \\ &\quad V_4 \frac{\sin^3 i}{\cos j} + V_5 \frac{\sin^5 i}{\cos j}; \\ R_{PSH} &= H_1 \sin i + H_2 \frac{\cos i \sin i}{\cos j} + H_3 \sin^3 i + \\ &\quad H_4 \frac{\cos i \sin^3 i}{\cos j}, \end{aligned} \tag{3}$$

where $V_k$ and $H_k$ contain linear combinations of the following parameters: isotropic contrast parameters $\Delta\rho/\bar{\rho}$, $\Delta\beta/\bar{\beta}$, and Thomsen-type anisotropic parameters $\gamma_I^{(S)}$, $\epsilon_I^{(1)}$, $\epsilon_I^{(2)}$, $\delta_I^{(1)}$, $\delta_I^{(2)}$ and $\delta_I^{(3)}$ $(I = 1, 2)$ introduced by Tsvankin (1997). The coefficients $V_k$ and $H_k$ are also functions of the azimuth $\psi$ and angle $\kappa$, which is the angle between vertical symmetry planes of the incidence and reflecting orthorhombic halfspaces. According to equations (3), different coefficients $V_k$ and $H_k$ control $R_{PSV}$ and $R_{PSH}$ components for different ranges of incidence angles.

Since isotropic, VTI and HTI media are special cases of the orthorhombic symmetry, equations (3) can be immediately used for any combination of isotropic, VTI, HTI and orthorhombic halfspaces above and below the interface. Only the anisotropic medium parameters in equations (3) have to be adapted for a specific model as shown by Tsvankin (1997).

## $R_{PSV}$ and $R_{PSH}$ approximations for small incidence angles

In practical AVO analysis, reflection coefficients are usually obtained for relatively small incidence angles. Keeping only low angle terms in equation (3), we arrive at

$$\begin{aligned} R_{PSV} &= \sin i \left\{ -\frac{1+2g}{2}\frac{\Delta\rho}{\bar{\rho}} - 2g\frac{\Delta\beta}{\bar{\beta}} + \frac{1}{2(1+g)}\delta_2^{(2)} \times \right. \\ &\cos^2(\psi-\kappa) + \left[\frac{1}{2(1+g)}\delta_2^{(1)} - 2g\gamma_2^{(S)}\right]\sin^2(\psi-\kappa) - \\ &\left. \frac{1}{2(1+g)}\delta_1^{(2)}\cos^2\psi - \left[\frac{1}{2(1+g)}\delta_1^{(1)} - 2g\gamma_1^{(S)}\right]\sin^2\psi \right\}, \\ R_{PSH} &= \sin i \left\{ \left[\frac{1}{4(1+g)}(\delta_2^{(2)} - \delta_2^{(1)}) + g\gamma_2^{(S)}\right] \times \right. \\ &\left. \sin 2(\psi-\kappa) - \left[\frac{1}{4(1+g)}(\delta_1^{(2)} - \delta_1^{(1)}) + g\gamma_1^{(S)}\right]\sin 2\psi \right\}. \end{aligned} \tag{4}$$

Here, $g \equiv \bar{\beta}/\bar{\alpha}$ is the background velocity ratio.

Equations (4) provide a good approximation to equations (3) for incidence angles up to $15^\circ - 20^\circ$. Equations (4) can be used for fast rough estimates of medium parameters, as well as for a linear inversion of reflection coefficients using a wide range of azimuths. Equations (3) also show that for small incidence angles $PS$ reflection coefficients do not depend on the parameters $\epsilon^{(1)}$, $\epsilon^{(2)}$ and $\delta^{(3)}$.

## Discussion of general expressions

It can be shown that the equations (2) and (3) reduce to the expressions derived by Rüger (1996) for the vertical symmetry planes of HTI media, if we set $\psi = 0^\circ$ or $\psi = 90^\circ$, and $\kappa = 0$ (notice that the polarization angle $\Phi$ is equal to $\psi$ for these azimuths).

General equations (2) and (3) suggest that application of $PS$-wave reflection coefficients in AVO analysis will be more problematic than that of $R_{PP}$ coefficients. The first apparent complication comes from the fact that the final equations (3) include both the incidence and reflection phase angles $i$ and $j$ related by Snell's law in anisotropic media. This inconvenience, however, can be easily overcome by using the weak-contrast, weak-anisotropy assumption. The term $\cos^{-1} j$ in equations (3) can be well-approximated by

$$\frac{1}{\cos j} \approx \frac{1}{\sqrt{1 - \frac{\bar{\beta}^2}{\bar{\alpha}^2}\sin^2 i}} \approx 1 + \frac{1}{2}\frac{\bar{\beta}^2}{\bar{\alpha}^2}\sin^2 i. \tag{5}$$

Numerical tests show that such an approximation works well for a whole set of $\bar{\alpha}/\bar{\beta}$ ratios, and for incidence angles corresponding to the offset to depth ratios conventionally used in AVO.

The second complication is due to the non-linearity of the polarization angle $\Phi$ in equations (2). Clearly, $\Phi$ should be eliminated in coefficients $R_{PS_1}$ and $R_{PS_2}$ in order to make them useful for practical applications. Fortunately, this can be done even for general anisotropy. According to equations (2), the elimination is achieved by a projection of the $R_{PS_1}$ and $R_{PS_2}$ coefficients (corresponding to the directions of the $PS_1$- and $PS_2$-wave polarization vectors) onto the $SV$ and $SH$ directions (defined by incidence and azimuth angles $i$ and $\psi$). Of course, the projection would fail for strongly anisotropic media, but equations (2) should not be used in such cases. The polarization vectors of the reflected $PS_1$- and $PS_2$ waves can be obtained from Alford-type polarization analysis. Such an analysis must always precede the $PS$-wave AVO analysis in order to extract $PS$-wave amplitudes from data.

Linear combinations of the $R_{PS_1}$ and $R_{PS_2}$ projections described above yield the components $R_{PSV}$ and $R_{PSH}$ which are linear in elastic medium parameters, and can be treated in a similar fashion as $P$-wave reflection coefficients discussed previously in the literature. Naturally, a relatively simple (linear) joint inversion of $R_{PSV}$, $R_{PSH}$ and $R_{PP}$ for medium parameters may provide more complete and more stable solution.

Finally, it should be emphasized that $R_{PS_1}$ and $R_{PS_2}$ approximations should not be used in the vicinity of $S$-wave singularities where plane-wave reflection coefficients fail to adequately represent the amplitude signatures.

## Numerical examples

Here, I compute the reflection coefficients $R_{PS_1}$ and $R_{PS_2}$ for two different models: VTI over HTI (model 1) and orthorhombic over orthorhombic (model 2). The upper halfspace represents the incidence medium. In model 2, the vertical symmetry planes of the reflecting orthorhombic halfspace are rotated by the angle $\kappa = 30^\circ$ with respect to the symmetry planes of the incidence medium. The components $R_{PSV}$ and $R_{PSH}$ of the exact reflection coefficients have been obtained by the projection of the exact $PS$ reflection coefficients described above. Then the absolute error is computed as the difference between the exact coefficients and the approximate values determined from equations (3).

Model 1 (Figure 1) shows a good agreement between the exact and approximate expressions for the whole azimuthal range and up to relatively large incidence angles for both $R_{PSV}$ and $R_{PSH}$. For the more complicated model 2 (Figure 2), $R_{PSV}$ is still approximated very well up to the incidence angle 35°. The accuracy of $R_{PSH}$, however, is much lower. Except for large azimuths ($85^\circ - 90^\circ$), the absolute error is comparable to (and even higher than) the values of the exact $R_{PSH}$ even for small incidence angles. This should not be surprising since in weakly anisotropic media $R_{PSH}$ is expected to be small and is probably less stable than $R_{PSV}$. However, it also suggests that the $R_{PSH}$ approximations should not be used for quantitative estimates of the medium parameters in more complicated models.

## Conclusion

The expressions for displacement $PS$-wave reflection coefficients given here can be applied for a plane interface separating two arbitrarily anisotropic halfspaces. Similarly to existing $R_{PP}$ approximations, their applicability is limited to weakly anisotropic media and small contrasts of the medium parameters across the interface.

As expected, the formulae for $R_{PS_1}$ and $R_{PS_2}$ coefficients are more complicated than those for $R_{PP}$. Also, even the first-order approximations for $PS$ reflection coefficients are not purely linear functions of the medium parameters. In order to extract and analyze the linear components, it is necessary to project the displacements of the $PS_1$ and $PS_2$ waves onto the $SV$ and $SH$ directions. Thus, the analysis of $PS$ reflection coefficients is more involved than that for $PP$ coefficients and requires high-quality data.

The final expressions can be applied in the linear inversion for the anisotropic parameters, in particular in combination with $R_{PP}$ reflection coefficients. As suggested by numerical tests it is preferable to use the $R_{PSV}$ component. However, due to its sensitivity to the azimuthal variation, the component $R_{PSH}$ can be used in detecting of the azimuths of the symmetry planes in HTI and orthorhombic media.

## Acknowledgments

I thank Ilya Tsvankin and Vladimir Grechka for their valuable advise during this research and for their suggestions in improving the manuscript. My special thanks to Ivan Pšenčík for many stimulating discussions.

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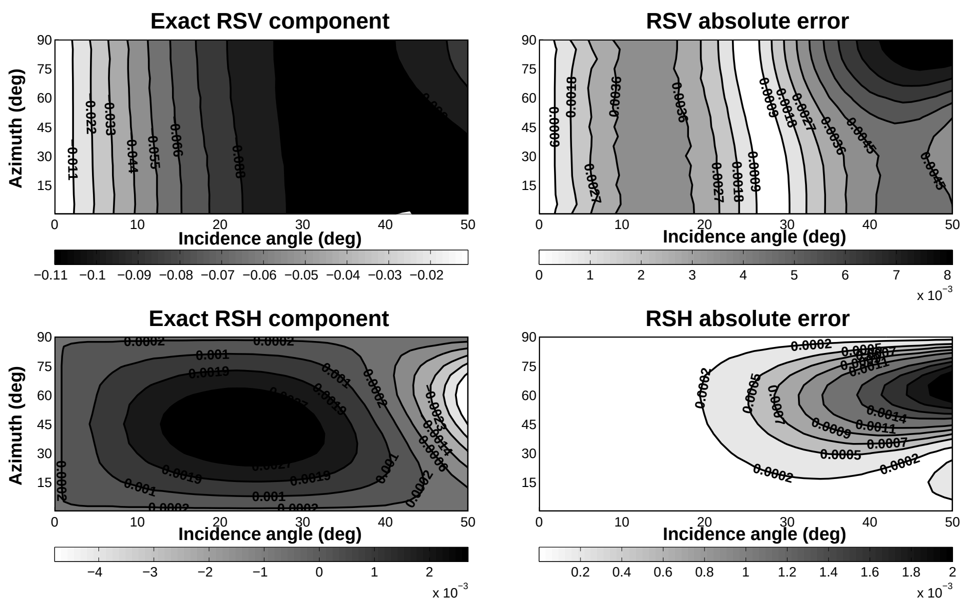


**Figure 1.** Model 1: components $R_{PSV}$ and $R_{PSH}$ of the exact $PS$-wave reflection coefficients for an VTI/HTI interface (left-hand side), and corresponding absolute errors (right-hand side), computed as functions of the incidence angle and azimuth. The absolute error is defined as the difference between the exact and approximate components [given by equations (3)]. The medium parameters are as follows: VTI incidence halfspace: $V_{P0} = 4.35\ km/s$, $V_{S0} = 2.57\ km/s$, $\epsilon = 0.09$, $\delta = 0.16$, $\gamma = 0.11$ and $\rho = 2.5$; HTI reflecting halfspace: $V_{P0} = 4.93\ km/s$, $V_{S0} = 2.89\ km/s$, $\epsilon^{(V)} = -0.11$, $\delta^{(V)} = -0.12$, $\gamma^{(V)} = -0.05$ and $\rho = 2.7$. Both velocities $V_{P0}$ correspond to $\sqrt{A_{33}}$ and $V_{S0}$- to $\sqrt{A_{44}}$.

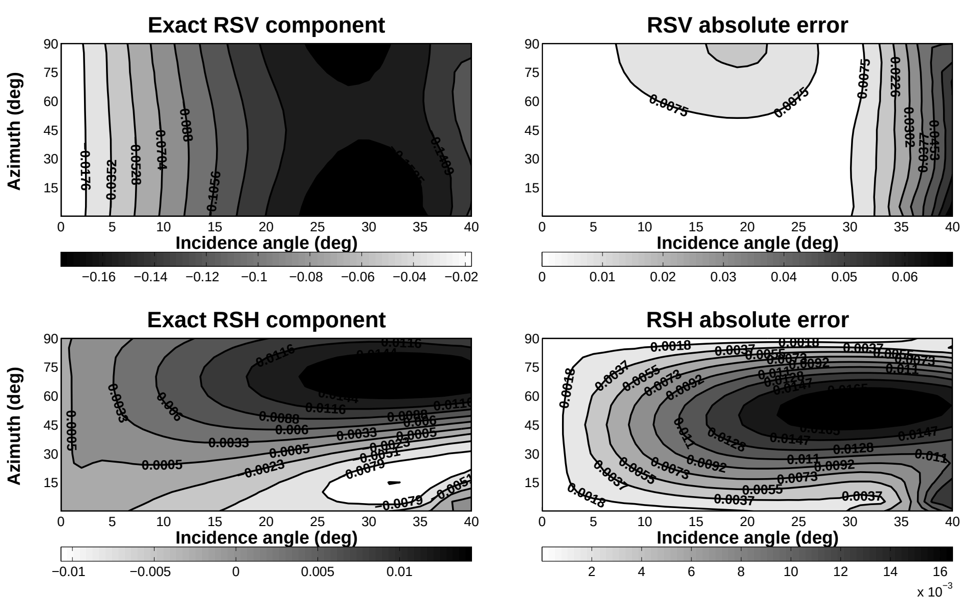


**Figure 2.** Model 2: components $R_{PSV}$ and $R_{PSH}$ of the exact $PS$-wave reflection coefficients for an orthorhombic/orthorhombic interface (left-hand side), and the corresponding absolute errors (right-hand side), computed as functions of the incidence angle and azimuth. The absolute error is defined as in Figure 1. The vertical planes of the two halfspaces are rotated by an angle $\kappa = 30°$ with respect to each other. The medium parameters are as follows: incidence halfspace: $V_{P0} = 2.85\ km/s$, $V_{S0} = 1.64\ km/s$, $\epsilon^{(1)} = 0.33$, $\epsilon^{(2)} = 0.26$, $\delta^{(1)} = 0.08$, $\delta^{(2)} = -0.08$, $\delta^{(3)} = -0.11$, $\gamma^{(1)} = 0.18$, $\gamma^{(2)} = 0.05$ and $\rho = 2.2$; reflecting halfspace: $V_{P0} = 3.5\ km/s$, $V_{S0} = 2.15\ km/s$, $\epsilon^{(1)} = 0.25$, $\epsilon^{(2)} = 0.15$, $\delta^{(1)} = 0.05$, $\delta^{(2)} = -0.1$, $\delta^{(3)} = 0.15$, $\gamma^{(1)} = 0.28$, $\gamma^{(2)} = 0.15$ and $\rho = 2.5$. Both velocities $V_{P0}$ correspond to $\sqrt{A_{33}}$ and $V_{S0}$- to $\sqrt{A_{44}}$.