Introduction

True-amplitude one-way wave equations (Zhang, 1993; Zhang et al. 2003) yield high quality solutions for forward modeling problems and inversion. A shortcoming of these equations is that they yield poor results *near* and fail *at* horizontal propagation. In theory, the separation into equations for up-going and down-going wave equations has a pathology in the neighborhood of horizontal propagation where the terminology, upward and downward, loses meaning.
Zhang et al. (2007) derived a new equation from the two-way wave equation. It involves the first derivative term in time, but can handle all the possible wave phenomena, including the turning waves, and provides a new way of doing reverse-time migration, called "Explicit Marching" (EM) method. Unlike the conventional finite-difference methods, EM does not suffer from stability and numerical dispersion problems. Since the pseudo-differential operator involved in EM is non-singular, the new equation can be numerically solved efficiently.
Here, we analyze this one-way like equations in *time* from a theoretical point of view. The methodology we use is similar to that we have been using to analyze the one-way wave equation in space. We apply asymptotic ray theory in time using smoothness of the functions as a basis for ordering one-way wave equations in *time*. We show that the eikonal equation for traveltime and the transport equation for leading order amplitude in the asymptotic solution of our new wave equation is equivalent to the eikonal and transport equations for the two-way wave equation. We further demonstrate through analysis of the Green's function that source problems for the two-way wave equations become initial or final value problems for the one-way wave equations. We further demonstrate that boundary data for the two-way wave equations become source problems for the one-way wave equation. We provide formulas for forward modeling of Kirchhoff-approximate data on a surface at depth and for back-projection of observed data at receivers. Thus, we provide the necessary tools for inversion—true-amplitude wave equation migration inversion. We can only report on this list of results here. Details of the theory are available in the preprint, Bleistein et al (2008).

One-way wave equations in time and progressing waves

The one-way wave equations that we will discuss below are

$$\mathcal{L}_{\pm} W = \frac{1}{\mathrm{v}}\frac{\partial W}{\partial t} \pm \sqrt{\nabla^2} W = 0. \tag{1}$$

We introduce a symbolic correspondence for the square root of the Laplacian appearing here:

$$\nabla \leftrightarrow i\boldsymbol{k}, \qquad \boldsymbol{k} = (k_x, k_y, k_z), \tag{2}$$

and

$$\sqrt{(\nabla)^2} \leftrightarrow \sqrt{(i\boldsymbol{k})^2} = ik. \tag{3}$$

If the wave speed v were constant, we could think of $\boldsymbol{k}$ as just the components of the spatial Fourier transform. In fact, this is still reasonable when v is not constant, but only in a leading order asymptotic sense for "large" values of $|\boldsymbol{k}| = k$. By imitating the methodology of Zhang et al [2003], we can show that

$$ik = i\,|k_z|\,[I(\boldsymbol{k}) + 1], \;\; I(\boldsymbol{k}) = \frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1 - s^2 k_T^2}}{k_z^2 + s^2 k_T^2} ds, \qquad \boldsymbol{k}_T = (k_x, k_y). \tag{4}$$

Both the factor $i\,|k_z|$ and the integral $I(\boldsymbol{k})$ require interpretation as pseudo-differential operators. We interpret $i\,|k_z|$ as follows.

$$k_z > 0,\;\; i\,|k_z| = ik_z \leftrightarrow \frac{\partial}{\partial z};\, k_z < 0,\;\; i\,|k_z| = -ik_z \leftrightarrow -\frac{\partial}{\partial z}. \tag{5}$$

As for the integral $I(k)$ in equation (4), the multiplication in the numerator of the integrand merely means differentiation. The symbolic operator in the denominator is interpreted as the *inverse* of a differential operator; that is, convolution with a Green's function or solution to an appropriate differential equation. Thus, we introduce an auxiliary function $W_T(x, t, s)$ that satisfies the differential equation,

$$\frac{\partial^2 W_T}{\partial z^2} + s^2 \nabla_T^2 W_T = \nabla_T^2 W, \;\; \nabla_T = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right). \tag{6}$$

This is exactly what we need to interpret $I(k)W$ with I and ik defined below equation (4). We now have a compound of $\pm$ signs due to the signs in the differential equation (1) and the signs of (5). Therefore, below we proceed only with $\mathcal{L}_+$ of equation (1) and then state results for $\mathcal{L}_-$, allowing $\pm$ to correspond to the signs of $i\,|k_z|$ in equation (5). Then, for the forward going wave equation (1), we write

$$\mathcal{L}_+ W \;\; = \;\; \frac{1}{\mathrm{v}}\frac{\partial W}{\partial t} \pm \frac{\partial W}{\partial z} \pm \frac{\partial}{\partial z}\frac{1}{\pi}\int_{-1}^{1} \sqrt{1-s^2}\, W_T\, ds = 0, \tag{7}$$

Here, the upper sign corresponds to downgoing waves and the lower sign corresponds to upgoing waves. Those signs are opposite ($\mp$) for $\mathcal{L}_-$).
We introduce a sequence of progressing wave functions $F_0[\tau(x) - t], F_1[\tau(x) - t], \ldots$, with the property that

$$F'_{n+1} = F_n, \;\; n = 0, 1, 2, \ldots\;. \tag{8}$$

Here the prime $\{'\}$ means derivative with respect to total argument of the function. With this definition, each wave function is smoother than its predecessor. We then assume that the solution W can be written as a series in these functions as follows.

$$W(x,t) = B_0(x) F_0[\tau(x) \mp t] + B_1(x) F_1[\tau(x) \mp t] + \ldots\;. \tag{9}$$

For leading order asymptotics, that is, to determine the governing equations for τ and B_0 we will not need any further terms in the series.
The procedure for asymptotic analysis of the one-way wave equations (1) is as follows. Write down progressing wave series for both W and W_T. Use the spatial differential equation (6) to determine the coefficients of the series for W_T in terms of the coefficients of the series for W and functions of s. Substitute the series for W_T into the integral in the one-way wave equations (7) just above, and carry out the integrals with respect to s. What results is a progressing wave series totally in terms of the traveltime and amplitudes of the progressing wave representation for W. The most singular part of this equation will be the coefficient of F'_0 and the next order will be the coefficient of F_0, itself. Setting those two coefficients equal to zero yields the eikonal equation and transport equation that we seek, namely,

$$\sqrt{p^2} = 1/\mathrm{v} \tag{10}$$

and

$$\nabla \cdot \left[\mathrm{v} B_0^2 p\right] = 0, \tag{11}$$

where $p = \nabla\tau$. Our care with signs has led to the same eikonal equation for upward and downward propagating waves, thereby not distinguishing them by direction. The transport equation conserves energy in ray tubes as it should. In contrast to the one-way equations in depth, there is no need for an additional zeroeth order pseudo-differential operator in this wave equation to achieve this conservation. Exactly the same eikonal and transport equation are obtained for the operator $\mathcal{L}_-$.

Analytical/numerical solution technique

The analytical scheme used in the previous section to derive the asymptotic solutions of the one-way wave equation (1) is not a viable solution technique. Instead, we use a spectral method. To this end, we introduce the spatial Fourier transform,

$$\mathcal{F}[W(x,t)] = \tilde{W}(k,t) = \int_{-\infty}^{\infty} W(x,t)\exp\{-ik\cdot x\}d^3x, \tag{12}$$

and rewrite the one-way wave equations (1) as

$$\frac{1}{\mathrm{v}}\frac{\partial W}{\partial t} \pm \mathcal{F}^{-1}[ik\mathcal{F}[W(x,t)] = 0. \tag{13}$$

That is, we effect the square root operator by computing the forward Fourier transform of W, multiplying by ik, and computing the inverse transform.
The Green's functions are solutions of the initial value problems

$$\mathcal{L}_\pm G_\pm = \frac{1}{\mathrm{v}}\frac{\partial G_\pm}{\partial t} \pm \sqrt{\nabla^2}G_\pm = 0,\ \ G_\pm(x,0) = \mp\frac{\mathrm{v}(0)}{ik}\delta(x). \tag{14}$$

For homogeneous media, the solutions of these equations are

$$G_\pm(x,t) \;\; = \;\; \pm H(\pm t)\frac{1}{4\pi r}\left\{\delta(r/\mathrm{v}-t) - \delta(r/\mathrm{v}+t) + \frac{i}{\pi}\left[-\frac{1}{r/\mathrm{v}-t} + \frac{1}{r/\mathrm{v}+t}\right]\right\} \tag{15}$$

The real parts of $G_\pm$ are the causal/anti-causal Green's functions of the two-way wave equation. The imaginary parts here yield the analytic solutions that we claimed as being typical of the solution of one-way wave equations. The complex values arise from the implementation of the operator ik. This operator yields analytic solutions whenever it is applied. Thus, the solution obtained from a problem with real data will be the analytic solution with its real part being the solution that we really want.
We can show that the adjoints (†) of the differential operators are

$$\mathcal{L}^\dagger_\pm W = -\mathcal{L}_\mp W = -\left\{\frac{\partial W}{\partial t} \mp [ikW]\right\}. \tag{16}$$

Using these adjoints, we can obtain a Green's identity,

$$\int dV' \int_{t_-}^{t_+} dt' \left\{U\mathcal{L}_\pm W - W\mathcal{L}^\dagger_\pm U\right\}(x',t') = \int \frac{dV'}{\mathrm{v}(x')}U(x',t')W(x',t')\Big|_{t_-}^{t_+}. \tag{17}$$

This identity allows us to model downward continuation of observed surface data, $D(x',y,t')$ at $z'=0$ as

$$U(x,t) = -\int_{z'=0} \frac{dx'dy'}{\mathrm{v}(x')}\int_t^{t_+} dt' D(x',y',t')[ikG_-(x',x,t',t)], \tag{18}$$

with the source of the delta function here being at (x,t) rather than at $(0,0)$ as in differential equations (14) for the Green's functions.
The Green's identity (17) also allows us to model upward propagation from a reflector S when we use Kirchhoff-approximate boundary data:

$$U(x,t) = \int_S dS' \int_{t_-}^{t} dt' D(x',t')[ikG_+(x',x,t',t)]. \tag{19}$$

Detailed derivations of all results presented here can be found in Bleistein et al (2008).

Applications to reverse time migration

Equations (1) have been applied to reverse time migration and leads to a new algorithm called Explicit Marching (EM) method (Zhang et al., 2007). Unlike the conventional finite-difference algorithm, it is guaranteed numerically stable and dose not suffer from numerical dispersion problems. To show how the EM works, we apply it to the 2004 BP 2D data set (Billette and Brandsberg-Dahl, 2005). This is a high quality dataset generated by finite-difference modeling with shot spacing of 50m, receiver spacing of 12.5m and 15000m maximum offset. For such a data set, the EM method handles complex velocity fairly well and gives good delineation of the salt boundaries (Figure 1) especially the steeply dipping salt flanks and the overturned salt edges, which require high angle propagation or turning waves to image clearly.

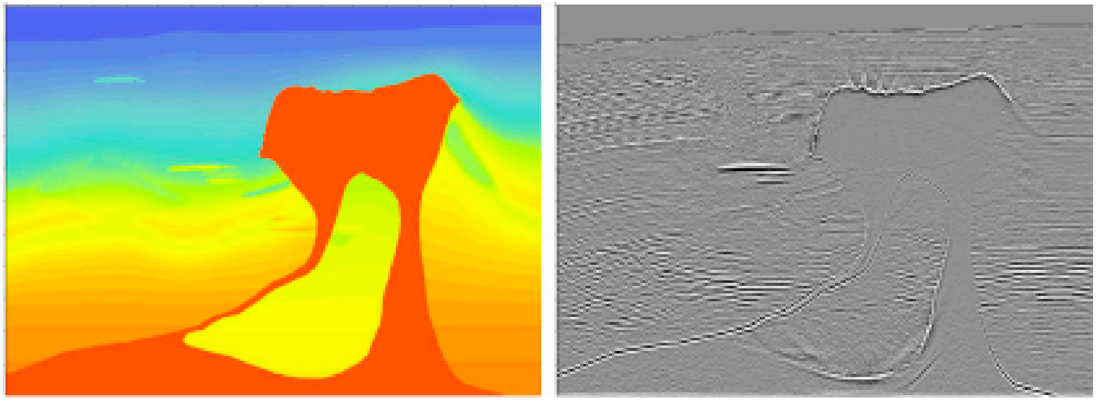

Fig. 1: BP 2004 velocity model (left) and its reverse-time migration images with two-step explicit marching method (right).

Conclusions

We have proved that the solutions of one-way wave equations in time asymptotically approximate the solutions to the two-way wave equation to leading order for forward or reverse time propagation. This provides a solid theoretical base for Explicit Marching algorithm. We present initial value problem for Green's functions. The Kirchhoff approximation of asymptotic ray theory in frequency domain applies to progressing waves in time domain. A Green's identity relating solutions of the one-way wave equations and their adjoint is derived. This allows us to develop Kirchhoff integral representations from propagation of surface data into the Earth and the propagation of reflection data to the upper surface, which provides the necessary tools for deriving an asymptotic solution of the seismic inverse problem.

References

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